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SOME CHARACTERIZATIONS ON CRITICAL METRICS FOR QUADRATIC CURVATURE FUNCTIONS

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ABSTRACT. Under some integral conditions, we classify closed n-dimensional manifolds of which the metrics are critical for quadratic curvature functions. Moreover, under some curvature conditions, we also obtain that a critical metric must be Einstein.

1. Introduction

Let $\mathscr{M}_1(M^n)$ be the space of equivalence classes of smooth Riemannian metrics of volume one on closed Riemannian manifold M^n, $n \geq 3$. It is well known that Einstein metrics are critical for the Einstein-Hilbert functional

$$\mathcal{H} = \int_M R dV$$

on $\mathscr{M}_1(M^n)$. Then, it is natural to study canonical metrics which arise as solutions of the Euler-Lagrange equations for more general curvature functionals or even high order curvature functionals. In [2], Berger commenced the study of Riemannian functionals which are quadratic in curvature functionals (see [3, Chapter 4] and [23] for surveys). In particular, if $n > 4$, it is not true that Einstein metrics are always critical points for this functional on $\mathscr{M}_1(M^n)$. Therefore, problems arise: when are Einstein metrics critical points for this quadratic curvature functional? In [6], Catino considers the curvature functional

$$\mathcal{F}_t = \int_M |Ric|^2 \, dv + t \int_M R^2 \, dv \tag{1.1}$$

defined for some constant $t \in \mathbb{R}$ (with $t = -\infty$ formally corresponding to the functional $\int_M R^2 \, dv$). Here Ric and R denote the Ricci curvature and the scalar curvature, respectively. It has been observed in [3] that every Einstein metric is critical for $\mathcal{F}_t$ on $\mathscr{M}_1(M^n)$, for all $t \in \mathbb{R}$ (this can be obtained by virtue of the formula (2.3)). Of course, there exist critical metrics which are not necessarily Einstein (for instance, see [3, Chapter 4] and [20]). It is natural to ask under what conditions a critical metric for $\mathcal{F}_t$ must be Einstein. For example, one can assume

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some curvature conditions or some integral conditions. For some development in this direction, see [1,10,11,13,14,19,24] and the references therein.

In this paper, we will give some new characterizations on critical metrics for $\mathcal{F}_t$ on $\mathscr{M}_1(M^n)$. Throughout this paper, we denote by E the traceless Ricci tensor. Our main results are as follows:

Theorem 1.1. *Let M^n be a closed manifold of dimension $n \geq 3$ with positive scalar curvature and g be a critical metric for $\mathcal{F}_t$ on $\mathscr{M}_1(M^n)$. Then*

(i) for $n = 3$, there exists $\varepsilon_0 > 0$ ($\varepsilon_0 \approx 0.3652$) such that for $-\varepsilon_0 \leq t < -\frac{1}{6}$ and

$$\frac{1+6t}{3}R + \frac{4}{\sqrt{6}}|E| < 0,$$

we have that M^3 is of constant positive sectional curvature.

(ii) For $n \geq 5$ there exist $\varepsilon_n > 0$ and $\eta_n > 0$ such that for $-\frac{1}{2} < t < -\varepsilon_n$ or $-\eta_n < t < -\frac{n}{4(n-1)}$, if

$$\frac{2(n-2)+2n(n-1)t}{n(n-1)}R + \frac{4}{\sqrt{n(n-1)}}|E| + \sqrt{\frac{2(n-2)}{n-1}}|W| < 0, \tag{1.2}$$

we have that M^n is Einstein. In particular, if the Weyl curvature tensor $W = 0$, then M^n is also of constant positive sectional curvature.

Next, we will give the following integral inequality under the condition that the curvature tensor is harmonic which is stronger than that the scalar is constant.

Theorem 1.2. *Let M^n be a closed manifold with harmonic curvature tensor. If g is a critical metric for $\mathcal{F}_t$ on $\mathscr{M}_1(M^n)$ with positive scalar curvature and $n \neq 4$, then*

$$\begin{aligned}\int_M \Big[&\frac{2[(n-2)+n(n-1)t]}{n(n-1)}R + \frac{4}{\sqrt{n(n-1)}}|E| \\ &+ \sqrt{\frac{2(n-2)}{n-1}}\,|W|\Big]|E|^{\frac{n-2}{n}} \geq 0\end{aligned} \tag{1.3}$$

and equality occurs if and only if

(1) either M^n is Einstein;

(2) or M^n is isometrically covered by $\mathbb{R}^1 \times \mathbb{S}^{n-1}$ with a product metric. In this case, we have $t = -\frac{1}{n-1}$ and $R = \sqrt{n(n-1)}|E|$.

Remark 1.1. Catino proved (see Theorem 1.5 in [6]) that for $n = 3$, if $t \in [-\frac{1}{3}, -\frac{1}{6})$ and

$$\frac{1+6t}{3}R + \frac{4}{\sqrt{6}}|E| < 0,$$

then M^3 is of constant positive sectional curvature. We note that $[-\frac{1}{3}, -\frac{1}{6}) \subset [-\varepsilon_0, -\frac{1}{6})$. Thus, the result for $n = 3$ in Theorem 1.1 generalizes Theorem 1.5 in [6]. On the other hand, the result for $n \geq 5$ in Theorem 1.1 is new.

2. Proof of results

First, we generalize Lemma 5.1 in [6] into arbitrary dimension.

Proposition 2.1. *Let M^n be a closed manifold of dimension $n \geq 3$ with nonnegative scalar curvature. If g is a critical metric for $\mathcal{F}_t$ on $\mathscr{M}_1(M^n)$, then*

$$\begin{aligned}\int_M \Big[&\frac{1}{2}\langle\nabla|E|^2,\nabla R\rangle + R|\nabla E|^2 - \frac{(n-2)(1+2t)}{2n}R|\nabla R|^2 \\ &- (1+2t)E(\nabla R,\nabla R)\Big]\,dv \\ \leq \int_M & R|E|^2\Big[\frac{2[(n-2)+n(n-1)t]}{n(n-1)}R + \frac{4}{\sqrt{n(n-1)}}|E| + \sqrt{\frac{2(n-2)}{n-1}}|W|\Big]\,dv.\end{aligned} \tag{2.1}$$

In particular, if the scalar curvature R in (2.1) is a positive constant and $n \neq 4$, then

$$\begin{aligned}\int_M |\nabla E|^2\,dv \leq \int_M |E|^2\Big[&\frac{2[(n-2)+n(n-1)t]}{n(n-1)}R + \frac{4}{\sqrt{n(n-1)}}|E| \\ &+ \sqrt{\frac{2(n-2)}{n-1}}|W|\Big]\,dv,\end{aligned} \tag{2.2}$$

and equality occurs if and only if

(1) either M^n is Einstein;

(2) or M^n is isometrically covered by $\mathbb{R}^1 \times \mathbb{S}^{n-1}$ with a product metric. In this case, we have $t = -\frac{1}{n-1}$ and $R = \sqrt{n(n-1)}|E|$.

Proof. It has been shown in [6, Proposition 2.1] by Catino that a metric g is critical for $\mathcal{F}_t$ on $\mathscr{M}_1(M^n)$ if and only if it satisfies the following equations:

$$\begin{aligned}\Delta E_{ij} =& (1+2t)\nabla^2_{ij}R - \frac{1+2t}{n}(\Delta R)g_{ij} - 2R_{ikjl}E_{kl} \\ &- \frac{2+2nt}{n}RE_{ij} + \frac{2}{n}|E|^2 g_{ij},\end{aligned} \tag{2.3}$$

$$[n+4(n-1)t]\Delta R = (n-4)[|Ric|^2 + tR^2 - \lambda], \tag{2.4}$$

where $\lambda = \mathcal{F}_t(g)$. Recall that for $n \geq 3$, the Weyl curvature tensor is defined by

$$\begin{aligned}W_{ijkl} =& R_{ijkl} - \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}) \\ &+ \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}) \\ =& R_{ijkl} - \frac{1}{n-2}(E_{ik}g_{jl} - E_{il}g_{jk} + E_{jl}g_{ik} - E_{jk}g_{il}) \\ &- \frac{R}{n(n-1)}(g_{ik}g_{jl} - g_{il}g_{jk}).\end{aligned} \tag{2.5}$$

Hence, it is easy to see from (2.3) that

$$
\begin{aligned}
\frac{1}{2}\Delta|E|^2 =&|\nabla E|^2 + E_{ij}\Delta E_{ij}\\
=&|\nabla E|^2 + (1+2t)E_{ij}\nabla^2_{ij}R - 2R_{ikjl}E_{kl}E_{ij} - \frac{2+2nt}{n}R|E|^2\\
=&|\nabla E|^2 + (1+2t)E_{ij}\nabla^2_{ij}R - \frac{2(n-2)+2n(n-1)t}{n(n-1)}R|E|^2\\
&+\frac{4}{n-2}E_{ij}E_{jk}E_{ki} - 2W_{ikjl}E_{kl}E_{ij}.
\end{aligned}
\tag{2.6}
$$

Multiplying both sides of (2.6) with R and integrating it, we obtain

$$
\begin{aligned}
&\int_M \Big[\frac{1}{2}\langle\nabla|E|^2,\nabla R\rangle + R|\nabla E|^2 - \frac{(n-2)(1+2t)}{2n}R|\nabla R|^2\\
&\quad - (1+2t)E(\nabla R,\nabla R)\Big]\,dv\\
&=\int_M \Big[\frac{2[(n-2)+n(n-1)t]}{n(n-1)}R^2|E|^2 - \frac{4}{n-2}RE_{ij}E_{jk}E_{ki} + 2RW_{ikjl}E_{kl}E_{ij}\Big]\,dv,
\end{aligned}
\tag{2.7}
$$

where we used the second Bianchi identity $E_{ij,j} = \frac{n-2}{2n}R_{,i}$.

We recall the following inequality which was first proved by Huisken (cf. [17, Lemma 3.4]):

$$
|W_{ikjl}E_{ij}E_{kl}| \le \sqrt{\frac{n-2}{2(n-1)}}|W||E|^2,
\tag{2.8}
$$

and

$$
E_{ij}E_{jk}E_{ki} \ge -\frac{n-2}{\sqrt{n(n-1)}}|E|^3,
\tag{2.9}
$$

with equality in (2.9) at some point $p \in M$ if and only if E can be diagonalized at p and the eigenvalue multiplicity of E is at least $n-1$. If $|E| \neq 0$ and the equality in (2.9) occurs, then $n-1$ of eigenvalues which are equal must be positive (see also [22] or Lemma 5.1 in [14]). Therefore, for $R \ge 0$, we have

$$
\begin{aligned}
&\frac{2(n-2)+2n(n-1)t}{n(n-1)}R^2|E|^2 - \frac{4}{n-2}RE_{ij}E_{jk}E_{ki} + 2RW_{ikjl}E_{kl}E_{ij}\\
&\le R|E|^2\Big[\frac{2[(n-2)+n(n-1)t]}{n(n-1)}R + \frac{4}{\sqrt{n(n-1)}}|E| + \sqrt{\frac{2(n-2)}{n-1}}|W|\Big].
\end{aligned}
\tag{2.10}
$$

The desired estimate (2.1) follows from (2.7) and (2.10) immediately.

If the equality in (2.1) occurs, then the two inequalities (2.8) and (2.9) now must both be equalities. Hence, as stated in the lines following (2.9), E has, at each point p, an eigenvalue of multiplicity $n-1$ or n. For $n=3$, it is well known that $W=0$ and (2.8) is an equality. When $n \ge 4$, writing $E_{ij} = ag_{ij} + bv_iv_j$ at p, with two scalars a, b and a vector v, we see that the left-hand side of (2.8) is zero at p. This shows that M^n, $n \ge 4$, must be conformally flat or Einstein due to the equality in (2.8).

In particular, if the scalar curvature R is a positive constant, the inequality (2.2) follows from (2.1) directly. Furthermore, if the equality in (2.2) occurs and M^n is not Einstein (that is, $E \neq 0$), we have $W = 0$ according to the arguments above, which shows from (2.2) that

$$\int_M |\nabla E|^2 \, dv = \int_M |E|^2 \Big[\frac{2[(n-2)+n(n-1)t]}{n(n-1)} R + \frac{4}{\sqrt{n(n-1)}} |E|\Big] \, dv. \tag{2.11}$$

When $n = 3$ or $n \geq 5$, we have from (2.4) that $|Ric|^2$ is constant and hence $|E|^2$ is also constant. Therefore, we obtain that eigenvalues of Ricci curvature are constant from E a traceless tensor and hence M^n has parallel Ricci curvature. In particular, in this case, $\nabla E = 0$ and from the de Rham decomposition theorem, M^n splits as a product of two Einstein manifolds $N^1 \times N^{n-1}$, where N^{n-1} is an Einstein manifold. Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of Ricci curvature and $\lambda_2 = \cdots = \lambda_n$. Since one factor N^1 of $N^1 \times N^{n-1}$ has dimension $n = 1$, we have $\lambda_1 = 0$. In particular, we have

$$(\lambda_2 + \cdots + \lambda_n)^2 = (n-1)(\lambda_2^2 + \cdots + \lambda_n^2) \tag{2.12}$$

which shows that $R^2 = (n-1)|Ric|^2$ and hence $R = \sqrt{n(n-1)}|E|$. In this case, (2.11) becomes

$$\frac{2n[1+(n-1)t]}{\sqrt{n(n-1)}} \int_M |E|^3 \, dv = 0, \tag{2.13}$$

which shows $t = -\frac{1}{n-1}$. The proof of Proposition 2.1 is completed. □

Remark 2.1. We can also obtain $t = -\frac{1}{n-1}$ by the relationship between eigenvalues of Ricci curvature. In fact, using $E_{ij} = R_{ij} - \frac{R}{n} g_{ij}$, (2.3) can be written as (see the formula (2.1) in [6])

$$\frac{1}{n}(|Ric|^2 + tR^2) g_{ij} - R_{ikjl}R_{kl} - tRR_{ij} = 0. \tag{2.14}$$

Since the Ricci curvature is parallel, from the Ricci identity, we have

$$0 = R_{ik,jp} - R_{ik,pj} = R_{lijp}R_{kl} + R_{lkjp}R_{il},$$

which gives $R_{ikjl}R_{kl} = R_{ik}R_{jl}$ and (2.14) becomes

$$\frac{1}{n}(|Ric|^2 + tR^2) g_{ij} - R_{ik}R_{jl} - tRR_{ij} = 0.$$

In particular, every eigenvalue λ of the Ricci curvature satisfies

$$\frac{1}{n}(|Ric|^2 + tR^2) - \lambda^2 - tR\lambda = 0,$$

which is equivalent to

$$n\lambda^2 + ntR\lambda - (|Ric|^2 + tR^2) = 0.$$

Solving this quadratic equation with respect to λ yields

$$\lambda = \frac{-ntR \pm \sqrt{(n^2t^2 + 4nt)R^2 + 4n|Ric|^2}}{2n}.$$

Then we have

$$\begin{cases} \lambda_1 = \frac{-ntR - \sqrt{(n^2t^2+4nt)R^2+4n|Ric|^2}}{2n}, \\ \lambda_2 = \cdots = \lambda_n = \frac{-ntR + \sqrt{(n^2t^2+4nt)R^2+4n|Ric|^2}}{2n}, \end{cases} \tag{2.15}$$

or

$$\left\{\begin{array}{l}\lambda_1=\frac{-ntR+\sqrt{(n^2t^2+4nt)R^2+4n|Ric|^2}}{2n},\\ \lambda_2=\cdots=\lambda_n=\frac{-ntR-\sqrt{(n^2t^2+4nt)R^2+4n|Ric|^2}}{2n}.\end{array}\right. \tag{2.16}$$

By a direct computation, by using (2.15) combining with $\lambda_1+\cdots+\lambda_n=R$ or (2.16) combining with $\lambda_1+\cdots+\lambda_n=R$, we can obtain $t=-\frac{1}{n-1}$ from $\lambda_1=0$. In this case, $R=\sqrt{n(n-1)}|E|$.

Next, we will prove Theorem 1.2.

Proof of Theorem 1.2. Using the second Bianchi identity, we have

$$R_{jkil,l}=R_{ij,k}-R_{ik,j}. \tag{2.17}$$

Hence, for the harmonic curvature tensor, we have that the Ricci curvature is Codazzi. Thus, we derive

$$R_{,i}=R_{jj,i}=R_{ij,j}=\frac{1}{2}R_{,i} \tag{2.18}$$

which shows that $R_{,i}=0$ and the scalar curvature is constant. In particular, the trace-less tensor E_{ij} is also a Codazzi tensor and satisfies the following sharp inequality (for a proof, see for instance [12], this inequality was first observed by Bourguignon [4]):

$$|\nabla E|^2\geq\frac{n+2}{n}|\nabla|E||^2. \tag{2.19}$$

Inserting (2.8), (2.9) and (2.19) into (2.6) yields

$$\begin{aligned}\frac{1}{2}\Delta|E|^2\geq&\frac{n+2}{n}|\nabla|E||^2-\frac{2[(n-2)+n(n-1)t]}{n(n-1)}R|E|^2\\&-\frac{4}{\sqrt{n(n-1)}}|E|^3-\sqrt{\frac{2(n-2)}{n-1}}|W||E|^2.\end{aligned} \tag{2.20}$$

Let

$$\Omega_0=\{p\in M:\ |E|\neq 0\}.$$

By virtue of Lemma 2.2 in [5] (or see [18, Theorem 1.8]), we have $\mathrm{Vol}(M\backslash\Omega_0)=0$. For any $\varepsilon>0$, we define $\Omega_\varepsilon=\{p\in M:|E|\geq\varepsilon\}$ and

$$f_\varepsilon(p)=\left\{\begin{array}{ll}|E|(p) & \text{if}\ \ p\in\Omega_\varepsilon;\\ \varepsilon & \text{if}\ \ p\in M\backslash\Omega_\varepsilon.\end{array}\right.$$

Then at the regular value ε of $|E|$, integration by parts gives

$$\begin{aligned}&\int_M\bigg(-\frac{1}{2}\Delta|E|^2+\frac{n+2}{n}|\nabla|E||^2\bigg)f_\varepsilon^{-\frac{n+2}{n}}\\&=-\frac{n+2}{n}\int_M\langle\nabla|E|,\nabla f_\varepsilon\rangle|E|f_\varepsilon^{-\frac{n+2}{n}-1}+\frac{n+2}{n}\int_M|\nabla|E||^2f_\varepsilon^{-\frac{n+2}{n}}\end{aligned} \tag{2.21}$$

which tends to the zero as $\varepsilon\to 0$, where in the last equality we used $f_\varepsilon=|E|$ on Ω_ε and $\nabla f_\varepsilon=0$ on $M\backslash\Omega_\varepsilon$. Multiplying both sides of inequality (2.20) by $f_\varepsilon^{-\frac{n+2}{n}}$

and applying (2.21), we have

$$
\begin{aligned}
0 \leq & \int_M \Big[\frac{2[(n-2)+n(n-1)t]}{n(n-1)} R|E|^2 + \frac{4}{\sqrt{n(n-1)}}|E|^3 \\
& + \sqrt{\frac{2(n-2)}{n-1}}\,|W||E|^2\Big] f_\varepsilon^{-\frac{n+2}{n}} \\
= & \int_M |E|^{\frac{n-2}{n}} \Big[\frac{2[(n-2)+n(n-1)t]}{n(n-1)} R + \frac{4}{\sqrt{n(n-1)}}|E| \\
& + \sqrt{\frac{2(n-2)}{n-1}}\,|W|\Big] |E|^{\frac{n+2}{n}} f_\varepsilon^{-\frac{n+2}{n}}.
\end{aligned} \tag{2.22}
$$

Taking $\varepsilon \to 0$ in (2.22), we have $|E|^{\frac{n+2}{n}} f_\varepsilon^{-\frac{n+2}{n}} \to 1$ a.e. on M and the desired estimate (1.3) follows. Then, we can obtain our results by using Proposition 2.1. □

Next, we will estimate the left-hand side of (2.1). By using some sharp inequalities, we can prove the following result:

Proposition 2.2. *Let M^n be a closed manifold of dimension $n \geq 3$ with positive scalar curvature and g be a critical metric for $\mathcal{F}_t$ on $\mathscr{M}_1(M^n)$. Then, for $n = 3$ and $t > -\frac{3}{8}$,*

$$
\begin{aligned}
& \int_M \frac{|\nabla E|^2}{R} \Big[R^2 - \frac{(5+16t)^2}{2(1+2t)(3+8t)}|E|^2\Big]\, dv \\
& \qquad \leq \int_M R|E|^2 \Big[\frac{1+6t}{3} R + \frac{4}{\sqrt{6}}|E|\Big]\, dv;
\end{aligned} \tag{2.23}
$$

for $n \geq 5$ and $-\frac{1}{2} < t < -\frac{n}{4(n-1)}$,

(2.24)
$$
\begin{aligned}
& \int_M \frac{|\nabla E|^2}{R} \Big\{R^2 + \frac{[3n-4+8(n-1)t]^2}{2(n-4)(1+2t)[n+4(n-1)t]}|E|^2\Big\}\, dv \\
& \leq \int_M R|E|^2 \Big[\frac{2(n-2)+2n(n-1)t}{n(n-1)} R + \frac{4}{\sqrt{n(n-1)}}|E| + \sqrt{\frac{2(n-2)}{n-1}}|W|\Big]\, dv.
\end{aligned}
$$

Proof. From the Bochner formula, we have

$$
\begin{aligned}
\frac{1}{2}\Delta|\nabla R|^2 =& |\nabla^2 R|^2 + \mathrm{Ric}(\nabla R, \nabla R) + \langle \nabla \Delta R, \nabla R\rangle \\
=& |\nabla^2 R|^2 + E(\nabla R, \nabla R) + \frac{1}{n} R|\nabla R|^2 + \langle \nabla \Delta R, \nabla R\rangle
\end{aligned}
$$

which gives

$$
\begin{aligned}
\int_M E(\nabla R, \nabla R)\, dv =& \int_M [-|\nabla^2 R|^2 + (\Delta R)^2 - \frac{1}{n} R|\nabla R|^2]\, dv \\
\leq & \frac{n-1}{n}\int_M (\Delta R)^2\, dv - \frac{1}{n}\int_M R|\nabla R|^2\, dv,
\end{aligned} \tag{2.25}
$$

where in the inequality of (2.25) we used the Cauchy inequality $|\nabla^2 R|^2 \geq \frac{1}{n}(\Delta R)^2$. On the other hand, from the traced equation of critical metrics (2.4) with $t \neq -\frac{n}{4(n-1)}$, one has

$$\begin{aligned}\int_M (\Delta R)^2\, dv =& \int_M R\Delta^2 R\, dv \\ =& \frac{n-4}{n+4(n-1)t}\int_M R(\Delta|Ric|^2\, dv + t\Delta R^2)\, dv \\ =& -\frac{n-4}{n+4(n-1)t}\int_M (\langle\nabla|Ric|^2, \nabla R\rangle + 2tR|\nabla R|^2)\, dv \\ =& -\frac{n-4}{n+4(n-1)t}\int_M \Big(\langle\nabla|E|^2, \nabla R\rangle + \frac{2(1+nt)}{n}R|\nabla R|^2\Big)\, dv.\end{aligned} \tag{2.26}$$

Inserting (2.26) into (2.25) yields

$$\begin{aligned}\int_M E(\nabla R, \nabla R)\, dv \leq& -\frac{(n-1)(n-4)}{n[n+4(n-1)t]}\int_M \langle\nabla|E|^2, \nabla R\rangle\, dv \\ &- \Big[\frac{2(n-1)(n-4)(1+nt)}{n^2[n+4(n-1)t]} + \frac{1}{n}\Big]\int_M R|\nabla R|^2\, dv.\end{aligned} \tag{2.27}$$

Since $t > -\frac{1}{2}$, we have by (2.1) that

$$\begin{aligned}&\int_M \Big[\frac{1}{2}\langle\nabla|E|^2, \nabla R\rangle + R|\nabla E|^2 - \frac{(n-2)(1+2t)}{2n}R|\nabla R|^2 \\ &\quad - (1+2t)E(\nabla R, \nabla R)\Big]\, dv \\ &\quad \geq \Big[\frac{(n-1)(n-4)(1+2t)}{n[n+4(n-1)t]} + \frac{1}{2}\Big]\int_M \langle\nabla|E|^2, \nabla R\rangle\, dv \\ &\quad + \int_M R|\nabla E|^2\, dv + (1+2t)\Big[\frac{2(n-1)(n-4)(1+nt)}{n^2[n+4(n-1)t]} - \frac{n-4}{2n}\Big]\int_M R|\nabla R|^2\, dv.\end{aligned} \tag{2.28}$$

On the other hand, for any positive constant ϵ, it holds that

$$\begin{aligned}&\Big[\frac{(n-1)(n-4)(1+2t)}{n[n+4(n-1)t]} + \frac{1}{2}\Big]\langle\nabla|E|^2, \nabla R\rangle \\ \geq& -2\Big|\frac{(n-1)(n-4)(1+2t)}{n[n+4(n-1)t]} + \frac{1}{2}\Big||E||\nabla|E|||\nabla R| \\ \geq& -2\,\Big|\frac{(n-1)(n-4)(1+2t)}{n[n+4(n-1)t]} + \frac{1}{2}\Big||E||\nabla E||\nabla R| \\ \geq& -\,\Big|\frac{(n-1)(n-4)(1+2t)}{n[n+4(n-1)t]} + \frac{1}{2}\Big|\Big[\epsilon R|\nabla R|^2 + \frac{1}{\epsilon}\frac{|E|^2}{R}|\nabla E|^2\Big],\end{aligned} \tag{2.29}$$

where we use Kato inequality $|\nabla E| \geq |\nabla |E||$ in the third line. Note that under the assumption that $n \geq 5$ and $-\frac{1}{2} < t < -\frac{n}{4(n-1)}$, or $n = 3$ and $t > -\frac{3}{8}$, we have both

$$(1+2t)\Big[\frac{2(n-1)(n-4)(1+nt)}{n^2[n+4(n-1)t]} - \frac{n-4}{2n}\Big] = -\frac{(n-4)(n-2)^2(1+2t)}{2n^2[n+4(n-1)t]} > 0.$$

Hence, there exists a positive constant ϵ_0 such that

$$\begin{aligned}(1+2t)\Big[\frac{2(n-1)(n-4)(1+nt)}{n^2[n+4(n-1)t]} - \frac{n-4}{2n}\Big]& \\ -\Big|\frac{(n-1)(n-4)(1+2t)}{n[n+4(n-1)t]} + \frac{1}{2}\Big|\epsilon_0 = 0.&\end{aligned} \tag{2.30}$$

Inserting (2.29) with ϵ_0 into (2.28) yields

$$\begin{aligned}&\int_M \Big[\frac{1}{2}\langle \nabla |E|^2, \nabla R\rangle + R|\nabla E|^2 - \frac{(n-2)(1+2t)}{2n} R|\nabla R|^2 \\
&\qquad - (1+2t)E(\nabla R, \nabla R)\Big]\,dv \\
&\qquad \geq \int_M \frac{|\nabla E|^2}{R}\Big[R^2 - \Big|\frac{(n-1)(n-4)(1+2t)}{n[n+4(n-1)t]} + \frac{1}{2}\Big|\frac{1}{\epsilon_0}|E|^2\Big]\,dv \\
&\qquad = \int_M \frac{|\nabla E|^2}{R}\Bigg[R^2 - \frac{\Big(\frac{(n-1)(n-4)(1+2t)}{n[n+4(n-1)t]} + \frac{1}{2}\Big)^2}{(1+2t)\Big[\frac{2(n-1)(n-4)(1+nt)}{n^2[n+4(n-1)t]} - \frac{n-4}{2n}\Big]}|E|^2\Bigg]\,dv \\
&\qquad = \int_M \frac{|\nabla E|^2}{R}\Big[R^2 + \frac{[3n-4+8(n-1)t]^2}{2(n-4)(1+2t)[n+4(n-1)t]}|E|^2\Big]\,dv.\end{aligned} \tag{2.31}$$

We complete the proof of Proposition 2.2 by combining (2.31) with (2.1). □

Proof of Theorem 1.1. Now with the help of Proposition 2.2, we will complete the proof of Theorem 1.1.

When $n = 3$, if $-\varepsilon_0 \leq t < -\frac{1}{6}$, then

$$\frac{(1+6t)^2}{24} \leq \frac{2(1+2t)(3+8t)}{(5+16t)^2}, \tag{2.32}$$

where $-\varepsilon_0$ is just the unique negative root of the corresponding equation of the above inequality and $\varepsilon_0 \approx 0.3652$. Therefore, under the assumption

$$\frac{1+6t}{3}R + \frac{4}{\sqrt{6}}|E| < 0, \tag{2.33}$$

the inequalities (2.23), (2.32) and (2.33) imply

$$\begin{aligned}0 &\leq \int_M \frac{|\nabla E|^2}{R}\Big[R^2 - \frac{(5+16t)^2}{2(1+2t)(3+8t)}|E|^2\Big]\,dv \\
&\leq \int_M R|E|^2\Big[\frac{1+6t}{3}R + \frac{4}{\sqrt{6}}|E|\Big]\,dv \leq 0\end{aligned} \tag{2.34}$$

which gives that $E = 0$ and M^3 is Einstein. Thus, M^3 is of constant positive sectional curvature.

When $n \geq 5$ and $-\frac{1}{2} < t < -\frac{n}{4(n-1)}$, under the assumption that

$$\frac{2(n-2)+2n(n-1)t}{n(n-1)}R + \frac{4}{\sqrt{n(n-1)}}|E| + \sqrt{\frac{2(n-2)}{n-1}}|W| < 0,$$

we have

$$\frac{2(n-2)+2n(n-1)t}{n(n-1)}R + \frac{4}{\sqrt{n(n-1)}}|E| < 0. \tag{2.35}$$

Let

$$f(t) = [3n-4+8(n-1)t]^2[(n-2)+n(n-1)t]^2+4n(n-1)2(n-4)(1+2t)[n+4(n-1)t].$$

Clearly,

$$f(-\frac{n}{4(n-1)}) = f(-\frac{1}{2}) > 0$$

and

$$f(-\frac{3n-4}{8(n-1)}) < 0.$$

Therefore, we can find the roots $-\varepsilon_n$ and $-\eta_n$ of $f(t) = 0$ such that for $-\frac{1}{2} < t < -\varepsilon_n$ or $-\eta_n < t < -\frac{n}{4(n-1)}$, the inequality

$$-\frac{[3n-4+8(n-1)t]^2}{2(n-4)(1+2t)[n+4(n-1)t]} \leq \frac{4n(n-1)}{[(n-2)+n(n-1)t]^2},$$

holds true. Then noticing that $-\frac{n}{4(n-1)} < -\frac{n-2}{n(n-1)}$, (2.24) and (2.35) imply

$$\begin{aligned} 0 \leq & \int_M \frac{|\nabla E|^2}{R}\Big[R^2 + \frac{[3n-4+8(n-1)t]^2}{2(n-4)(1+2t)[n+4(n-1)t]}|E|^2\Big]\,dv \\ \leq & \int_M R|E|^2\Big[\frac{2(n-2)+2n(n-1)t}{n(n-1)}R + \frac{4}{\sqrt{n(n-1)}}|E| + \sqrt{\frac{2(n-2)}{n-1}}|W|\Big]\,dv \\ \leq & 0. \end{aligned}$$

This gives $E = 0$ and M^n is Einstein. In particular, in this case, if $W = 0$, then (2.5) shows that M^n is of constant positive sectional curvature. The proof of Theorem 1.1 is complete. □

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