

NEW MAXIMAL CURVES AS RAY CLASS FIELDS OVER DELIGNE-LUSZTIG CURVES

DANE C. SKABELUND

(Communicated by Romyar T. Sharifi)

ABSTRACT. We construct new covers of the Suzuki and Ree curves which are maximal with respect to the Hasse-Weil bound over suitable finite fields. These covers are analogues of the Giulietti-Korchmáros curve, which covers the Hermitian curve and is maximal over a base field extension. We show that the maximality of these curves implies that of certain ray class field extensions of each of the Deligne-Lusztig curves. Moreover, we show that the Giulietti-Korchmáros curve is equal to the above-mentioned ray class field extension of the Hermitian curve.

1. INTRODUCTION

A smooth, geometrically irreducible, projective algebraic curve (henceforth *curve*) X defined over a finite field \mathbb{F}_q is called maximal (over \mathbb{F}_q) if it meets the Hasse-Weil upper bound for the number of \mathbb{F}_q -rational points for its genus, that is, if

$$\#X(\mathbb{F}_q) = q + 1 + 2gq^{1/2}.$$

This is equivalent to saying that the L -polynomial of X over \mathbb{F}_q is of the form $L_X(T) = (1 + q^{1/2}T)^{2g}$. Interest in studying maximal curves was sparked in the 1980s by their application in the theory of algebraic geometry codes. An interesting observation, often attributed to Serre (see [15, Prop. 6]), is that if $X \rightarrow Y$ is a non-constant map of curves defined over \mathbb{F}_q , then there is divisibility of L -polynomials in the opposite direction. This implies that any curve covered by an \mathbb{F}_q -maximal curve is also \mathbb{F}_q -maximal. This observation motivates the question of determining which \mathbb{F}_q -maximal curves are largest with respect to the partial order on the set of \mathbb{F}_q -maximal curves induced by coverings of one curve by another.

It was shown by Ihara [13] that an \mathbb{F}_q -maximal curve has genus at most $q_0(q_0 - 1)/2$, where $q = q_0^2$. Since this bound is attained by the Hermitian curve $H = H_{q_0}$ defined by

$$y^{q_0} + y = x^{q_0+1},$$

the curve H is not covered by any other \mathbb{F}_q -maximal curve.

Other examples of maximal curves include the Suzuki and Ree curves. These curves are defined over finite fields \mathbb{F}_q of odd prime power order in characteristic 2 and 3, respectively. Each has an optimal number of \mathbb{F}_q -rational points for its genus, and becomes maximal over a suitable extension of the base field. Together with the Hermitian curve they constitute the Deligne-Lusztig curves associated to the simple groups of type 2A_2 , 2B_2 , and 2G_2 [2], [7]. It has been shown that the smallest

Received by the editors June 29, 2016 and, in revised form, March 30, 2017.
2010 *Mathematics Subject Classification*. Primary 11G20; Secondary 14H25.

examples of the Suzuki and Ree curves are not Galois covered by the Hermitian curve maximal over the same field [17]. Each of the Deligne-Lusztig curves can be described as a composite of Artin-Schreier extensions over \mathbb{P}^1 with very similar ramification structure, and Lauter has shown [16] that their function fields admit a unified description as ray class fields over $\mathbb{F}_q(x)$ with conductor supported at a single rational point.

In 2009, Giulietti and Korchmáros [6] discovered a new family of curves maximal over \mathbb{F}_{q^3} for $q = q_0^2$ a square, each of which is not covered by the Hermitian curve $H_{q_0^3}$. We denote this ‘‘GK-curve’’ by $\tilde{H} = \tilde{H}(q)$. The curve \tilde{H} has genus $\frac{1}{2}q(q - q_0 - 1)(q_0 + 1)$ and its function field may be defined by the equations

$$y^{q_0} + y = x^{q_0+1}, \quad t^m = x^q - x,$$

where $m = q - q_0 + 1$. These equations describe the curve \tilde{H} as the fiber product of the cover $H \rightarrow \mathbb{P}_x^1$ with a Kummer extension of \mathbb{P}_x^1 of degree m .

The results in this paper are motivated by the observation that the abelian cover $\tilde{H} \rightarrow H$ is tamely ramified at each \mathbb{F}_q -rational point of H , and that every other \mathbb{F}_{q^3} -rational point of H splits completely in \tilde{H} . Thus, the function field $\mathbb{F}_{q^3}(\tilde{H})$ is contained in the maximal abelian extension of $\mathbb{F}_{q^3}(H)$ of conductor $\mathfrak{m} = \sum_{P \in H(\mathbb{F}_q)} P$ in which each point of $\Sigma = H(\mathbb{F}_{q^3}) \setminus H(\mathbb{F}_q)$ splits. The corresponding curve H_{rcf} is also maximal over \mathbb{F}_{q^3} and is in fact equal to \tilde{H} , as we show in section 5. We note that the splitting set Σ consists of the second short orbit under the action of $\text{Aut}(H) = \text{PGU}(3, q_0)$, the first being the set of \mathbb{F}_q -rational points. This provides an intrinsic alternate description of the GK-curve as arising from the Hermitian curve in a natural way. Moreover, it motivates our analogous construction of maximal covers of the Suzuki and Ree curves.

We outline the remaining contents of the paper. In section 2 we give background information about the Deligne-Lusztig curves. In sections 3 and 4 we introduce cyclic covers \tilde{S} and \tilde{R} of the Suzuki and Ree curves analogous to the GK-curve, and show that these are maximal over a suitable base field. In section 5, we consider for each Deligne-Lusztig curve X a cover $X_{\text{rcf}} \rightarrow \tilde{X}$ obtained from X via a ray class field construction, and show that the maximality of \tilde{X} implies that of X_{rcf} . We then show that the cover $X_{\text{rcf}} \rightarrow X$ is cyclic of a prescribed form. This allows us to show that $H_{\text{rcf}} = \tilde{H}$, and to verify computationally that $X_{\text{rcf}} = \tilde{X}$ for small values of q when X is one of the Suzuki or Ree curves. We leave as an open problem whether $X_{\text{rcf}} = \tilde{X}$ in general, but give a bound on the degree of the cover $X_{\text{rcf}} \rightarrow \tilde{X}$.

The curves X_{rcf} can be thought of as arising via a two-step ray class field construction over \mathbb{P}^1 . First, apply the construction of Lauter allowing one \mathbb{F}_q -rational point to ramify and splitting all other rational points. Then extend the base field before taking another ray class field in which the \mathbb{F}_q -rational points are allowed to ramify tamely, while all other points rational over the new base field are caused to split.

2. THE DELIGNE-LUSZTIG CURVES

In this section we collect some facts about the Deligne-Lusztig curves which will be used later. In particular, we note that the Hermitian, Suzuki, and Ree curves have exactly two short orbits under the action of their full automorphism group, each consisting of all points of certain degrees. These orbits form the sets of points

which ramify and split in the covers constructed in later sections. The results in this section should be well known; we include proofs of some statements for the sake of completeness.

A point of a curve X is fixed by some automorphism of X exactly if it is ramified in the quotient by the full automorphism group. Thus, the Riemann-Hurwitz formula may be applied to the cover $X \rightarrow X/\text{Aut}(X)$ to study these points. For G a finite subgroup of automorphisms of X , this formula may be written in the form

$$2g(X) - 2 = \#H \left(2g(X/G) - 2 + \sum_{\mathfrak{p} \in X/G} \frac{d(\mathfrak{p})}{e(\mathfrak{p})} \deg \mathfrak{p} \right),$$

where $d(\mathfrak{p})$ and $e(\mathfrak{p})$ are the different exponent and ramification index of \mathfrak{p} in $X \rightarrow X/G$.

2.1. The Hermitian curve. Let $q = q_0^2$ be a prime power. The Hermitian curve $H = H_{q_0}$ has an affine plane model defined by

$$y^{q_0} + y = x^{q_0+1}.$$

It has genus $q_0(q_0 - 1)/2$ and is maximal over \mathbb{F}_q , with $\#X(\mathbb{F}_q) = q^{3/2} + 1$. Its automorphism group $\text{PGU}(3, q_0)$ is of size $(q^{3/2} + 1)q^{3/2}(q - 1)$.

Proposition 2.1 ([4, Proposition 2.2]). *The Hermitian curve H_{q_0} has exactly two short orbits under the action of its full automorphism group. One is non-tame of size $q^{3/2} + 1$, consisting of the \mathbb{F}_q -rational points. The other is tame of size $\frac{1}{3}q^{3/2}(q - 1)(q_0 + 1)$, consisting of all points of degree 3.*

2.2. The Suzuki curve. For $s \geq 1$ and $q = 2q_0^2 = 2^{2s+1}$, the Suzuki curve S/\mathbb{F}_q has an affine model defined by

$$y^q + y = x^{q_0}(x^q + x)$$

and has genus $q_0(q - 1)$. Its automorphism group $\text{Sz}(q)$, which has size $(q^2 + 1)q^2(q - 1)$, acts doubly transitively on the $q^2 + 1$ rational points. The Suzuki curve is maximal over \mathbb{F}_{q^4} .

Proposition 2.2. *The Suzuki curve has exactly two short orbits under the action of its full automorphism group. One is non-tame of size $q^2 + 1$, consisting of the \mathbb{F}_q -rational points. The other is tame of size $\frac{1}{4}q^2(q - 1)(q + 2q_0 + 1)$, consisting of all points of degree 4.*

Proof. The automorphism group $G = \text{Sz}(q)$ transitive on the $q^2 + 1$ points of $S(\mathbb{F}_q)$ with point stabilizer of order $q^2(q - 1)$. Fix a rational point \mathfrak{P}_∞ on S lying above a point \mathfrak{p}_∞ in $S/G \cong \mathbb{P}^1$. Then

$$\#G_\infty = e(\mathfrak{P}_\infty | \mathfrak{p}_\infty) = q^2(q - 1).$$

Let G_i denote the (lower) ramification groups at \mathfrak{P}_∞ . Then ([11, §12/2])

$$\begin{aligned} \#G_1 &= q^2, \\ \#G_2 &= \dots = \#G_{2q_0+1} = q, \\ \#G_{2q_0+2} &= 1. \end{aligned}$$

Therefore, the different exponent at \mathfrak{p}_∞ is

$$\begin{aligned} d(\mathfrak{p}_\infty) &= (q^2(q-1) - 1) + (q^2 - 1) + 2q_0 \cdot (q-1) \\ &= q^3 + 2qq_0 - 2q_0 - 2. \end{aligned}$$

From the Hurwitz formula, it follows that

$$\sum_{\mathfrak{p} \neq \mathfrak{p}_\infty} \frac{d(\mathfrak{p})}{e(\mathfrak{p})} \deg \mathfrak{p} = \frac{q - 2q_0}{q - 2q_0 + 1} < 1.$$

Now $d(\mathfrak{p}) \geq e(\mathfrak{p}) - 1$, with equality if and only if \mathfrak{p} is tamely ramified. Thus the inequality above implies that there is exactly one place \mathfrak{p} of S/G other than \mathfrak{p}_∞ which is ramified in $S \rightarrow S/G$, and that $\deg \mathfrak{p} = 1$. Moreover, \mathfrak{p} is tamely ramified with $e(\mathfrak{p}) = q - 2q_0 + 1$.

Let \mathfrak{P} be a prime of S lying over \mathfrak{p} . Then the inertia group $I = I(\mathfrak{P}|\mathfrak{p})$ is cyclic of order $q - 2q_0 + 1$. There is a unique conjugacy class of cyclic subgroups of this order (I is a Singer subgroup; see [12]). The decomposition group D of \mathfrak{P} has size $(\deg \mathfrak{P})[G : I]$. Since $N_G(I)$ is the unique maximal subgroup containing I , $D \subset N_G(I)$ and $\deg \mathfrak{P}$ divides $[N_G(I) : I] = 4$. But \mathfrak{P} does not have degree 1 because it is not conjugate to \mathfrak{P}_∞ , and S has no points of degree 2, so $\deg \mathfrak{P} = 4$ and $D = N_G(I)$. Thus, the point \mathfrak{P} has orbit of size $[G : D] = \frac{1}{4}q^2(q-1)(q+2q_0+1)$. Since this is equal to the number of points on S of degree 4, these points form a single orbit under the action of G . \square

2.3. The Ree curve. For $s \geq 1$ and $q = 3q_0^2 = 3^{2s+1}$, the Ree curve R/\mathbb{F}_q may be defined by the affine equations

$$\begin{aligned} y^q - y &= x^{q_0}(x^q - x), \\ z^q - z &= x^{2q_0}(x^q - x). \end{aligned}$$

The curve R has $q^3 + 1$ points rational over \mathbb{F}_q , genus $\frac{3}{2}q_0(q-1)(q+q_0+1)$, and automorphism group $\text{Ree}(q)$ of size $(q^3 + 1)q^3(q-1)$. The Ree curve is maximal over \mathbb{F}_{q^6} .

Proposition 2.3. *The Ree curve has exactly two short orbits under the action of its full automorphism group. One is non-tame of size $q^3 + 1$, consisting of the \mathbb{F}_q -rational points. The other is tame of size $\frac{1}{6}q^3(q-1)(q+1)(q+3q_0+1)$, consisting of all points of degree 6.*

Proof. The proof of this proposition is similar to that of Proposition 2.2. The only pieces of information needed are the sizes of the ramifications groups at a rational point of R in $R \rightarrow R/\text{Aut}(R)$, and a list of the maximal subgroups of $\text{Aut}(R)$. These may be found in [8] and on page 648 of [11], respectively. \square

3. THE CURVE \tilde{S}

In this section, we introduce a cover $\tilde{S} \rightarrow S$ of the Suzuki curve which is analogous to the GK-curve $\tilde{H} \rightarrow H$ and show that it is maximal over \mathbb{F}_{q^4} . Fix $s \geq 1$, let $q = 2q_0^2 = 2^{2s+1}$, and recall the definition of the Suzuki curve S from section 2. Let \tilde{S} be a smooth model of the curve with function field described by

$$\begin{aligned} y^q + y &= x^{q_0}(x^q + x), \\ t^m &= x^q + x, \end{aligned}$$

where $m = q - 2q_0 + 1$. The curve \tilde{S} may be described as the normalization of the fiber product of the covers $S \rightarrow \mathbb{P}_x^1$ and $C_m \rightarrow \mathbb{P}_x^1$, where C_m is the curve described by the second equation above.

Let $F = \mathbb{F}_q(x)$. The function field $\mathbb{F}_q(\tilde{S})$ is the composite of $\mathbb{F}_q(C_m) = F(t)$ and $\mathbb{F}_q(S) = F(y)$. Each place of F of degree 1 is ramified in $F(t)$, with ramification index m , and no other places are ramified. Also, the place ∞ corresponding to $1/x$ is the only place ramified in $F(y)$, with ramification index q . Therefore, the only places ramified in $F(t, y)/F(y)$ are the $q^2 + 1$ rational places, and each is tamely ramified with ramification index m . Thus, the Hurwitz formula gives

$$g_{\tilde{S}} = 1 + m(g_S - 1) + \frac{1}{2}(q^2 + 1)(m - 1) = \frac{1}{2}(q^3 - 2q^2 + q).$$

Theorem 3.1. *The curve \tilde{S} is maximal over \mathbb{F}_{q^4} .*

We prove this by means of the following “Natural Embedding Theorem” of Korchmáros and Torres.

Theorem 3.2 (Korchmáros-Torres [14]). *A smooth geometrically irreducible projective curve is maximal over \mathbb{F}_{ℓ^2} if and only if it admits an embedding as a curve of degree $\ell + 1$ in a non-degenerate Hermitian variety defined over \mathbb{F}_{ℓ^2} .*

Here a Hermitian variety means a projective hypersurface defined by the vanishing of

$$\sum_{0 \leq i, j \leq n} a_{ij} x_i x_j^\alpha,$$

where α is the involutive automorphism $w \mapsto w^\ell$ of \mathbb{F}_{ℓ^2} , and $A = [a_{ij}]$ is some “Hermitian” matrix satisfying $A^t = A^\alpha$.

Given an \mathbb{F}_{ℓ^2} -maximal curve X , there is a concrete construction described in [14] for an embedding as in Theorem 3.2. First take a basis f_1, \dots, f_m for the linear series $L((\ell + 1)P_0)$, where P_0 is any rational point of X . Then there is a unique point $(z_0 : \dots : z_m) \in \mathbb{P}_{\mathbb{F}_{\ell^2}(X)}^m$ satisfying

$$z_1^\ell f_1 + \dots + z_m^\ell f_m = 0.$$

After a linear change of variables, and possibly taking a projection, the functions z_i then give desired embedding.

Define functions $z = y^{2q_0} + x^{2q_0+1}$ and $w = xy^{2q_0} + z^{2q_0}$ on \tilde{S} . These satisfy the relations

$$(1) \quad z^q + z = x^{2q_0}(x^q + x), \quad w^q + w = y^{2q_0}(x^q + x).$$

Moreover, if ∞ denotes the pole of the function x , then

$$\begin{aligned} -v_\infty(x) &= qm = q^2 - 2qq_0 + q, \\ -v_\infty(y) &= -(1 + \frac{1}{2q_0})v_\infty(x) = q^2 - qq_0 + q_0, \\ -v_\infty(z) &= -(1 + \frac{1}{q_0})v_\infty(x) = q^2 - q + 2q_0, \\ -v_\infty(w) &= -(1 + \frac{1}{q_0} + \frac{1}{q})v_\infty(x) = q^2 + 1, \\ -v_\infty(t) &= -\frac{q}{m}v_\infty(x) = q^2. \end{aligned}$$

Since the semigroup generated by these numbers has genus $g(\tilde{S})$, the pole orders of these functions generate the Weierstrass semigroup of \tilde{S} at ∞ . In particular, the linear series $L((q^2 + 1)\infty)$ has a basis \mathcal{B} of monomials in $1, x, y, z, w, t$. This fact was useful for finding the equation of the particular Hermitian variety in which we wish to embed \tilde{S} , since it allows a search for the functions z_i mentioned above to be phrased as a linear algebra problem over a vector space with basis $b_i^{q^2k}b_j$ with $b_i, b_j \in \mathcal{B}$. Such a computation performed in Magma [1] resulted in the discovery of equation (2) below.

Lemma 3.3. *Every automorphism of S lifts to an automorphism of \tilde{S} defined over \mathbb{F}_{q^4} .*

Proof. The group $G = \text{Aut } \mathbb{F}_q(\tilde{S})$ is generated by the stabilizer of the point ∞ , which consists of automorphisms ψ_{abc} taking

$$\begin{aligned} x &\mapsto ax + b \\ y &\mapsto a^{q_0+1}y + b^{q_0}x + c, \end{aligned}$$

for $a \in \mathbb{F}_q^\times$ and $b, c \in \mathbb{F}_q$, along with an involution ϕ defined by $\phi(x) = z/w$ and $\phi(y) = y/w$, which swaps ∞ with another rational point (see [9] and [10]).

To extend $\psi = \psi_{abc}$ to \tilde{S} , we need

$$\psi(t)^m = \psi(x)^q - \psi(x) = a(x^q + x).$$

Fix a generator α of \mathbb{F}_q^\times and an m th root β of α which is contained in \mathbb{F}_{q^4} since m divides $q^4 - 1$. Then we may take $\psi(t) = a^{1/m}t$, where $a^{1/m}$ is chosen consistently with the choice of α and β .

The automorphism ϕ may be lifted to an automorphism of \tilde{R} by $\phi(t) = t/w$. Indeed, ϕ so defined satisfies

$$\phi(t)^m = \phi(x)^q + \phi(x) = (z/w)^q + z/w.$$

To verify this, first multiply the desired equality by w^{q+1} and use (1) to obtain

$$\begin{aligned} t^m w^{2q_0} &= z^q w + z w^q \\ &= w(z^q + z) + z(w^q + w) \\ &= w x^{2q_0}(x^q + x) + z y^{2q_0}(x^q + x). \end{aligned}$$

Thus the desired equation is equivalent to $w^{2q_0} = w x^{2q_0} + z y^{2q_0}$, which may now be verified by using the definitions of z and w . □

Lemma 3.4. *The map $\pi = (1 : x : t : z : w)$ defines a smooth embedding of the curve \tilde{S} in \mathbb{P}^4 .*

Proof. Let X_0, \dots, X_4 be homogeneous coordinates on \mathbb{P}^4 . We first check that $\pi(\tilde{S})$ has no singular points on the affine piece $X_0 = 1$. Here the equations

$$\begin{aligned} z^q + z &= x^{2q_0}(x^q + x), \\ w^q + w &= (x^{2q_0+1} + z)(x^q + x), \\ t^m &= x^q + x, \end{aligned}$$

give a matrix of derivatives of rank 3. It remains to check that $\pi(\tilde{S})$ has no singular points on the hyperplane $X_0 = 0$. Since the function defining π are in $L((q^2 + 1)\infty)$ and

$$v_\infty(w) = -(q^2 + 1) < v_\infty(f)$$

for $f \in \{1, x, z, t\}$, the only point in $\pi(\tilde{S}) \cap Z(X_0)$ is $P_\infty := \pi(\infty) = (0 : 0 : 0 : 0 : 1)$.

Let ϕ be the automorphism of \tilde{R} mentioned in the previous lemma, which acts on the image $\pi(\tilde{S}) \subset \mathbb{P}^4$ via the permutation (04)(13) of homogeneous coordinates on \mathbb{P}^4 . Since the point $P_0 = (1 : 0 : 0 : 0 : 0) \in \pi(\tilde{S})$ is non-singular, so is the point $\phi(P_0) = P_\infty$. □

Proof of Theorem 3.1. We claim that

$$(2) \quad w^{q^2} + w + z^{q^2}x + x^{q^2}z = t^{q^2+1},$$

so that the image of the map $\pi = (1 : x : t : z : w)$ lies on the Hermitian hypersurface

$$X_0X_4^{q^2} + X_0^{q^2}X_4 + X_1^{q^2}X_3 + X_1X_3^{q^2} = X_2^{q^2+1}$$

in \mathbb{P}^4 . By Theorem 3.2 this will complete the proof of the theorem. Writing $f = x^q + x$ for convenience, we use (1) to rewrite the terms on the left hand side of the desired equation as

$$\begin{aligned} w^{q^2} + w &= (y^{2q_0}f)^q + y^{2q_0}f \\ &= (y^{2q_0} + x^{2q_0+q} + x^{q+2q_0})f^q + y^{2q_0}f \end{aligned}$$

and

$$\begin{aligned} z^{q^2}x + x^{q^2}z &= (z^{q^2} + z)x + (x^{q^2} + x)z \\ &= ((x^{2q_0}f)^q + x^{2q_0}f)x + (f^q + f)(x^{2q_0+1} + y^{2q_0}) \\ &= (x^{2q_0+1} + x^{2q_0+1} + y^{2q_0})f^q + y^{2q_0}f. \end{aligned}$$

Adding these gives

$$(x^{2q_0+q} + x^{q+2q_0} + x^{2q_0+1} + x^{2q_0+1})f^q = f^{q+2q_0+1} = t^{q^2+1},$$

and so the claim is proven. □

4. THE CURVE \tilde{R}

Fix $s \geq 1$, and let $q = 3q_0^2 = 3^{2s+1}$, and recall the definition of the Ree curve R from section 2. In this section, we construct a cover $\tilde{R} \rightarrow R$ which is also maximal over \mathbb{F}_{q^6} . Let \tilde{R} be a smooth model of the curve with function field described by

$$\begin{aligned} y^q - y &= x^{q_0}(x^q - x), \\ z^q - z &= x^{2q_0}(x^q - x), \\ t^m &= x^q - x, \end{aligned}$$

where $m = q - 3q_0 + 1$. The curve \tilde{R} may be described as the normalization of the fiber product of the covers $R \rightarrow \mathbb{P}_x^1$ and $C_m \rightarrow \mathbb{P}_x^1$, where C_m is the curve described by the third equation above.

Let $F = \mathbb{F}_q(x)$. The function field $\mathbb{F}_q(\tilde{R})$ is the composite of $\mathbb{F}_q(C_m) = F(t)$ and $\mathbb{F}_q(S) = F(y, z)$. Each place of F of degree 1 is ramified in $F(t)$ with ramification index m , and no other places are ramified. Also, the place ∞ corresponding to $1/x$ is the only place ramified in $F(y, z)$, with ramification index q^2 . Therefore, the

only places ramified in $F(t, y, z)/F(y, z)$ are the $q^3 + 1$ rational places, and each is tamely ramified with ramification index m . Thus, the Hurwitz formula gives

$$g_{\tilde{R}} = 1 + m(g_R - 1) + \frac{1}{2}(q^3 + 1)(m - 1) \\ = \frac{1}{2}(q^4 + 6q^3q_0 + 2q^3 - 2q^2 - 6qq_0 - 3q + 2).$$

Theorem 4.1. *The curve \tilde{R} is maximal over \mathbb{F}_{q^6} .*

We proceed as with the curve \tilde{S} by embedding \tilde{R} in a Hermitian hypersurface. In [18], Pedersen defines ten functions w_1, \dots, w_{10} on the Ree curve by

$$(3) \quad \begin{aligned} w_1 &= x^{3q_0+1} - y^{3q_0}, & w_6 &= v^{3q_0} - w_2^{3q_0} + xw_4^{3q_0}, \\ w_2 &= xy^{3q_0} - z^{3q_0}, & w_7 &= w_2 + v, \\ w_3 &= xz^{3q_0} - w_1^{3q_0}, & w_8 &= w_6^{3q_0} + xw_7^{3q_0}, \\ w_4 &= xw_2^{q_0} - yw_1^{q_0}, & w_9 &= w_4w_2^{q_0} - yw_6^{q_0}, \\ v &= xw_3^{q_0} - zw_1^{q_0}, & w_{10} &= zw_6^{q_0} - w_3^{q_0}w_4. \\ w_5 &= yw_3^{q_0} - zw_1^{q_0}, \end{aligned}$$

From the appendix of [18], these satisfy

$$(4) \quad \begin{aligned} w_1^q - w_1 &= x^{3q_0}(x^q - x), & w_4^q - w_4 &= (w_2 - xw_1)^{q_0}(x^q - x), \\ w_2^q - w_2 &= y^{3q_0}(x^q - x), & w_6^q - w_6 &= w_4^{3q_0}(x^q - x), \\ w_3^q - w_3 &= z^{3q_0}(x^q - x), & w_8^q - w_8 &= w_7^{3q_0}(x^q - x). \end{aligned}$$

Furthermore, if ∞ denotes the unique pole of x in \tilde{R} , then it follows from the above that

$$\begin{aligned} -v_\infty(x) &= q^3 - 3q^2q_0 + q^2, & -v_\infty(w_6) &= q^3 - q + 3q_0, \\ -v_\infty(w_1) &= q^3 - 2q^2 + 3qq_0, & -v_\infty(w_8) &= q^3 + 1, \\ -v_\infty(w_2) &= q^3 - q^2 + q, & -v_\infty(t) &= q^3. \\ -v_\infty(w_3) &= q^3 - 3qq_0 + 2q, \end{aligned}$$

Duursma and Eid show in [3] that the 14 functions $1, x, y, z, w_1, \dots, w_{10}$ define a smooth embedding of the Ree curve in \mathbb{P}^{13} and give a system of 105 equations describing this image, a few of which are referred to in this section.

Lemma 4.2. *Every automorphism of R lifts to an automorphism of \tilde{R} defined over \mathbb{F}_{q^6} .*

Proof. We use the concrete description of $G = \text{Aut } \mathbb{F}_q(R)$ found in [18]. The group G is generated by the stabilizer G_∞ of the point ∞ and an involution ϕ which swaps ∞ with another rational point. The stabilizer G_∞ consists of automorphisms ψ_{abcd} taking

$$\begin{aligned} x &\mapsto ax + b \\ y &\mapsto a^{q_0+1}y + ab^{q_0}x + c \\ z &\mapsto a^{2q_0+1}z - a^{q_0+1}b^{q_0}y + ab^{2q_0}x + d, \end{aligned}$$

for $a \in \mathbb{F}_q^\times$ and $b, c, d \in \mathbb{F}_q$. To extend $\psi = \psi_{abcd}$ to \tilde{R} , we need

$$\psi(t)^m = \psi(x)^q - \psi(x) = a(x^q - x).$$

Fix a generator α of \mathbb{F}_q^\times and an m th root β of α which is contained in \mathbb{F}_{q^6} since m divides $q^6 - 1$. Then we may take $\psi(t) = a^{1/m}t$, where $a^{1/m}$ is chosen consistently with the choice of α and β .

The involution $\phi \in \text{Aut } \mathbb{F}_q(R)$ mentioned above sends

$$x \mapsto w_6/w_8, \quad y \mapsto w_{10}/w_8, \quad z \mapsto w_9/w_8.$$

We claim that ϕ extends to \tilde{R} via $\phi(t) = t/w_8$. To verify this, we show that

$$\phi(t)^m = \phi(x)^q - \phi(x) = (w_6/w_8)^q - w_6/w_8.$$

Upon multiplying through by w_8^{q+1} , using (4), and then dividing by $x^q - x$, this is seen to be equivalent to

$$w_8^{3q_0} = w_8 w_4^{3q_0} - w_6 w_7^{3q_0}.$$

But this is one of the equations appearing in Lemma 4.3 of [3]. □

Lemma 4.3. *The map $\pi = (1 : x : w_1 : w_2 : t : w_3 : w_6 : w_8)$ defines a smooth embedding of the curve \tilde{R} in \mathbb{P}^7 .*

Proof. Let X_0, \dots, X_7 be homogeneous coordinates on \mathbb{P}^7 . We first check that $\pi(\tilde{R})$ has no singular points on the affine piece $X_0 = 1$. Here we have, from (3) and (4),

$$\begin{aligned} w_1^q - w_1 &= x^{3q_0}(x^q - x), \\ w_2^q - w_2 &= (x^{3q_0+1} - w_1)(x^q - x), \\ w_3^q - w_3 &= (x^{3q_0+2} - xw_1 - w_2)(x^q - x), \\ w_6^q - w_6 &= (x^{3q_0}w_2^q - w_1^q x^{3q_0+1} + w_1^{q+1})(x^q - x), \\ w_8^q - w_8 &= (w_2^{3q_0} + x^{3q_0}w_3^q - x^{6q_0+1}w_1^q + x^{3q_0}w_1^{q+1} + w_2w_1^q)(x^q - x). \end{aligned}$$

These equations, along with $t^m = x^q - x$, give a matrix of derivatives of rank 6. It remains to show that $\pi(\tilde{R})$ has no singular points lying on the hyperplane $X_0 = 0$. Since each of the functions defining π are in $L((q^3 + 1)\infty)$, and

$$v_\infty(w_8) = -(q^3 + 1) < v_\infty(f)$$

for $f \in \{1, x, w_1, w_2, w_3, w_6, t\}$, the only point of $\pi(\tilde{R}) \cap Z(X_0)$ is $P_\infty := \pi(\infty) = (0 : 0 : 0 : 0 : 0 : 0 : 0 : 1)$.

Let ϕ be the involution in $\text{Aut } \mathbb{F}_q(\tilde{R})$ defined in the previous lemma, which takes $x \mapsto w_6/w_8$ and $t \mapsto t/w_8$. Since the automorphism ϕ also sends

$$\begin{aligned} w_1 &\mapsto w_3/w_8, & w_2 &\mapsto w_2/w_8, & w_3 &\mapsto w_1/w_8, \\ w_6 &\mapsto x/w_8, & w_8 &\mapsto w_8/w_8, \end{aligned}$$

it acts on the image $\pi(\tilde{R}) \subset \mathbb{P}^7$ via the permutation (07)(16)(25) of homogeneous coordinates on \mathbb{P}^7 . Since the point $P_0 = (1 : 0 : 0 : 0 : 0 : 0 : 0 : 0) \in \pi(\tilde{S})$ is non-singular, so is the point $\phi(P_0) = P_\infty$. □

Proof of Theorem 4.1. We show that

$$w_8^{q^3} + w_8 + xw_6^{q^3} + x^{q^3}w_6 + w_1w_3^{q^3} + w_1^{q^3}w_3 + w_2^{q^3+1} = t^{q^3+1},$$

so that the image of the map $\pi = (1 : x : w_1 : w_2 : t : w_3 : w_6 : w_8)$ in \mathbb{P}^7 lies on the Hermitian hypersurface

$$X_0 X_7^{q^3} + X_0^{q^3} X_7 + X_1 X_6^{q^3} + X_1^{q^3} X_6 + X_2 X_5^{q^3} + X_2^{q^3} X_5 + X_3^{q^3+1} = X_4^{q^3+1}.$$

The desired result will then follow by Theorem 3.2. Since

$$t^{q^3+1} = (x^q - x)^{\frac{q^3+1}{m}} = (x^q - x)^{q^2+3q_0+2q+3q_0+1},$$

our verification may be done completely inside of the function field $\mathbb{F}_q(x, y, z)$ of R . Writing $f = x^q - x$ for convenience, we use (4) to rewrite the terms on the left hand side of the desired equation as

$$\begin{aligned} w_8^{q^3} + w_8 &= (w_7^{3q_0})^{q^2} f^{q^2} + (w_7^{3q_0})^q f^q + w_7^{3q_0} f - w_8, \\ xw_6^{q^3} + x^{q^3} w_6 &= (w_6^{q^3} - w_6)x + (x^{q^3} - x)w_6 - xw_6 \\ &= ((w_4^{3q_0})^{q^2} x + w_6) f^{q^2} \\ &\quad + ((w_4^{3q_0})^q x + w_6) f^q + (w_4^{3q_0} x + w_6) f - xw_6, \\ w_1 w_3^{q^3} + w_1^{q^3} w_3 &= (w_3^{q^3} - w_3)w_1 + (w_1^{q^3} - w_1)w_3 - w_1 w_3 \\ &= ((z^{3q_0})^{q^2} w_1 + (x^{3q_0})^{q^2} w_3) f^{q^2} \\ &\quad + ((z^{3q_0})^q w_1 + (x^{3q_0})^q w_3) f^q + (z^{3q_0} w_1 + x^{3q_0} w_3) f - w_1 w_3, \\ w_2^{q^3+1} &= (w_2^{q^3} - w_2)w_2 + w_2^2 \\ &= (y^{3q_0})^{q^2} w_2 f^{q^2} + (y^{3q_0})^q w_2 f^q + y^{3q_0} w_2 f + w_2^2. \end{aligned}$$

Collecting terms involving common powers of f gives

$$A_{-1} + A_0 f + A_1 f^q + A_2 f^{q^2},$$

where

$$A_{-1} = -w_8 - xw_6 - w_1 w_3 + w_2^2$$

and

$$A_i = (w_7^{q^i})^{3q_0} + (w_4^{q^i})^{3q_0} x + (z^{q^i})^{3q_0} w_1 + (y^{q^i})^{3q_0} w_2 + (x^{q^i})^{3q_0} w_3 + w_6$$

for $i = 0, 1, 2$. We claim that $A_{-1} = A_0 = A_1 = 0$. Indeed, the quadric A_{-1} and each of the three terms in the expression

$$\begin{aligned} A_0 &= (x^{3q_0} w_3 - z^{3q_0} w_1 - w_7^{3q_0} + w_2^{3q_0}) \\ &\quad - (w_4^{3q_0} x + z^{3q_0} w_1 - y^{3q_0} w_2) \\ &\quad - (w_4^{3q_0} x + w_7^{3q_0} + w_2^{3q_0} - w_6) \end{aligned}$$

are among the relations listed in Lemma 4.3 of [3], so $A_{-1} = 0 = A_0$. Now using (4), we obtain

$$\begin{aligned} A_0^q - A_1 &= (w_4^q)^{3q_0} (x^q - x) + (w_6^q - w_6) \\ &\quad + (z^q)^{3q_0} (w_1^q - w_1) + (x^q)^{3q_0} (w_3^q - w_3) + (y^q)^{3q_0} (w_2^q - w_2) \\ &= (w_4^q + w_4 + z^q x + x^q z + y^{q+1})^{3q_0} (x^q - x). \end{aligned}$$

Further simplification reveals that

$$B_1 := w_4^q + w_4 + z^q x + x^q z + y^{q+1} = 0,$$

and so $A_1 = 0$.

It remains to show that $A_2 = (x^q - x)^{3qq_0+2q+3q_0+1}$. By (4),

$$\begin{aligned} A_1^q - A_2 &= (w_4^q)^{3q_0}(x^q - x) + (w_6^q - w_6) \\ &\quad + (z^{q^2})^{3q_0}(w_1^q - w_1) + (x^{q^2})^{3q_0}(w_3^q - w_3) + (y^{q^2})^{3q_0}(w_2^q - w_2) \\ &= (w_4^q + w_4 + z^{q^2}x + x^{q^2}z + y^{q^2+1})^{3q_0}(x^q - x). \end{aligned}$$

Thus it suffices to show that

$$B_2 := w_4^q + w_4 + z^{q^2}x + x^{q^2}z + y^{q^2+1} = -(x^q - x)^{q+2q_0+1}.$$

To do this, we use (4) again, obtaining

$$\begin{aligned} B_1^q - B_2 &= (w_4^q - w_4) + z^{q^2}(x^q - x) + x^{q^2}(z^q - z) + y^{q^2}(y^q - y) \\ &= (w_2^{q_0} - w_1^{q_0}x^{q_0} + z^{q^2} + x^{q^2}x^{2q_0} + y^{q^2}x^{q_0})(x^q - x) \\ &= [(xy^{3q_0} - z^{3q_0})^{q_0} - (x^{3q_0+1} - y^{3q_0})^{q_0}x^{q_0} \\ &\quad + z^{q^2} + x^{q^2}x^{2q_0} + y^{q^2}x^{q_0}](x^q - x) \\ &= [(z^q - z)^q + (y^q - y)^qx^{q_0} + (x^q - x)^qx^{2q_0}](x^q - x) \\ &= (x^{2qq_0} + x^{qq_0+q_0} + x^{2q_0})(x^q - x)^{q+1} \\ &= (x^q - x)^{q+2q_0+1}. \end{aligned}$$

□

5. RAY CLASS FIELDS

In this section, we let X/\mathbb{F}_q denote one the Deligne-Lusztig curves H , S , or R , and let $d = 3, 4$, or 6 , respectively. Then $\#X(\mathbb{F}_q) = q^{d/2} + 1$, and \tilde{X} is maximal over \mathbb{F}_{q^d} , and the cover $\tilde{X} \rightarrow X$ is of degree $m = q - \lfloor d/2 \rfloor q_0 + 1$. Moreover, each \mathbb{F}_q -rational point of X is totally ramified in \tilde{X} and these are the only points ramified in $\tilde{X} \rightarrow X$. The following lemma implies then that every point of X of degree d splits completely in \tilde{X} over \mathbb{F}_{q^d} .

Lemma 5.1. *Let X/\mathbb{F}_q be a curve which is maximal over \mathbb{F}_{q^d} , and let $f: Y \rightarrow X$ be a tame cover of degree $m > 1$ defined over \mathbb{F}_q . Suppose that f is totally ramified at each \mathbb{F}_q -rational point of X and unramified elsewhere. Then any two of the following conditions implies the third:*

- (i) Y is maximal over \mathbb{F}_{q^d} ,
- (ii) $\#X(\mathbb{F}_q) = q^{d/2} + 1$,
- (iii) $\#(f^{-1}(P) \cap Y(\mathbb{F}_{q^d})) = m$ for every $P \in X(\mathbb{F}_{q^d}) \setminus X(\mathbb{F}_q)$.

Proof. Let N_r denote the number of \mathbb{F}_{q^r} -rational points of X . From the Hurwitz genus formula,

$$2g_Y - 2 = m(2g_X - 2) + N_1(m - 1).$$

Also, since X is maximal over \mathbb{F}_{q^d} we have $N_d = q^d + 1 + 2g_Xq^{d/2}$. Then

$$\begin{aligned} \#Y(\mathbb{F}_{q^d}) &\leq q^d + 1 + 2g_Yq^{d/2} \\ &= q^d + 1 + q^{d/2}[m(2g_X - 2) + N_1(m - 1) + 2] \\ &= N_1 + m(N_d - N_1) + q^{d/2}(m - 1)[N_1 - (q^{d/2} + 1)]. \end{aligned}$$

If Y is maximal over \mathbb{F}_{q^d} , then equality holds above, and so (ii) is satisfied if and only if (iii) is. On the other hand, if both (ii) and (iii) hold, then

$$\#Y(\mathbb{F}_{q^d}) = N_1 + m(N_d - N_1) = q^d + 1 + 2g_Y q^{d/2},$$

and Y is maximal over \mathbb{F}_{q^d} . □

Define the divisor \mathfrak{m} as the sum of all \mathbb{F}_q -rational points of X and let Σ be the set of the points of X of degree d over \mathbb{F}_q . Then there is a curve $X_{\text{rcf}} \rightarrow X$ whose function field is the ray class field over $K = \mathbb{F}_{q^d}(X)$ of conductor \mathfrak{m} in which each place in Σ splits completely. Since $\tilde{K} = \mathbb{F}_{q^d}(\tilde{X})$ is an abelian extension of K satisfying these ramification and splitting conditions, \tilde{K} is contained in $K_{\text{rcf}} = \mathbb{F}_{q^d}(X_{\text{rcf}})$, and the cover $X_{\text{rcf}} \rightarrow X$ factors through \tilde{X} .

Theorem 5.2. *The curve X_{rcf} is maximal over \mathbb{F}_{q^d} .*

Proof. Write $Y = X_{\text{rcf}}$. By Lemma 5.1 it suffices to show that $Y \rightarrow X$ is totally ramified at each point of $X(\mathbb{F}_q)$. Let k be the degree of the cover $Y \rightarrow \tilde{X}$ and write $N = \#\tilde{X}(\mathbb{F}_q) = q^{d/2} + 1$. For $P \in \tilde{X}$, let e_P denote the ramification index of P in Y . Then

$$\#Y(\mathbb{F}_{q^d}) = km\#\Sigma + N - r,$$

where r is the number of \mathbb{F}_q -rational points of \tilde{X} with $e_P < k$. On the other hand, the Hasse-Weil bound and Riemann-Hurwitz give

$$\begin{aligned} \#Y(\mathbb{F}_{q^d}) &\leq q^d + 1 + 2g_Y q^{d/2} \\ &= (q^{d/2} + 1)^2 + (2g_Y - 2)q^{d/2} \\ &= N^2 + kq^{d/2}(2g_{\tilde{X}} - 2) + q^{d/2} \deg \text{Diff } Y/\tilde{X}. \end{aligned}$$

Now

$$\deg \text{Diff } Y/\tilde{X} = k \sum_{P \in \tilde{X}(\mathbb{F}_q)} \left(1 - \frac{1}{e_P}\right) = N(k - 1) - \sum_{e_P > 1} \left(\frac{k}{e_P} - 1\right).$$

Combining all this with the facts $m\#\Sigma = q^{d/2}(q - 1)N$ and $2g_{\tilde{X}} - 2 = (q - 2)N$ and doing some rearranging yields

$$(5) \quad \sum_{e_P > 1} \left(\frac{k}{e_P} - 1\right) \leq 1 + rq^{-d/2} \leq 2 + q^{-d/2}.$$

Since e_P divides k , each non-zero summand on the left hand side of (5) is at least 1, so we conclude that $e_P < k$ for at most two $P \in X(\mathbb{F}_q)$. In particular, if $r > 0$, then either $r = 1$ and $e_P = k/3$ for a single P , or $r \in \{1, 2\}$ and $e_P \geq k/2$ for all P . But either of these cases gives a contradiction in (5). □

Corollary 5.3. *The cover $X_{\text{rcf}} \rightarrow X$ is cyclic. In particular, the function field K_{rcf} is of the form $K((x^q - x)^{1/mk})$ for some k dividing $(q^{d/2} + 1)/m$.*

Proof. The first statement follows from the fact that any tame abelian extension L/K which is totally ramified at some place is cyclic. Indeed, if not, then by replacing K with a larger subfield of L we may assume that $\text{Gal}(L/K) \cong (\mathbb{Z}/r\mathbb{Z})^2$ for some $r > 1$. Then L is a composite of two cyclic Kummer extensions $K_i = K(v_i)$ with $v_i^r = f_i \in K$. Since L/K is totally ramified at some place P , it follows that $a_i = v_P(f_i)$ is invertible mod r for $i = 1, 2$ (see [19, Prop. 3.7.3]). Choose j so

that $ja_1 \equiv a_2 \pmod r$. Then $v = v_1^j/v_2 \in L$ is a root of $v^r = f_1^j/f_2$, and $K(v)/K$ is unramified at P since $v_P(f_1^j/f_2) = ja_1 - a_2 \equiv 0 \pmod r$.

Let $J_{\mathfrak{m}}$ denote the ray class group over K of conductor \mathfrak{m} . Then $J_{\mathfrak{m}}$ fits into an exact sequence

$$1 \rightarrow \mathbb{F}_{q^d}^\times \rightarrow \mathcal{O}_{\mathfrak{m}}^\times \rightarrow J_{\mathfrak{m}} \rightarrow J \rightarrow 1,$$

where $J = \text{Jac}(X)(\mathbb{F}_{q^d})$ and

$$\mathcal{O}_{\mathfrak{m}}^\times = \prod_{P \in \mathfrak{m}} (\mathcal{O}_P/\mathfrak{m}_P)^\times \cong (\mathbb{F}_{q^d}^\times)^{\#X(\mathbb{F}_{q^2})} \cong (\mathbb{Z}/(q^d - 1)\mathbb{Z})^{q^{d/2}+1}.$$

Furthermore, we have $J \cong (\mathbb{Z}/(q^{d/2} + 1)\mathbb{Z})^{2g_X}$ since X is maximal over \mathbb{F}_{q^d} . Thus $J_{\mathfrak{m}}$ has exponent dividing $q^d - 1$, and so does its quotient $\text{Gal}(K_{\text{rcf}}/K)$. From the discussion above, $L = K((x^q - x)^{1/(q^d - 1)})$ is the largest abelian extension of K of exponent dividing $q^d - 1$ in which each \mathbb{F}_q -rational place of K is totally ramified. Since K_{rcf} is such an extension, we have $\bar{K} \subset K_{\text{rcf}} \subset L$, and so K_{rcf} is of the form $K((x^q - x)^{1/mk})$ for some k dividing $(q^d - 1)/m$.

This shows that X_{rcf} covers the curve C_{mk} given by $u^{mk} = x^q - x$. Since X_{rcf} is maximal over \mathbb{F}_{q^d} , so is C_{mk} . But a theorem of Garcia and Tafazolian [5] states that a curve of the form $u^r = x^q - x$ may be maximal over \mathbb{F}_{q^d} only for r dividing $q^{d/2} + 1$. Thus k divides $(q^{d/2} + 1)/m$, as desired. \square

It follows from the proof of Corollary 5.3 that a sufficient condition for X_{rcf} to be equal to \tilde{X} is that none of the curves C_r defined by $u^r = x^q - x$ are maximal over \mathbb{F}_{q^d} for r a proper multiple of m . This is the case when $X = H$, as we show in the following proposition.

Proposition 5.4. *Let $q = q_0^2$ be a square, and let r be a multiple of $m = q - q_0 + 1$ which divides $q^{3/2} + 1 = (q_0 + 1)m$. Then the curve C_r defined by $u^r = x^q - x$ is maximal over \mathbb{F}_{q^3} if and only if $r = m$.*

Proof. Let $r = mk$ for some k dividing $q_0 + 1$. Then since $g(C_r) = \frac{1}{2}(r - 1)(q - 1)$, C_r is maximal over \mathbb{F}_{q^3} only if

$$\#C_r(\mathbb{F}_{q^3}) = q^3 + 1 + (r - 1)(q - 1)q^{3/2}.$$

Let Tr denote the field trace from \mathbb{F}_{q^3} to \mathbb{F}_q . Since r divides $q^3 - 1$, the field \mathbb{F}_{q^3} contains the r th roots of unity. Therefore, each $\alpha \in \mathbb{F}_{q^3}^{\times r}$ has exactly r r th roots in \mathbb{F}_{q^3} , and the number of solutions of $\alpha = \beta^q - \beta$ is either q or 0 , depending on whether $\text{Tr}(\alpha) = 0$ or not. Thus, every element of $\mathbb{F}_{q^3}^{\times r} \cap \ker \text{Tr}$ contributes rq points to $C_r(\mathbb{F}_{q^3})$. Along with the q points corresponding to $u = 0$ and the point at infinity, this gives

$$\#C_r(\mathbb{F}_{q^3}) = q + 1 + rq \cdot \#(\mathbb{F}_{q^3}^{\times r} \cap \ker \text{Tr}).$$

Therefore, C_r is maximal over \mathbb{F}_{q^3} if and only if

$$(6) \quad \#(\mathbb{F}_{q^3}^{\times r} \cap \ker \text{Tr}) = (q - 1)(q_0 + 1/k).$$

Since the curve C_m is maximal over \mathbb{F}_{q^3} , we have $\#(\mathbb{F}_{q^3}^{\times m} \cap \ker \text{Tr}) = (q-1)(q_0+1)$. Let α be an element of $\mathbb{F}_{q^3}^{\times m} \cap \ker \text{Tr}$, so that $\alpha^{(q^3-1)/m} = 1$ and $\text{Tr}(\alpha) = 0$. Then

$$\begin{aligned} 0 &= \alpha^{(q^3-1)/m-q^2} \text{Tr}(\alpha) \\ &= \alpha^{(q^3-1)/m-q^2} (\alpha + \alpha^q + \alpha^{q^2}) \\ &= \alpha^{(q-1)q_0} + \alpha^{(q-1)(q_0+1)} + 1, \end{aligned}$$

so there is at most $(q-1)(q_0+1)$ such α . We conclude that $\mathbb{F}_{q^3}^{\times m} \cap \ker \text{Tr}$ consists of the roots of the polynomial

$$f(T) = T^{(q-1)(q_0+1)} + T^{(q-1)q_0} + 1.$$

We claim that the trace zero elements of $\mathbb{F}_{q_0^3}^{\times m}$ are evenly distributed among the cosets of the multiplicative subgroup $W = \mathbb{F}_{q_0^3}^{\times} \subset \mathbb{F}_{q^3}^{\times m}$ of index q_0+1 . To see this, first note that the polynomial $f(T)$ admits the factorization

$$f(T) = \prod_{\zeta^{q_0+1}=1} (T^{q-1} + \zeta T^{q_0-1} + 1).$$

Fix a generator β of $\mathbb{F}_{q_0^3}^{\times m}$, so that $\zeta = \beta^{q^{3/2}-1}$ is a primitive (q_0+1) th root of unity. Then we claim that each root of $f(T)$ lying in the coset $\beta^{-i}W$ is a root of $f_i(T) = T^{q-1} + \zeta^i T^{q_0-1} + 1$. For if $\alpha \in \beta^{-i}W$, then $(\alpha\beta^i)^{q_0^3-1} = 1$ and

$$\begin{aligned} \alpha^{(q-1)q_0} f_i(\alpha) &= \alpha^{(q-1)q_0} (\alpha^{q-1} + \zeta^i \alpha^{q_0-1} + 1) \\ &= \alpha^{(q-1)(q_0+1)} + \beta^{(q_0^3-1)i} \alpha^{q_0^3-1} + \alpha^{(q-1)q_0} = f(\alpha). \end{aligned}$$

It follows that $\#(\mathbb{F}_{q^3}^r \cap \ker \text{Tr}) = (q-1)(q_0+1)/k$. Comparing with (6), we see that C_r is maximal only if $k = 1$. □

Corollary 5.5. *The Giulietti-Korchmáros curve \tilde{H} is equal to H_{rcf} .*

The analogue of Proposition 5.4 does not hold in the situation corresponding to S_{rcf} and R_{rcf} . Indeed, in the Suzuki case $q^{d/2} + 1 = q^2 + 1 = (q + q_0 + 1)m$, and the curve $u^{q^2+1} = x^q + x$ is covered by the Hermitian curve $u^{q^2+1} = x^{q^2} + x$, hence is maximal over \mathbb{F}_{q^4} . In the Ree case, there are also proper multiples r of m such that the curve $u^r = x^q - x$ is maximal over \mathbb{F}_{q^6} .

In any case, Corollary 5.3 allows one to verify computationally that $X_{\text{rcf}} = \tilde{X}$ for small values of q by checking that $K((x^q - x)^{1/m\ell})$ is not maximal over \mathbb{F}_{q^d} for any prime ℓ dividing $(q^{d/2} + 1)/m$. Our computations in Magma have shown that $S_{\text{rcf}} = \tilde{S}$ for $q = 2^{2s+1}$ with $1 \leq s \leq 6$, and that $R_{\text{rcf}} = \tilde{R}$ for $q = 27$. We leave as an open problem the question of whether $X_{\text{rcf}} = \tilde{X}$ in general. The following bound on the degree of the cover $X_{\text{rcf}} \rightarrow \tilde{X}$ follows from Theorem 5.2.

Corollary 5.6. *The degree k of the cover $X_{\text{rcf}} \rightarrow \tilde{X}$ satisfies*

$$k \leq \frac{q^{d/2} - 3}{q - 2}.$$

Proof. Since the curve X_{rcf} is maximal over \mathbb{F}_{q^d} its genus is at most $q^{d/2}(q^{d/2}-1)/2$ [13]. The desired bound follows immediately by combining this with the Hurwitz formula applied to the cover $X_{\text{rcf}} \rightarrow \tilde{X}$, and the fact that $2g_{\tilde{X}} - 2 = (q-2)(q^{d/2} + 1)$. □

Remark. This bound is slightly better than the bound $k \leq (q^{d/2} + 1)/m$ from Corollary 5.3. When X is the Suzuki or Ree curve, it gives

$$k \leq q + 2,$$

$$k \leq q^2 + 2q + 4,$$

respectively. Current results on the genus spectrum of maximal curves may be used to reduce these bounds by a factor of 3.

REFERENCES

- [1] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, Computational algebra and number theory (London, 1993). MR1484478
- [2] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (2) **103** (1976), no. 1, 103–161. MR0393266
- [3] Abdulla Eid and Iwan Duursma, *Smooth embeddings for the Suzuki and Ree curves*, Algorithmic arithmetic, geometry, and coding theory, Contemp. Math., vol. 637, Amer. Math. Soc., Providence, RI, 2015, pp. 251–291. MR3364452
- [4] Arnaldo Garcia, Henning Stichtenoth, and Chao-Ping Xing, *On subfields of the Hermitian function field*, Compositio Math. **120** (2000), no. 2, 137–170. MR1739176
- [5] Arnaldo Garcia and Saeed Tafazolian, *On additive polynomials and certain maximal curves*, J. Pure Appl. Algebra **212** (2008), no. 11, 2513–2521. MR2440263
- [6] Massimo Giulietti and Gábor Korchmáros, *A new family of maximal curves over a finite field*, Math. Ann. **343** (2009), no. 1, 229–245. MR2448446
- [7] Johan P. Hansen, *Deligne-Lusztig varieties and group codes*, Coding theory and algebraic geometry (Luminy, 1991), Lecture Notes in Math., vol. 1518, Springer, Berlin, 1992, pp. 63–81. MR1186416
- [8] Johan P. Hansen and Jens Peter Pedersen, *Automorphism groups of Ree type, Deligne-Lusztig curves and function fields*, J. Reine Angew. Math. **440** (1993), 99–109. MR1225959 (94h:14024)
- [9] Johan P. Hansen and Henning Stichtenoth, *Group codes on certain algebraic curves with many rational points*, Appl. Algebra Engrg. Comm. Comput. **1** (1990), no. 1, 67–77. MR1325513
- [10] Hans-Wolfgang Henn, *Funktionenkörper mit grosser Automorphismengruppe*, J. Reine Angew. Math. **302** (1978), 96–115. MR511696
- [11] J. W. P. Hirschfeld, G. Korchmáros, and F. Torres, *Algebraic curves over a finite field*, Princeton Series in Applied Mathematics, Princeton University Press, Princeton, NJ, 2008. MR2386879 (2008m:14040)
- [12] Bertram Huppert and Norman Blackburn, *Finite groups. III*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 243, Springer-Verlag, Berlin-New York, 1982. MR662826
- [13] Yasutaka Ihara, *Some remarks on the number of rational points of algebraic curves over finite fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), no. 3, 721–724 (1982). MR656048
- [14] Gábor Korchmáros and Fernando Torres, *Embedding of a maximal curve in a Hermitian variety*, Compositio Math. **128** (2001), no. 1, 95–113. MR1847666
- [15] Gilles Lachaud, *Sommes d'Eisenstein et nombre de points de certaines courbes algébriques sur les corps finis*, C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), no. 16, 729–732. MR920053
- [16] Kristin Lauter, *Deligne-Lusztig curves as ray class fields*, Manuscripta Math. **98** (1999), no. 1, 87–96. MR1669591
- [17] M. Montanucci and G. Zini, *Some Ree and Suzuki curves are not Galois covered by the Hermitian curve*, (2016), Pre-print, [arXiv:1603.06706](https://arxiv.org/abs/1603.06706).
- [18] Jens Peter Pedersen, *A function field related to the Ree group*, Coding theory and algebraic geometry (Luminy, 1991), Lecture Notes in Math., vol. 1518, Springer, Berlin, 1992, pp. 122–131. MR1186420 (93k:14036)

- [19] Henning Stichtenoth, *Algebraic function fields and codes*, 2nd ed., Springer Publishing Company, Incorporated, 2008.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801
E-mail address: `skabelu2@illinois.edu`