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STRICT S-NUMBERS OF THE VOLTERRA OPERATOR

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ABSTRACT. For Volterra operator $V: L^1(0,1) \to C[0,1]$ and summation operator $\sigma: \ell^1 \to c$, we obtain exact values of Approximation, Gelfand, Kolmogorov and Isomorphism numbers.

1. INTRODUCTION AND MAIN RESULTS

Compact operators and their sub-classes (nuclear operators, Hilbert-Schmidt operators, etc.) play a crucial role in many different areas of Mathematics. These operators are studied extensively but somehow less attention is devoted to operators which are non-compact but close to the class of compact operators. In this work we will focus on two such operators.

First, consider the Volterra operator V, given by

$$Vf(t) = \int_0^t f(s)\,\mathrm{d}s, \quad (0 \le t \le 1), \quad \text{for } f \in L^1(0,1). \tag{1.1}$$

When V is regarded as an operator from L^p into L^q, $(1 < p, q < \infty)$, it is a compact operator, however in the limiting case, when V maps L^1 into the space C of continuous functions on the closed unit interval, the operator is bounded, with the operator norm $\|V\| = 1$, but non-compact. It is worth mentioning that, despite being non-compact or even weakly non-compact, this operator possesses some good properties as being strictly singular (follows from [2], or see [11]). This makes Volterra operator, in the above-mentioned limiting case, an interesting example of a non-compact operator "close" to the class of compact operators. The focus of our paper will be on obtaining exact values of strict s-numbers for this operator.

Volterra operator was already extensively studied. Let us briefly recall those results related to our work. The first credit goes to V. I. Levin [12], who computed explicitly the norm of V between two L^p spaces $(1 < p < \infty)$ and described the extremal function which is connected with the function $\sin_p$. Later on, E. Schmidt in [18] extended this result for $V: L^p \to L^q$, where $1 < p, q < \infty$. This operator was also studied in the context of Approximation theory [3, 10, 16, 17, 19, 20]. Later a weighted version of this operator was studied in connection with Brownian motion [13], Spectral theory [5, 6] and Approximation theory [9].

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Recently, sharp estimates for Bernstein numbers of V in the limiting case were obtained in [11, Theorem 2.2], more specifically,

$$b_n(V) = \frac{1}{2n-1} \quad \text{for } n \in \mathbb{N}, \tag{1.2}$$

and also the estimates for the essential norm can be found in the recent preprint [1].

In our paper, we will compute exact values of all the remaining strict s-numbers, i.e., Approximation, Gelfand, Kolmogorov and Isomorphism numbers denoted by a_n, c_n, d_n and i_n respectively (for the exact definitions see Section 2). Our main result reads as follows.

Theorem 1.1. *Let* $V\colon L^1(0,1) \to C[0,1]$ *be defined as in* (1.1). *Then*

$$a_n(V) = c_n(V) = d_n(V) = \frac{1}{2} \quad \textit{for } n \geq 2 \tag{1.3}$$

and

$$i_n(V) = \frac{1}{2n-1} \quad \textit{for } n \in \mathbb{N}. \tag{1.4}$$

If we also include the result (1.2) concerning the Bernstein numbers, we see that all the strict s-numbers of V split between two groups. The upper half (1.3) remains bounded from below while the lower half (1.4) converges to zero. This phenomenon for this operator was already observed; for instance compare [3] and [2], and for the weighted version see [8] and [7].

The similar results continue to hold for the sequence spaces and for the discrete analogue of V, namely for the operator $\sigma\colon \ell^1 \to c$, defined as

$$\sigma(\mathbf{x})_k = \sum_{j=1}^{k} x_j, \quad (k \in \mathbb{N}), \quad \text{for } \mathbf{x} \in \ell^1, \tag{1.5}$$

where we denoted $\mathbf{x} = \{x_j\}_{j=1}^{\infty}$ for brevity. The operator is well defined and bounded with the operator norm $\|\sigma\| = 1$. It is shown in [11, Theorem 3.2] that

$$b_n(\sigma) = \frac{1}{2n-1} \quad \text{for } n \in \mathbb{N}.$$

We have the next result.

Theorem 1.2. *Let* $\sigma\colon \ell^1 \to c$ *be the operator from* (1.5). *Then*

$$a_n(\sigma) = c_n(\sigma) = d_n(\sigma) = \frac{1}{2} \quad \textit{for } n \geq 2$$

and

$$i_n(\sigma) = \frac{1}{2n-1} \quad \textit{for } n \in \mathbb{N}.$$

The proofs are provided at the end of Section 3.

2. Background material

We shall fix the notation in this section, although we mostly work with standard notions from functional analysis.

2.1. **Normed linear spaces.** For normed linear spaces X and Y, we denote by $B(X,Y)$ the set of all bounded linear operators acting between X and Y. For any $T \in B(X,Y)$, we use just $\|T\|$ for its operator norm, since the domain and target spaces are always clear from the context. By B_X, we mean the closed unit ball of X and, similarly, S_X stands for the unit sphere of X. It is well known that B_X is compact if and only if X is finite-dimensional.

Let Z be a closed subspace of the normed space X. The quotient space X/Z is the collection of the sets $[x] = x + Z = \{x + z;\ z \in Z\}$ equipped with the norm

$$\|[x]\|_{X/Z} = \inf\{\|x - z\|_X;\ z \in Z\}.$$

We sometimes adopt the notation $\|x\|_{X/Z}$ when no confusion is likely to happen. Recall the notion of canonical map $Q_Z\colon X \to X/Z$, given by $Q_Z(x) = [x]$. A normed linear space X is said to have the lifting property if for every $\varepsilon > 0$ and for each linear operator T mapping X into a quotient space Y/N of an arbitrary normed linear space Y, there is a lifting $\widehat{T}$ of T from X into Y with $\|\widehat{T}\| \leq (1+\varepsilon)\|T\|$.

By the Lebesgue space L^1, we mean the set of all real-valued, Lebesgue integrable functions on $(0,1)$ identified almost everywhere and equipped with the norm

$$\|f\|_1 = \int_0^1 |f(s)|\,\mathrm{d}s.$$

The space of real-valued, continuous functions on $[0,1]$, denoted by C, enjoys the norm

$$\|f\|_\infty = \sup_{0\leq t\leq 1} |f(t)|.$$

The discrete counterpart to L^1 is the space of all summable sequences, ℓ^1, where

$$\|\mathbf{x}\|_1 = \sum_{j=1}^{\infty} |x_j|$$

and, similarly to the space C, we denote by c the space of all convergent sequences endowed with the norm

$$\|\mathbf{x}\|_\infty = \sup_{j\in\mathbb{N}} |x_j|.$$

Here and in the latter, we use the abbreviation $\mathbf{x} = \{x_j\}_{j=1}^{\infty}$ for the sequences and we write them in bold font. Note that we also consider only real-valued sequence spaces.

All the above-mentioned spaces are complete, i.e., they form Banach spaces. Furthermore the spaces L^1 and ℓ^1 have the lifting property (see [17, p. 36]).

2.2. **s-numbers.** Let X and Y be Banach spaces. To every operator $T \in B(X,Y)$, one can attach a sequence of non-negative numbers $s_n(T)$ satisfying for every $n \in \mathbb{N}$ the following conditions:

(S1) $\|T\| = s_1(T) \geq s_2(T) \geq \cdots \geq 0$,
(S2) $s_n(T+S) \leq s_n(T) + \|S\|$ for every $S \in B(X,Y)$,
(S3) $s_n(B \circ T \circ A) \leq \|B\| s_n(T) \|A\|$ for every $A \in B(X_1, X)$ and $B \in B(Y, Y_1)$,
(S4) $s_n(\mathrm{Id}\colon \ell_n^2 \to \ell_n^2) = 1$,
(S5) $s_n(T) = 0$ whenever $\operatorname{rank} T < n$.

The number $s_n(T)$ is then called the n-th s-number of the operator T. When (S4) is replaced by a stronger condition

(S6) $s_n(\mathrm{Id}\colon E \to E) = 1$ for every Banach space E, $\dim E = n$,

we say that $s_n(T)$ is the n-th strict s-number of T.

Note that the original definition of s-numbers, which was introduced by Pietsch in [14], uses the condition (S6) which was later modified to accommodate a wider class of s-numbers (like Weyl, Chang and Hilbert numbers). For a detailed account of s-numbers, one is referred for instance to [15], [4] or [9].

We shall briefly recall some particular strict s-numbers. Let $T \in B(X,Y)$ and $n \in \mathbb{N}$. Then the n-th Approximation, Gelfand, Kolmogorov, Isomorphism and Bernstein numbers of T are defined by

$$\begin{aligned} a_n(T) &= \inf_{\substack{F\in B(X,Y)\\ \operatorname{rank} F<n}} \|T-F\|,\\ c_n(T) &= \inf_{\substack{M\subseteq X\\ \operatorname{codim} M<n}} \sup_{x\in B_M} \|Tx\|_Y,\\ d_n(T) &= \inf_{\substack{N\subseteq Y\\ \dim N<n}} \sup_{x\in B_X} \|Tx\|_{Y/N},\\ i_n(T) &= \sup \|A\|^{-1}\|B\|^{-1}, \end{aligned}$$

where the supremum is taken over all Banach spaces E with $\dim E \geq n$ and $A \in B(Y,E)$, $B \in B(E,X)$ such that $A \circ T \circ B$ is the identity map on E and

$$b_n(T) = \sup_{\substack{M\subseteq X\\ \dim M\geq n}} \inf_{x\in S_M} \|Tx\|_Y,$$

respectively.

Let us briefly recall some general inequalities between the above-introduced quantities. It is known that Approximation numbers are the largest s-numbers while the Isomorphism numbers are the smallest among all strict s-numbers. Next, for any $T \in B(X,Y)$ and any $n \in \mathbb{N}$ we have that

$$i_n(T) \leq b_n(T) \leq \max\{c_n(T), d_n(T)\} \leq a_n(T).$$

Moreover, if X has lifting property, then $a_n(T) = d_n(T)$ for every $n \in \mathbb{N}$ (see [15] and [9]).

3. Proofs

Lemma 3.1. *Let $n \in \mathbb{N}$. Then we have the following lower bounds of the Isomorphism numbers of V and σ:*

(i)
$$i_n(V) \geq \frac{1}{2n-1};$$

(ii)
$$i_n(\sigma) \geq \frac{1}{2n-1}.$$

Proof. (i) Let $n \in \mathbb{N}$ be fixed. We shall construct a pair of maps A and B such that the chain

$$\ell^1_{w,n} \xrightarrow{B} L^1 \xrightarrow{V} C \xrightarrow{A} \ell^1_{w,n}$$

forms the identity on $\ell^1_{w,n}$. Here $\ell^1_{w,n}$ is the n-dimensional weighted space ℓ^1 with the norm given by

$$\|\mathbf{x}\|_{\ell^1_{w,n}} = \sum_{k=1}^{n} w_k |x_k|.$$

For the purpose of this proof, we choose $w_k = 2$ for $1 \le k \le n-1$ and $w_n = 1$.

Now, define $A\colon C \to \ell^1_{w,n}$ by

$$(Af)_k = (2n-1)\, f\left(\frac{2k-1}{2n-1}\right), \quad (1 \le k \le n), \quad \text{for } f \in C.$$

Obviously, A is bounded with the operator norm

$$\|A\| = (2n-1)^2.$$

In order to construct the mapping B, consider the partition of the unit interval into subintervals $I_1, I_2, \ldots, I_{2n-1}$ of the same length, i.e.

$$I_k = \left[\frac{k-1}{2n-1}, \frac{k}{2n-1}\right] \quad \text{for } 1 \le k \le 2n-1,$$

and define

$$B(\mathbf{x}) = \sum_{k=1}^{n-1} x_k \big(\chi_{I_{2k-1}} - \chi_{I_{2k}}\big) + x_n \chi_{I_{2n-1}}.$$

Clearly $B(\mathbf{x})$ is integrable for every $\mathbf{x} \in \ell^1_{w,n}$, B is bounded and the operator norm satisfies

$$\|B\| = \frac{1}{2n-1}.$$

One can observe that the composition $A \circ V \circ B$ is the identity mapping on $\ell^1_{w,n}$ and by the very definition of the n-th Isomorphism number we have

$$i_n(V) \ge \|A\|^{-1}\|B\|^{-1} = \frac{1}{2n-1}$$

which completes the proof.

As for the discrete case (ii), we consider the chain

$$\ell^\infty_n \xrightarrow{B} \ell^1 \xrightarrow{\sigma} c \xrightarrow{A} \ell^\infty_n$$

where ℓ^∞_n stands for the n-dimensional space ℓ^∞, A is given by

$$A(\mathbf{x})_k = x_{2k-1}, \quad (1 \le k \le n), \quad \text{for } \mathbf{x} \in c$$

and B satisfies

$$B(\mathbf{y}) = (y_1, -y_1, \ldots, y_{n-1}, -y_{n-1}, y_n, 0, 0, \ldots) \quad \text{for } \mathbf{y} \in \ell^\infty_n.$$

Both A and B are bounded, the composition $A \circ \sigma \circ B$ forms the identity on ℓ^∞_n and thus

$$i_n(\sigma) \ge \|A\|^{-1}\|B\|^{-1} = \frac{1}{2n-1},$$

since $\|A\| = 1$ and $\|B\| = 2n-1$. □

Lemma 3.2. *Let $n \ge 2$; then the estimates of Gelfand numbers of V and σ read as*

(i)

$$c_n(V) \ge \frac{1}{2};$$

(ii)

$$c_n(\sigma) \geq \frac{1}{2}.$$

Proof. (i) Let $n \geq 2$ and $\varepsilon > 0$ be fixed. By the definition of the n-th Gelfand number, we can find a subspace M in L^1 having $\operatorname{codim} M < n$ and satisfying

$$c_n(V) + \varepsilon \geq \sup_{f \in B_M} \|Vf\|_\infty. \tag{3.1}$$

The proof will be finished once we show that the supremum in (3.1) is at least one half.

Let us define the trial functions

$$f_k = 2^{k+1} \chi_{(2^{-k-1}, 2^{-k})}, \quad (k \in \mathbb{N}). \tag{3.2}$$

There is $\|f_k\|_1 = 1$ for every $k \in \mathbb{N}$ and $\|f_k - f_l\|_1 = 2$ and $\|Vf_k - Vf_l\|_\infty = 1$ for distinct k and l. Note that the quotient space L^1/M is of finite dimension thus, the projected sequence $\{[f_k]\}$ is bounded and hence there is a Cauchy subsequence, which we denote $\{[f_k]\}$ again. Now, let $\eta > 0$ be fixed. We have

$$\|f_k - f_l\|_{L^1/M} < \eta \tag{3.3}$$

for k and l sufficiently large. Let us denote $f = \frac{1}{2}(f_k - f_l)$ for these k and l. Thanks to (3.3) and the definition of quotient norm, one can find a function $g \in M$ such that

$$\|f - g\|_1 \leq \eta.$$

On setting

$$h = \frac{g}{1+\eta},$$

we have

$$\|h\|_1 \leq \frac{1}{1+\eta}\big(\|f\|_1 + \|f - g\|_1\big) \leq 1,$$

whence $h \in B_M$. Next

$$\begin{aligned} \|Vh\|_\infty &\geq \frac{1}{1+\eta}\big(\|Vf\|_\infty - \|V(f-g)\|_\infty\big) \\ &\geq \frac{1}{1+\eta}\left(\frac{1}{2} - \|V\|\|f - g\|_1\right) \\ &\geq \frac{1}{1+\eta}\left(\frac{1}{2} - \eta\right), \end{aligned}$$

thus

$$\sup_{f \in B_M} \|Vf\|_\infty \geq \frac{1}{1+\eta}\left(\frac{1}{2} - \eta\right)$$

and the lemma follows, since $\eta > 0$ was arbitrarily chosen.

The proof of the discrete counterpart (ii) is completely analogous, once we consider the canonical vectors $\mathbf{e}^k$ instead of f_k, hence we omit it. □

Lemma 3.3. *Let $n \geq 2$. Then we have the following upper bounds of the Approximation numbers of V and σ:*

(i)

$$a_n(V) \leq \frac{1}{2};$$

(ii)

$$a_n(\sigma) \leq \frac{1}{2}.$$

Proof. (i) Consider the one-dimensional operator $F\colon L^1 \to C$ given by

$$Ff(t) = \frac{1}{2}\int_0^1 f(s)\,\mathrm{d}s, \quad (0 \leq t \leq 1), \quad \text{for } f \in L^1.$$

Then F is a sufficient approximation of V. Indeed,

$$\begin{aligned}
\|Vf - Ff\|_\infty &= \sup_{0\leq t\leq 1}\left|\int_0^t f(s)\,\mathrm{d}s - \frac{1}{2}\int_0^1 f(s)\,\mathrm{d}s\right| \\
&= \sup_{0\leq t\leq 1}\left|\frac{1}{2}\int_0^t f(s)\,\mathrm{d}s - \frac{1}{2}\int_t^1 f(s)\,\mathrm{d}s\right| \\
&\leq \sup_{0\leq t\leq 1}\frac{1}{2}\int_0^t |f(s)|\,\mathrm{d}s + \frac{1}{2}\int_t^1 |f(s)|\,\mathrm{d}s \\
&= \frac{1}{2}\|f\|_1
\end{aligned}$$

and therefore

$$a_n(V) \leq \|V - F\| \leq \frac{1}{2}.$$

In order to show (ii), choose the operator

$$\varrho(\mathbf{x})_k = \frac{1}{2}\sum_{j=1}^{\infty} x_j, \quad (k \in \mathbb{N}), \quad \text{for } \mathbf{x} \in \ell^1.$$

This is a well-defined one-dimensional operator and, by the calculations similar to above, $\|\sigma - \varrho\| \leq 1/2$. The proof is complete. □

Now, we are at the position to prove the main results.

Proof of Theorem 1.1. Let $n \geq 2$ be fixed. Since $a_n(V)$ is the largest among all s-numbers, we immediately obtain the inequality $a_n(V) \geq c_n(V)$ and using Lemma 3.3 with Lemma 3.2, we get $\frac{1}{2} \geq a_n(V) \geq c_n(V) \geq \frac{1}{2}$. Due to the fact that the domain space L^1 has the lifting property, we have that Kolmogorov and Approximation numbers of V coincide and therefore $d_n(V) = a_n(V) = \frac{1}{2}$. This gives (1.3).

Next, let n be arbitrary. Due to $i_n(V)$ being the smallest strict s-number, we have that $i_n(V) \leq b_n(V)$. For the lower bound, we use Lemma 3.1, while for the upper, we make use of the result of Lefèvre, [11]. This gives (1.4). □

Proof of Theorem 1.2. The proof follows along exactly the same lines as that of Theorem 1.1 and hence omitted. □

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References

[1] I. A. Alam, G. Habib, P. Lefèvre, and F. Maalouf, *Essential norms of Volterra and Cesàro operators on Müntz spaces*, arXiv:1612.03218, 2016.

[2] J. Bourgain and M. Gromov, *Estimates of Bernstein widths of Sobolev spaces*, Geometric aspects of functional analysis (1987–88), Lecture Notes in Math., vol. 1376, Springer, Berlin, 1989, pp. 176–185, DOI 10.1007/BFb0090054. MR1008722

[3] M. Š. Birman and M. Z. Solomjak, *Piecewise polynomial approximations of functions of classes* ${W_p}^\alpha$ (Russian), Mat. Sb. (N.S.) **73 (115)** (1967), 331–355. MR0217487

[4] Bernd Carl and Irmtraud Stephani, *Entropy, compactness and the approximation of operators*, Cambridge Tracts in Mathematics, vol. 98, Cambridge University Press, Cambridge, 1990. MR1098497

[5] D. E. Edmunds and W. D. Evans, *Spectral theory and differential operators*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1987. Oxford Science Publications. MR929030

[6] David E. Edmunds and W. Desmond Evans, *Hardy operators, function spaces and embeddings*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2004. MR2091115

[7] D. E. Edmunds and J. Lang, *Approximation numbers and Kolmogorov widths of Hardy-type operators in a non-homogeneous case*, Math. Nachr. **279** (2006), no. 7, 727–742, DOI 10.1002/mana.200510389. MR2226408

[8] D. E. Edmunds and J. Lang, *Bernstein widths of Hardy-type operators in a non-homogeneous case*, J. Math. Anal. Appl. **325** (2007), no. 2, 1060–1076, DOI 10.1016/j.jmaa.2006.02.025. MR2270069

[9] Jan Lang and David Edmunds, *Eigenvalues, embeddings and generalised trigonometric functions*, Lecture Notes in Mathematics, vol. 2016, Springer, Heidelberg, 2011. MR2796520

[10] A. Kolmogoroff, *Über die beste Annäherung von Funktionen einer gegebenen Funktionenklasse* (German), Ann. of Math. (2) **37** (1936), no. 1, 107–110, DOI 10.2307/1968691. MR1503273

[11] Pascal Lefèvre, *The Volterra operator is finitely strictly singular from L^1 to L^∞*, J. Approx. Theory **214** (2017), 1–8, DOI 10.1016/j.jat.2016.11.001. MR3588527

[12] V. I. Levin, *On a class of integral inequalities*, Recueil Mathématiques **4** (1938), no. 46, 309–331.

[13] Mikhail A. Lifshits and Werner Linde, *Approximation and entropy numbers of Volterra operators with application to Brownian motion*, Mem. Amer. Math. Soc. **157** (2002), no. 745, viii+87, DOI 10.1090/memo/0745. MR1895252

[14] Albrecht Pietsch, *s-numbers of operators in Banach spaces*, Studia Math. **51** (1974), 201–223. MR0361883

[15] Albrecht Pietsch, *History of Banach spaces and linear operators*, Birkhäuser Boston, Inc., Boston, MA, 2007. MR2300779

[16] Allan Pinkus, *n-widths in approximation theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 7, Springer-Verlag, Berlin, 1985. MR774404

[17] Allan Pinkus, *n-widths of Sobolev spaces in L^p*, Constr. Approx. **1** (1985), no. 1, 15–62, DOI 10.1007/BF01890021. MR766094

[18] Erhard Schmidt, *Über die Ungleichung, welche die Integrale über eine Potenz einer Funktion und über eine andere Potenz ihrer Ableitung verbindet* (German), Math. Ann. **117** (1940), 301–326, DOI 10.1007/BF01450021. MR0003430

[19] V. M. Tihomirov and S. B. Babadžanov, *Diameters of a function class in an L_p-space* ($p \geq 1$) (Russian, with Uzbek summary), Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **11** (1967), no. 2, 24–30. MR0209747

[20] V. M. Tihomirov, *Diameters of sets in functional spaces and the theory of best approximations*, Russian Math. Surveys **15** (1960), no. 3, 75–111, DOI 10.1070/RM1960v015n03ABEH004093. MR0117489

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