PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 2, February 2018, Pages 791–802
http://dx.doi.org/10.1090/proc/13770

PFA AND PRECIPITOUSNESS OF THE NONSTATIONARY IDEAL

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(Communicated by Mirna Džamonja)

ABSTRACT. We apply Neeman's method of forcing with side conditions to show that PFA does not imply the precipitousness of the nonstationary ideal on ω_1.

INTRODUCTION

One of the main consequences of Martin's Maximum (MM) is that the nonstationary ideal on ω_1 (NS_{ω_1}) is saturated and hence also precipitous. This was already shown by Foreman, Magidor and Shelah in [2], where the principle MM was introduced. It is natural to ask if the weaker Proper Forcing Axiom (PFA) is sufficient to imply the same conclusion. In the 1990s the author adapted the argument of Shelah in [8, Chapter XVII], where PFA is shown to be consistent with the existence of a function $f : \omega_1 \to \omega_1$ dominating all the canonical functions below ω_2, to show that PFA does not imply the precipitousness of NS_{ω_1}. This result was also obtained independently by Shelah and perhaps other people, but since it was never published it was considered a folklore result in the subject.

Some twenty years later Neeman [7] introduced a method for iterating proper forcing by using conditions which consist of two components: the *working part*, which is a function of finite support, and the *side condition*, which is a finite $\in$-chain of models of one of two types. The interplay between the working parts and the side conditions allows one to show that the iteration of proper forcing notions is proper. Neeman used this new iteration technique to give another proof of the consistency of PFA as well as several other interesting applications.

In this paper we adapt Neeman's iteration technique to give another proof of the consistency of PFA together with NS_{ω_1} being nonprecipitous. Our modification consists of two parts. First, we consider a *decorated* version of the side condition poset. This version is already present in [7]. Its principal virtue is that it guarantees that the generic sequence of models added by the side condition part of the forcing is continuous. The second modification is more subtle. To each condition p we attach the *height function* ht_p which is defined on certain pairs of ordinals. In order for a condition q to extend p we require that ht_q extends ht_p. Now, if G is a generic filter, we can define the derived height function ht_G from which we can read off functions h_α, for $\alpha < \theta$, where θ is the length of the iteration. Each of these functions is defined on a club in ω_1 and takes values in ω_1. If $\mathcal{U}$ is a generic ultrafilter over

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Received by the editors August 8, 2016 and, in revised form, March 19, 2017.
2010 *Mathematics Subject Classification.* Primary 03E35, 03E55, 03E65; Secondary 03E05.
Key words and phrases. Proper forcing axiom, nonstationary ideal, side conditions.

$\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ we can consider $[h_\alpha]_{\mathcal{U}}$, the equivalence class of h_α modulo $\mathcal{U}$, as an element of $\mathrm{Ult}(V,\mathcal{U})$. The point is that the family $\{[h_\alpha]_{\mathcal{U}} : \alpha < \theta\}$ cannot be well ordered by $\leq_{\mathcal{U}}$ and hence $\mathrm{Ult}(V,\mathcal{U})$ will not be well founded. Now, our requirement on the height functions introduces some complications in the proof of Neeman's lemmas required to show the properness of the iteration. The main change concerns the pure side condition part of the forcing. Our side conditions consist of pairs $(\mathcal{M}_p, d_p)$ where $\mathcal{M}_p$ is the $\in$-chain of models and d_p is the decoration. If a model M occurs in $\mathcal{M}_p$ we can form the restriction $p|M$, which is simply $(\mathcal{M}_p \cap M, d_p \restriction M)$. Then $p|M$ is itself a side condition and belongs to M. The problem is that we do not know that p is stronger than $p|M$, simply because ht_p may not extend $\mathrm{ht}_{p|M}$. However, for many models M we will be able to find a reflection q of p inside M such that ht_q does extend $\mathrm{ht}_p \restriction M$ and we will then use q instead of $p|M$. This requires reworking some of the lemmas of [7]. Since our main changes involve the side condition part of the forcing, we present a detailed proof that after forcing with the pure side condition poset the nonstationary ideal on ω_1 is not precipitous. When we add the working parts we need to rework some of Neeman's iteration lemmas, but the modifications are mostly straightforward, so we only sketch the arguments and refer the reader to [7]. Finally, let us mention that precipitousness of ideals in forcing extensions was studied by Laver [6], and while we do not use directly results from that paper, some of our ideas were inspired by [6].

The paper is organized as follows. In §1 we recall some preliminaries about canonical functions and precipitous ideals. In §2 we introduce a modification of the pure side condition forcing with models of two types from [7]. In §3 we prove a factoring lemma for our modified pure side condition poset and use it to show that after forcing with this poset the nonstationary ideal is nonprecipitous. In §4 we introduce the working parts and show how to complete the proof of the main theorem.

In order to read this paper, a fairly good understanding of [7] is necessary and the reader will be referred to it quite often. Our notation is fairly standard and can be found in [8] and [1] to which we refer the reader for background information on precipitous ideals, proper forcing, and all other undefined concepts. Let us just mention that a family $\mathcal{F}$ of subsets of a set K is called *stationary* in K if for every function $f : K^{<\omega} \to K$ there is $M \in \mathcal{F}$ which is closed under f, i.e. such that $f[M^{<\omega}] \subseteq M$.

1. Preliminaries

We start by recalling the relevant notions concerning precipitous ideals from [5]. Suppose $\mathcal{I}$ is a κ-complete ideal on a cardinal κ which contains all singletons. Let $\mathcal{I}^+$ be the collection of all $\mathcal{I}$-positive subsets of κ, i.e. $\mathcal{I}^+ = \mathcal{P}(\kappa) \setminus \mathcal{I}$. We consider $\mathcal{I}^+$ as a forcing notion under inclusion. If $G_{\mathcal{I}}$ is a V-generic over $\mathcal{I}^+$, then $G_{\mathcal{I}}$ is an ultrafilter on $\mathcal{P}^V(\kappa)$ which extends the dual filter of $\mathcal{I}$. We can then form the generic ultrapower $\mathrm{Ult}(V, G_{\mathcal{I}})$ of V by $G_{\mathcal{I}}$ in the usual way, i.e. it is $(V^\kappa \cap V)/G_{\mathcal{I}}$. Recall that $\mathcal{I}$ is called *precipitous* if the maximal condition forces that this ultrapower is well founded. There is a convenient reformulation of this property in terms of games.

Definition 1.1. Let $\mathcal{I}$ be a κ-complete ideal on a cardinal κ which contains all singletons. The game $\mathcal{G}_{\mathcal{I}}$ is played between two players I and II as follows:

$$\begin{array}{c|ccccccccc} \mathrm{I}: & E_0 & & E_2 & & \cdots & E_{2n} & & & \cdots \\ \hline \mathrm{II}: & & E_1 & & E_3 & \cdots & & E_{2n+1} & & \cdots \end{array}.$$

We require that $E_n \in \mathcal{I}^+$ and $E_{n+1} \subseteq E_n$, for all n. The first player who violates these rules loses. If both players respect the rules, we say that I wins the game if $\bigcap_n E_n = \emptyset$. Otherwise, II wins.

Fact 1.2 (Galvin, Jech, Magidor [4]). The ideal $\mathcal{I}$ is precipitous if and only if player I does not have a winning strategy in $\mathcal{G}_{\mathcal{I}}$.

We will also need the notion of *canonical functions* relative to the nonstationary ideal, NS_{ω_1}. We recall the relevant definitions from [3]. Given $f, g : \omega_1 \to \mathrm{ORD}$ we let $f <_{\mathrm{NS}_{\omega_1}} g$ if $\{\alpha : f(\alpha) < g(\alpha)\}$ contains a club. Since NS_{ω_1} is countably complete, the quasi-order $<_{\mathrm{NS}_{\omega_1}}$ is well founded. For a function $f \in \mathrm{ORD}^{\omega_1}$, let $||f||$ denote the rank of f in this ordering. It is also known as the *Galvin-Hajnal norm* of f. By induction on α, the α-th *canonical function* f_α is defined (if it exists) as the $<_{\mathrm{NS}_{\omega_1}}$-least ordinal valued function greater than the f_ξ, for all $\xi < \alpha$. Clearly, if the α-th canonical function exists, then it is unique up to the equivalence $=_{\mathrm{NS}_{\omega_1}}$. One can show in ZFC that the α-th canonical function f_α exists, for all $\alpha < \omega_2$. One way to define f_α is to fix an increasing continuous sequence $(x_\xi)_{\xi<\omega_1}$ of countable sets with $\bigcup_{\xi<\omega_1} x_\xi = \alpha$ and let $f_\alpha(\xi) = \mathrm{o.t.}(x_\xi)$, for all ξ. The point is that if we wish to witness the nonwell-foundedness of the generic ultrapower we have to work with functions that are above the ω_2 first canonical functions. Our forcing is designed to introduce θ many such functions, where θ is the length of the iteration. From these functions we define a winning strategy for I in $\mathcal{G}_{\mathrm{NS}_{\omega_1}}$, which implies that NS_{ω_1} is not precipitous in the final model.

2. The side condition poset

We start by reviewing Neeman's side condition poset from [7]. We fix a transitive model $\mathcal{K} = (K, \in, \dots)$ of a sufficient fragment of ZFC, possibly with some additional functions or predicates. Let $\mathcal{S}$ denote a collection of countable elementary submodels of $\mathcal{K}$ and let $\mathcal{T}$ be a collection of transitive $W \prec \mathcal{K}$ such that $W \in K$. We say that the pair $(\mathcal{S}, \mathcal{T})$ is *appropriate* if $M \cap W \in \mathcal{S} \cap W$, for every $M \in \mathcal{S}$ and $W \in \mathcal{T}$. We are primarily interested in the case when K is equal to V_θ, for some inaccessible cardinal θ, let $\mathcal{S}$ consist of all countable submodels of V_θ and $\mathcal{T}$ consist of all the V_α such that $V_\alpha \prec V_\theta$ and α has uncountable cofinality. We present the more general version since it will be needed in the analysis of the factor posets of the side condition forcing.

Let us fix a transitive model $\mathcal{K}$ of a sufficient fragment of set theory and an appropriate pair $(\mathcal{S}, \mathcal{T})$. The *side condition poset* $\mathbb{M}_{\mathcal{S},\mathcal{T}}$ consists of finite $\in$-chains $\mathcal{M} = \{M_0, \dots, M_{n-1}\}$ of elements of $\mathcal{S} \cup \mathcal{T}$, closed under intersection. So for each $k < n$, $M_k \in M_{k+1}$, and if $M, N \in \mathcal{M}$, then also $M \cap N \in \mathcal{M}$. We will refer to elements of $\mathcal{M} \cap \mathcal{S}$ as *small* or *countable nodes* of $\mathcal{M}$, and to the elements of $\mathcal{M} \cap \mathcal{T}$ as *transitive* nodes of $\mathcal{M}$. We will write $\pi_{\mathcal{S}}(\mathcal{M})$ for $\mathcal{M} \cap \mathcal{S}$ and $\pi_{\mathcal{T}}(\mathcal{M})$ for $\mathcal{M} \cap \mathcal{T}$. Notice that $\mathcal{M}$ is totally ordered by the ranks of its nodes, so it makes sense to say, for example, that M is above or below N, when M and N are nodes of $\mathcal{M}$. The order on $\mathbb{M}_{\mathcal{S},\mathcal{T}}$ is reverse inclusion, i.e. $\mathcal{M} \leq \mathcal{N}$ iff $\mathcal{N} \subseteq \mathcal{M}$.

The *decorated* side condition poset $\mathbb{M}^{\mathrm{dec}}_{\mathcal{S},\mathcal{T}}$ consists of pairs p of the form $(\mathcal{M}_p, d_p)$, where $\mathcal{M}_p \in \mathbb{M}_{\mathcal{S},\mathcal{T}}$ and $d_p : \mathcal{M}_p \to K$ is such that $d_p(M)$ is a finite set which belongs to the successor of M in $\mathcal{M}_p$, if this successor exists. Sometimes our d_p will be only a partial function on $\mathcal{M}_p$. In this case, we identify it with the total function which assigns the empty set to all nodes on which d_p is not defined. The order on $\mathbb{M}^{\mathrm{dec}}_{\mathcal{S},\mathcal{T}}$ is given by letting $q \leq p$ iff $\mathcal{M}_p \subseteq \mathcal{M}_q$ and $d_p(M) \subseteq d_q(M)$, for every $M \in \mathcal{M}_p$. Suppose $p \in \mathbb{M}^{\mathrm{dec}}_{\mathcal{S},\mathcal{T}}$ and $Q \in \mathcal{M}_p$. Let $p|Q$ be the condition $(\mathcal{M}_p \cap Q, d_p \restriction Q)$. One can check that $p|Q$ is indeed a condition in $\mathbb{M}^{\mathrm{dec}}_{\mathcal{S},\mathcal{T}}$.

We first observe the following simple fact.

Lemma 2.1. *Suppose p is a condition in $\mathbb{M}^{\mathrm{dec}}_{\mathcal{S},\mathcal{T}}$ and M is a model in $\mathcal{S} \cup \mathcal{T}$ such that $p \in M$. Then there is a condition p^M extending p such that M is the top model of $\mathcal{M}_{p^M}$.*

Proof. We let $\mathcal{M}_{p^M}$ be the closure of $\mathcal{M}_p \cup \{M\}$ under intersection. Note that if $M \in \mathcal{T}$, then all the nodes of $\mathcal{M}_p$ are subsets of M and hence $\mathcal{M}_{p^M}$ is simply $\mathcal{M}_p \cup \{M\}$. On the other hand if M is countable we need to add nodes of the form $M \cap W$, where W is a transitive mode in $\mathcal{M}_p$. We define d_{p^M} by letting $d_{p^M}(N) = d_p(N)$, if $N \in \mathcal{M}_p$, and $d_{p^M}(N) = \emptyset$, if N is one of the new nodes. It is straightforward to check that p^M is as desired. $\square$

The main technical results about the (decorated) side condition poset are Corollaries 2.31 and 2.32 together with Claim 2.38 in [7]. We combine them here as one lemma.

Lemma 2.2 ([7]). *Let p be a condition in $\mathbb{M}^{\mathrm{dec}}_{\mathcal{S},\mathcal{T}}$, and let Q be a node in $\mathcal{M}_p$. Suppose that q is a condition in $\mathbb{M}^{\mathrm{dec}}_{\mathcal{S},\mathcal{T}}$ which belongs to Q and strengthens the condition $p|Q$. Then there is $r \in \mathbb{M}^{\mathrm{dec}}_{\mathcal{S},\mathcal{T}}$ with $r \leq p, q$ such that:*

1. *$\mathcal{M}_r$ is the closure under intersection of $\mathcal{M}_p \cup \mathcal{M}_q$,*
2. *$\mathcal{M}_r \cap Q = \mathcal{M}_q$,*
3. *the small nodes of $\mathcal{M}_r$ outside Q are of the form M or $M \cap W$, where M is a small node of $\mathcal{M}_p$ and W is a transitive node of $\mathcal{M}_q$.*

We now discuss our modification of Neeman's posets. Let $\theta = K \cap \mathrm{ORD}$. We will choose $\mathcal{S}$ and $\mathcal{T}$ to be stationary families of subsets of K. The stationarity of $\mathcal{S}$ guarantees that ω_1 is preserved in the generic extension and the stationarity of $\mathcal{T}$ guarantees that θ is preserved. All the cardinals in between will be collapsed to ω_1, thus θ becomes ω_2 in the final model. We plan to simultaneously add θ-many partial functions from ω_1 to ω_1. Each of the partial functions will be defined on a club in ω_1 and will be forced to dominate the θ first canonical functions in the generic extension. The decorated version of the pure side condition forcing gives us a natural way to represent the canonical function f_α, for cofinally many $\alpha < \theta$.

Before we introduce our version of the side condition poset let us make a definition.

Definition 2.3. Let $\mathcal{M}$ be a member of $\mathbb{M}_{\mathcal{S},\mathcal{T}}$. We define the partial function $h_{\mathcal{M}}$ from $\theta \times \omega_1$ as follows. The domain of $h_{\mathcal{M}}$ is the set of pairs (α, ξ) such that there is a countable node $M \in \mathcal{M}$ with $M \cap \omega_1 = \xi$ and $\alpha \in M$. If $(\alpha, \xi) \in \mathrm{dom}(h_{\mathcal{M}})$ we let

$$h_{\mathcal{M}}(\alpha, \xi) = \max\{\mathrm{o.t.}(M \cap \theta) : M \in \pi_{\mathcal{S}}(\mathcal{M}), \alpha \in M \text{ and } M \cap \omega_1 = \xi\}.$$

If $p \in \mathbb{M}^*_{\mathcal{S},\mathcal{T}}$ we let h_p denote $h_{\mathcal{M}_p}$. We are now ready to define our modified side condition poset $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$.

Definition 2.4. The poset $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$ consists of all conditions $p \in \mathbb{M}^{\mathrm{dec}}_{\mathcal{S},\mathcal{T}}$ such that

$(*)$ for every $M, N \in \pi_{\mathcal{S}}(\mathcal{M}_p)$, if $N \cap \omega_1 \in M$, then $\mathrm{o.t.}(N \cap \theta) \in M$.

The ordering is defined by letting $q \leq p$ iff $(\mathcal{M}_q, d_q) \leq_{\mathbb{M}^{\mathrm{dec}}_{\mathcal{S},\mathcal{T}}} (\mathcal{M}_p, d_p)$ and $h_p \subseteq h_q$.

Let us first observe that we have an analog of Lemma 2.1.

Lemma 2.5. *Suppose p belongs to $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$ and M is a model in $\mathcal{S} \cup \mathcal{T}$ such that $p \in M$. Then there is a condition p^M extending p such that M is the top node of $\mathcal{M}_{p^M}$.*

We now establish some elementary properties of conditions in $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$.

Lemma 2.6. *Suppose p is a condition in $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$ and $M \in \mathcal{M}_p$. Then $h_p \restriction M \in M$.*

Proof. If M is a transitive node, then this is immediate. Suppose M is countable. We will use the following.

Claim 2.7. Suppose N is a countable node in $\mathcal{M}_p$ and $N \cap \omega_1 \in M$. Then $N \cap M \in M$.

Proof. Since $\mathcal{M}_p$ is closed under intersection we have that $N \cap M \in \mathcal{M}_p$. Moreover, since $N \cap \omega_1 \in M$ we have that $N \cap M$ is below M. If there is no transitive node between $N \cap M$ and M, then $N \cap M \in M$. Otherwise, let W be the least transitive node above $N \cap M$. By closure under intersection again, $M \cap W \in \mathcal{M}_p$. Moreover, $N \cap M \subseteq M \cap W$ and the inclusion is proper. Therefore, $M \cap W$ is a countable node above $N \cap M$ and there is no transitive node between them. Therefore, $N \cap M \in M \cap W$ and so $N \cap M \in M$. □

Let us say that a node N of $\mathcal{M}_p$ is an *end node* of $\mathcal{M}_p$ if there is no node in $\mathcal{M}_p$ which is an end extension of N. The domain of the function h_p is the union of all the sets of the form $(N \cap \theta) \times \{\xi\}$, where N is a countable end node of $\mathcal{M}_p$ and $\xi = N \cap \omega_1$. Moreover, on $(N \cap \theta) \times \{\xi\}$ the function h_p is constant and equal to $\mathrm{o.t.}(N \cap \theta)$. Now, if $\xi \in M$, then by Claim 2.7 $N \cap M \in M$. Moreover, since $p \in \mathbb{M}^*_{\mathcal{S},\mathcal{T}}$ we have that $\mathrm{o.t.}(N \cap \theta) \in M$. It follows that $h_p \restriction M \in M$. □

We wish to have an analog of Lemma 2.2. If $p \in \mathbb{M}^*_{\mathcal{S},\mathcal{T}}$ and $Q \in \mathcal{M}_p$ we can let $p|Q$ be $(\mathcal{M}_p \cap Q, d_p \restriction Q)$. It is easy to check that $p|Q$ is a condition. However, we do not know that p extends $p|Q$ since h_p may not be an extension of $h_{p|Q}$. We must refine the notion of restriction in order to arrange this. In order to do this, let us enrich our initial structure $\mathcal{K}$ by adding predicates for $\mathcal{S}$ and $\mathcal{T}$. Let $\mathcal{K}^*$ denote the structure $(K, \in, \mathcal{S}, \mathcal{T}, \ldots)$. Note that our poset $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$ is definable in $\mathcal{K}^*$. Let $\mathcal{S}^*$ be the collection of all $M \in \mathcal{S}$ that are elementary in $\mathcal{K}^*$ and let $\mathcal{T}^*$ be the set of all $W \in \mathcal{T}$ that are elementary in $\mathcal{K}^*$. Note that $\mathcal{S}^*$ (respectively $\mathcal{T}^*$) is a relative club in $\mathcal{S}$ (respectively $\mathcal{T}$), hence if $\mathcal{S}$ (respectively $\mathcal{T}$) is stationary, then so is $\mathcal{S}^*$ (respectively $\mathcal{T}^*$).

Assume p is a condition and Q a node in p which belongs to $\mathcal{S}^* \cup \mathcal{T}^*$. Now, we know that $p|Q$ and $h_p \restriction Q$ belong to Q. Moreover, Q is elementary in $\mathcal{K}^*$ and $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$ is definable in this structure. Therefore, there is a condition $q \in Q$ such that $\mathcal{M}_p \cap Q \subseteq \mathcal{M}_q$, $d_p(R) \subseteq d_q(R)$ for all $R \in \mathcal{M}_p \cap Q$, and h_q extends $h_p \restriction Q$. We will call such q a *reflection* of p inside Q. Note that if q is a reflection of p inside

Q, then any condition $r \in Q$ which is stronger than q is also a reflection of p inside Q. Let us say that p *reflect* to Q if $p|Q$ is already a reflection of p to Q. Finally, let us say that p is *reflecting* if p reflects to W, for all nodes $W \in \mathcal{M}_p \cap \mathcal{T}^*$.

We now have a version of Lemma 2.2 for our poset.

Lemma 2.8. *Let p be a condition in $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$, and let Q be a node in $\mathcal{M}_p$ which belongs to $\mathcal{S}^* \cup \mathcal{T}^*$. Suppose that $q \in \mathbb{M}^*_{\mathcal{S},\mathcal{T}}$ is a reflection of p inside Q. Then there is $r \in \mathbb{M}^*_{\mathcal{S},\mathcal{T}}$ with $r \leq p, q$ such that:*

1. *$\mathcal{M}_r$ is the closure under intersection of $\mathcal{M}_p \cup \mathcal{M}_q$,*
2. *$\mathcal{M}_r \cap Q = \mathcal{M}_q$,*
3. *the small nodes of $\mathcal{M}_r$ outside Q are of the form M or $M \cap W$, where M is a small node of $\mathcal{M}_p$ and W is a transitive node of $\mathcal{M}_q$.*

Proof. Let r be the condition given by Lemma 2.2. We need to check that $\mathcal{M}_r$ satisfies $(*)$ and h_r extends h_p and h_q. If $Q \in \mathcal{T}^*$ this is straightforward, so let us assume that $Q \in \mathcal{S}^*$. We first verify that $(*)$ holds for $\mathcal{M}_r$. So, suppose N, M are countable nodes in $\mathcal{M}_r$ and $N \cap \omega_1 \in M$. We need to check that $\mathrm{o.t.}(N \cap \theta) \in M$. Recall that by (3) of Lemma 2.2 every small node R of $\mathcal{M}_r$ outside Q is either in $\mathcal{M}_p$ or of the form $R' \cap W$, for some small node $R' \in \mathcal{M}_p$ and a transitive node W of $\mathcal{M}_q$. In the latter case, note that $R \cap \omega_1 = R' \cap \omega_1$ and $\mathrm{o.t.}(R \cap \theta) \leq \mathrm{o.t.}(R' \cap \theta)$. Thus, we may assume that N and M belong to $\mathcal{M}_p \cup \mathcal{M}_q$. If N and M are both in $\mathcal{M}_p$ or in $\mathcal{M}_q$ this follows from the fact that $\mathcal{M}_p$ and $\mathcal{M}_q$ satisfy $(*)$. Now, suppose $N \in \mathcal{M}_q$ and $M \in \mathcal{M}_p \setminus \mathcal{M}_q$. Since $N \in Q$ it follows that $\mathrm{o.t.}(N \cap \theta) \in Q$, so if $M \cap \omega_1 \geq Q \cap \omega_1$, then we have $\mathrm{o.t.}(N \cap \theta) \in M$. If $M \cap \omega_1 < Q \cap \omega_1$, then by Claim 2.7, $M \cap Q \in Q$ and hence $M \cap Q \in \mathcal{M}_q$. Therefore our conclusion follows from the fact that $\mathcal{M}_q$ satisfies $(*)$. Now, suppose $M \in \mathcal{M}_q$ and $N \in \mathcal{M}_p \setminus \mathcal{M}_q$. Since $N \cap \omega_1 \in M$ and $M \in Q$ it follows that $N \cap \omega_1 \in Q$. By our assumption q is a reflection of p inside Q, so there is a node $N' \in \mathcal{M}_q$ such that $N' \cap \omega_1 = N \cap \omega_1$ and $\mathrm{o.t.}(N \cap \theta) \leq \mathrm{o.t.}(N' \cap \theta)$. Now, N' and M both belong to $\mathcal{M}_q$ which satisfies $(*)$, so $\mathrm{o.t.}(N' \cap \theta) \in M$. Therefore, in all cases $\mathrm{o.t.}(N \cap \theta) \in M$.

Now, we check that h_r extends h_p and h_q. Since every small node in $\mathcal{M}_r$ is either in $\mathcal{M}_p \cup \mathcal{M}_q$ or is of the form $M \cap W$, for some small node M of $\mathcal{M}_p$ and a transitive node $W \in \mathcal{M}_q$, we have that all end nodes of $\mathcal{M}_r$ are in $\mathcal{M}_p \cup \mathcal{M}_q$. Also, since q is a reflection of p inside Q, we have that h_q extends $h_p \restriction Q$ and $\mathrm{dom}(h_q) \subseteq Q$. It follows that $h_r = h_p \cup h_q$. This completes the proof of the lemma. □

We have a couple of immediate corollaries.

Corollary 2.9. *For every condition $p \in \mathbb{M}^*_{\mathcal{S},\mathcal{T}}$ there is a reflecting condition $q \leq p$ which has the same top model as p.*

Corollary 2.10. *$\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$ is $\mathcal{S}^* \cup \mathcal{T}^*$-strongly proper. In particular, if $\mathcal{S}$ is stationary, then $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$ preserves ω_1, and if $\mathcal{T}$ is stationary, then $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$ preserves θ.*

From now on, we assume that $\mathcal{S}$ and $\mathcal{T}$ are stationary families of subsets of K. Suppose that G is a V-generic filter over $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$. Let $\mathcal{M}_G$ denote $\bigcup\{\mathcal{M}_p : p \in G\}$. Then $\mathcal{M}_G$ is an $\in$-chain of models in $\mathcal{S} \cup \mathcal{T}$. Hence, $\mathcal{M}_G$ is totally ordered by $\in^*$, where $\in^*$ denotes the transitive closure of $\in$. If M, N are members of $\mathcal{M}_G$ with $M \in^* N$ let $(M, N)_G$ denote the interval consisting of all $P \in \mathcal{M}_G$ such that $M \in^* P \in^* N$. The following lemma is the main reason we are working with the decorated version of the side condition poset.

Lemma 2.11. *Suppose W and W' are two consecutive elements of $\mathcal{M}_G \cap \mathcal{T}$. Then the $\in$-chain $(W, W')_G$ is continuous.*

Proof. Note that $(W, W')_G$ consists entirely of countable models and therefore the membership relation is transitive on $(W, W')_G$. Suppose M is a limit member of $(W, W')_G$. We need to show that M is the union of the $\in$-chain $(W, M)_G$. Let $p \in G$ be a condition such that $M \in \mathcal{M}_p$ and p forces that M is a limit member of $(W, W')_G$. Given any $x \in M$ and a condition $q \leq p$ we show that there is $r \leq q$ and a countable node that $R \in \mathcal{M}_r \cap M$ such that $x \in R$. We may assume that there is a countable model in $\mathcal{M}_q$ between W and M. Let Q be the $\in^*$-largest such model. By increasing $d_q(Q)$ if necessary, we may assume that $x \in d_q(Q)$. Since p forces that M is a limit member of $\mathcal{M}_G$ so does q. Therefore, there exists $r \leq q$ such that $\mathcal{M}_r$ contains a countable node between Q and M. Let R be the $\in^*$-least such node. By the definition of the order relation on $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$ we must have that $d_q(Q) \in R$ and hence $x \in R$, as desired. □

Suppose W and W' are two consecutive members of $\mathcal{M}_G \cap \mathcal{T}$ and let $\beta = W' \cap \mathrm{ORD}$. We give a description of the β-th canonical function f_β in $V[G]$. First, since $\mathcal{S}$ is stationary it follows that every element of K belongs to some small node in $\mathcal{M}_G$. Also $\mathcal{M}_G$ is closed under intersections, hence W' is the union of the continuous $\in$-chain $(W, W')_G$. By Corollary 2.10 we know that $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$ preserves ω_1, so this chain is of length ω_1. Let $\{M_\xi : \xi < \omega_1\}$ be its increasing enumeration. Then we can let $f_\beta(\xi) = \mathrm{o.t.}(M_\xi \cap \beta)$, for all ξ. Note that $M_\xi \cap \omega_1 = \xi$, for club many ξ.

Now, let h_G denote $\bigcup\{h_p : p \in G\}$. Then h_G is a partial function from $\theta \times \omega_1$ to ω_1. Let $h_{G,\alpha}$ be a partial function from ω_1 to ω_1 defined by letting $h_{G,\alpha}(\xi) = h_G(\alpha, \xi)$, for every ξ such that $(\alpha, \xi) \in \mathrm{dom}(h_G)$. By Lemma 2.11 and the above remarks we have the following.

Corollary 2.12. *For all $\alpha < \theta$, the function $h_{G,\alpha}$ is defined on a club in ω_1. Moreover, $h_{G,\alpha}$ dominates under $<_{\mathrm{NS}_{\omega_1}}$ all the canonical functions f_β, for $\beta < \theta$.*

3. Factoring the side condition poset

We now let $\mathcal{K} = (V_\theta, \in, \ldots)$, for some inaccessible cardinal θ. Let T be the set of all $\alpha < \theta$ of uncountable cofinality such that $V_\alpha \prec \mathcal{K}$ and let $\mathcal{T} = \{V_\delta : \delta \in T\}$. Finally, let $\mathcal{S}$ be the set of all countable elementary submodels of $\mathcal{K}$. Clearly, the pair $(\mathcal{S}, \mathcal{T})$ is appropriate. Let $\mathcal{S}^*$ and $\mathcal{T}^*$ be defined as before and let $T^* = \{\alpha : V_\alpha \in \mathcal{T}^*\}$. We start by analyzing the factor posets of $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$. Suppose $\delta \in T^*$ and let $p_\delta = (\{V_\delta\}, \emptyset)$. Then, by Lemma 2.8, the map $i_\delta : \mathbb{M}^*_{\mathcal{S},\mathcal{T}} \cap V_\delta \to \mathbb{M}^*_{\mathcal{S},\mathcal{T}} \restriction p_\delta$ given by $i_\delta(p) = (\mathcal{M}_p \cup \{V_\delta\}, d_p)$ is a complete embedding. Fix a V-generic filter G_δ over $\mathbb{M}^*_{\mathcal{S},\mathcal{T}} \cap V_\delta$. Let $\mathcal{M}_{G_\delta}$ denote $\bigcup\{\mathcal{M}_p : p \in G_\delta\}$ and let h_{G_δ} be the derived height function, i.e. $h_{G_\delta} = \bigcup\{h_p : p \in G_\delta\}$. Let $\mathbb{Q}_\delta$ denote the factor forcing $\mathbb{M}^*_{\mathcal{S},\mathcal{T}} \restriction p_\delta / i_\delta[G_\delta]$. We can identify $\mathbb{Q}_\delta$ with the set of all conditions $p \in \mathbb{M}^*_{\mathcal{S},\mathcal{T}}$ such that $V_\delta \in \mathcal{M}_p$, p reflects to V_δ and $p|V_\delta \in G_\delta$.

We make the following definition in $V[G_\delta]$.

Definition 3.1. Let $\mathcal{S}_\delta$ be the collection of all $M \in \mathcal{S}$ such that $M \not\subseteq V_\delta$, $M \cap V_\delta \in \mathcal{M}_{G_\delta}$ and $\mathrm{o.t.}(M \cap \theta) \leq h_{G_\delta}(\alpha, M \cap \omega_1)$, for all $\alpha \in M \cap \delta$.

We also let $T_\delta = T \setminus (\delta + 1)$ and $\mathcal{T}_\delta = \{V_\gamma : \gamma \in T_\delta\}$. We define $\mathcal{S}^*_\delta$ and $\mathcal{T}^*_\delta$ as before. Clearly, the pair $(\mathcal{S}_\delta, \mathcal{T}_\delta)$ is appropriate. We show that $\mathbb{Q}_\delta$ is very close to being $\mathbb{M}^*_{\mathcal{S}_\delta,\mathcal{T}_\delta}$. More precisely, let $\mathbb{M}^*_\delta$ consist of all pairs p of the form $(\mathcal{M}_p, d_p)$ such that $\mathcal{M}_p \in \mathbb{M}_{\mathcal{S}_\delta,\mathcal{T}_\delta}$, $d_p : \mathcal{M}_p \cup \{V_\delta\} \to V_\theta$, $(\mathcal{M}_p, d_p \restriction \mathcal{M}_p) \in \mathbb{M}^*_{\mathcal{S}_\delta,\mathcal{T}_\delta}$, and V_δ and $d_p(V_\delta)$ belong to the least model of $\mathcal{M}_p$. So, formally we do not put V_δ as the least node of conditions p in $\mathbb{M}^*_\delta$, but we require the function d_p to be defined on $\mathcal{M}_p \cup \{V_\delta\}$. This puts a restriction on the nodes we are allowed to add below the least node of $\mathcal{M}_p$.

Lemma 3.2. *$\mathbb{Q}_\delta$ and $\mathbb{M}^*_\delta$ are equivalent forcing notions.*

Proof. Given a condition $p \in \mathbb{Q}_\delta$, let $\varphi(p) = (\mathcal{M}_p \setminus V_{\delta+1}, d_p \restriction (\mathcal{M}_p \setminus V_\delta))$. Clearly, the function φ is order preserving. To see that φ is onto, let $s \in \mathbb{M}^*_\delta$. Then $M \cap V_\delta \in \mathcal{M}_{G_\delta}$, for all small nodes $M \in \mathcal{M}_s$. Fix a condition $p \in G_\delta$ such that $M \cap V_\delta \in \mathcal{M}_p$, for every such M. Define a condition q by letting $\mathcal{M}_q = \mathcal{M}_p \cup \{V_\delta\} \cup \mathcal{M}_s$ and $d_q = d_p \cup d_s$. Since every small node of $\mathcal{M}_s$ is in $\mathcal{S}_\delta$ it follows that $\mathrm{ht}_q \restriction \delta \times \omega_1 = \mathrm{ht}_p$. Therefore, $q \in \mathbb{Q}_\delta$ and $\varphi(q) = s$. Finally, note that if $p, q \in \mathbb{Q}_\delta$, then p and q are compatible in $\mathbb{Q}_\delta$ iff $\varphi(p)$ and $\varphi(q)$ are compatible in $\mathbb{M}^*_\delta$. This implies that $\mathbb{Q}_\delta$ and $\mathbb{M}^*_\delta$ are equivalent forcing notions. □

By Lemma 3.2 and Corollary 2.10 we now have the following.

Corollary 3.3. *$\mathbb{Q}_\delta$ is $\mathcal{S}^*_\delta \cup \mathcal{T}^*_\delta$-strongly proper.*

Lemma 3.4. *$\mathcal{S}_\delta$ is a stationary family of countable subsets of V_θ.*

Proof. We argue in V via a density argument. Let $\dot{f}$ be an $\mathbb{M}^*_{\mathcal{S},\mathcal{T}} \cap V_\delta$-name for a function from $V_\theta^{<\omega}$ to V_θ and let $p \in \mathbb{M}^*_{\mathcal{S},\mathcal{T}} \cap V_\delta$. We find a condition $q \leq p$ and $M \in \mathcal{S}$ such that q forces that M belongs to $\dot{\mathcal{S}}_\delta$ and is closed under $\dot{f}$. For this purpose, fix a cardinal $\theta^* > \theta$ such that V_{θ^*} satisfies a sufficient fragment of ZFC. Let M^* be a countable elementary submodel of V_{θ^*} containing all the relevant parameters. It follows that $M \in \mathcal{S}^*$, where $M = M^* \cap V_\theta$. Let p^M be the condition given by Lemma 2.5. Since $\delta \in \mathcal{T}^*$ we can find a reflection q of p^M inside V_δ. We claim that q and M are as required. To see this, note that, since $M \cap V_\delta \in \mathcal{M}_q \cap \mathcal{S}^*$, by Lemma 2.8, q is $(M \cap V_\delta, \mathbb{M}^*_{\mathcal{S},\mathcal{T}} \cap V_\delta)$-strongly generic and hence also $(M^*, \mathbb{M}^*_{\mathcal{S},\mathcal{T}} \cap V_\delta)$-generic. It follows that q forces that $M^*[\dot{G}_\delta] \cap V_\theta = M$, and hence that M is closed under $\dot{f}$. On the other hand, $h_{p^M}(\alpha, M \cap \omega_1) = \mathrm{o.t.}(M \cap \theta)$, for all $\alpha \in M \cap \theta$. Since q is a reflection of p^M, we have $h_q(\alpha, M \cap \omega_1) = h_{p^M}(\alpha, M \cap \omega_1)$, for all $\alpha \in M \cap \delta$. Therefore, q forces M to belong to $\dot{\mathcal{S}}_\delta$. This completes the argument. □

We need to understand which stationary subsets of ω_1 in $V[G_\delta]$ will remain stationary in the final model. So, suppose E is a subset of ω_1 in $V[G_\delta]$. Let

$$\mathcal{S}_\delta(E) = \{M \in \mathcal{S}_\delta : M \cap \omega_1 \in E\}.$$

For $\rho \in T_\delta$ let $\mathcal{S}^\rho_\delta(E) = \mathcal{S}_\delta(E) \cap V_\rho$. Note that if $M \in \mathcal{S}_\delta(E)$ and $\rho \in T_\delta$, then $M \cap V_\rho \in \mathcal{S}^\rho_\delta(E)$. Therefore, if $\rho < \sigma$ and $\mathcal{S}^\sigma_\delta(E)$ is stationary in V_σ, then $\mathcal{S}^\rho_\delta(E)$ is stationary in V_ρ. Since θ is inaccessible, it follows that $\mathcal{S}_\delta(E)$ is stationary in V_θ iff $\mathcal{S}^\rho_\delta(E)$ is stationary in V_ρ, for all $\rho \in T_\delta$.

Lemma 3.5. *The maximal condition in $\mathbb{Q}_\delta$ decides if E remains stationary in ω_1. Namely, if $\mathcal{S}_\delta(E)$ is stationary in V_θ, then $\Vdash_{\mathbb{Q}_\delta} \check{E}$ is stationary, and if $\mathcal{S}_\delta(E)$ is nonstationary, then $\Vdash_{\mathbb{Q}_\delta} \check{E}$ is nonstationary.*

Proof. The first implication follows from Corollary 3.3 and the fact that $\mathcal{S}_\delta^*$ is a relative club in $\mathcal{S}_\delta$. For the second implication, suppose $\mathcal{S}_\delta(E)$ is nonstationary and fix a successor element of T_δ, say σ, such that $\mathcal{S}_\delta^\sigma(E)$ is nonstationary in V_σ. Let ρ be the predecessor of σ in T_δ and fix a condition $p \in \mathbb{Q}_\delta$ such that $V_\rho, V_\sigma \in \mathcal{M}_p$. Pick an arbitrary $V[G_\delta]$-generic filter G over $\mathbb{Q}_\delta$ containing p. Then we can identify G with a V-generic filter $\bar{G}$ over $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$ which extends G_δ and such that $V_\delta \in \mathcal{M}_{\bar{G}}$. Since $p \in \bar{G}$, we have that V_ρ and V_σ are consecutive elements of $\mathcal{M}_{\bar{G}} \cap \mathcal{T}$. By Lemma 2.11 we know that, in $V[\bar{G}]$, $\mathcal{S}_\delta \cap V_\sigma$ contains a club of countable subsets of V_σ. On the other hand, by our assumption, $\mathcal{S}_\delta^\sigma(E)$ is nonstationary. It follows that E is a nonstationary subset of ω_1 in $V[\bar{G}]$. Since G was an arbitrary generic filter containing p, it follows that $p \Vdash_{\mathbb{Q}_\delta} \check{E}$ is nonstationary in ω_1. □

Remark 3.6. One can show that if δ is inaccessible in V, then $\mathbb{Q}_\delta$ actually preserves stationary subsets of ω_1. To see this note that, under this assumption, for every subset E of ω_1 in $V[G_\delta]$ there is $\delta^* < \delta$ with $V_{\delta^*} \in \mathcal{M}_{G_\delta}$ such that $E \in V[G_{\delta^*}]$, where $G_{\delta^*} = G_\delta \cap V_{\delta^*}$. If, in the model $V[G_{\delta^*}]$, $\mathcal{S}_{\delta^*}(E)$ is nonstationary there is $\rho < \theta$ such that $\mathcal{S}^\rho_{\delta^*}(E)$ is nonstationary. By elementarity of V_δ in V_θ there is such $\rho < \delta$. But then, as in the proof of Lemma 3.5, we would have that E is nonstationary already in the model $V[G_\delta]$.

Suppose $E \in V[G_\delta]$ is a subset ω_1 and $\gamma < \delta$. Let

$$\mathcal{S}_\delta(E,\gamma) = \{M \in \mathcal{S}_\delta(E) : \gamma, \delta \in M \text{ and } \mathrm{o.t.}(M \cap \theta) < h_{G_\delta}(\gamma, M \cap \omega_1)\}.$$

Recall that if $M \in \mathcal{S}_\delta$, then $M \cap V_\delta \in G_\delta$. Hence, if $\gamma \in M$, then $(\gamma, M \cap \omega_1) \in \mathrm{dom}(h_{G_\delta})$.

Lemma 3.7. *Suppose that, in $V[G_\delta]$, E is a subset of ω_1 such that $\mathcal{S}_\delta(E)$ is stationary. Then $\mathcal{S}_\delta(E,\gamma)$ is stationary, for all $\gamma < \delta$.*

Proof. Work in $V[G_\delta]$ and let $\gamma < \delta$ and $f : V_\theta^{<\omega} \to V_\theta$ be given. We need to find a member of $\mathcal{S}_\delta(E,\gamma)$ which is closed under f. Since θ is inaccessible, we can first find $\sigma \in T_\delta$ such that V_σ is closed under f. We know that $\mathcal{S}_\delta(E)$ is stationary, hence we can find $M \in \mathcal{S}_\delta(E)$ which is closed under f and such that $\gamma, \delta, \sigma \in M$. It follows that $M \cap V_\sigma$ is also closed under f. Since $\sigma \in M$ we have that $\mathrm{o.t.}(M \cap \sigma) < \mathrm{o.t.}(M \cap \theta)$. Since $M \in \mathcal{S}_\delta(E)$ and $\gamma \in M$ we have that $\mathrm{o.t.}(M \cap \theta) \leq h_{G_\delta}(\gamma, M \cap \omega_1)$. Finally, $(M \cap V_\sigma) \cap \omega_1 = M \cap \omega_1$. It follows that $M \cap V_\sigma \in \mathcal{S}_\delta(E,\gamma)$, as desired. □

We now consider what happens in the final model $V[G]$, where G is V-generic over $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$. For an ordinal $\gamma < \theta$ let D_γ denote the domain of $h_{G,\gamma}$. Recall that, by Corollary 2.12, D_γ contains a club, for all γ. Given a subset E of ω_1 and $\gamma, \delta < \theta$ let

$$\varphi(E,\gamma,\delta) = \{\xi \in E \cap D_\gamma \cap D_\delta : h_{G,\delta}(\xi) < h_{G,\gamma}(\xi)\}.$$

Lemma 3.8. *Let G be V-generic over $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$. Suppose, in $V[G]$, that E is a stationary subset of ω_1 and $\gamma < \theta$. Then there is $\delta < \theta$ such that $\varphi(E,\gamma,\delta)$ is stationary.*

Proof. Since $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$ is $\mathcal{T}^*$-proper, we can find $\delta \in T^* \setminus (\gamma + 1)$ such that $V_\delta \in \mathcal{M}_G$ and $E \in V[G_\delta]$, where $G_\delta = G \cap V_\delta$. Since E remains stationary in $V[G]$, it follows that, in $V[G_\delta]$, $\mathcal{S}_\delta(E)$ is stationary. Work for a while in $V[G_\delta]$. We claim that the maximal condition in $\mathbb{Q}_\delta$ forces that $\dot{\varphi}(E,\gamma,\delta)$ is stationary, where $\dot{\varphi}(E,\gamma,\delta)$ is the canonical name for $\varphi(E,\gamma,\delta)$. To see this fix a $\mathbb{Q}_\delta$-name $\dot{C}$ for a club in

ω_1 and a condition $p \in \mathbb{Q}_\delta$. Let $\theta^* > \theta$ be such that $(V_{\theta^*}, \in)$ satisfies a sufficient fragment of ZFC. We know, by Lemma 3.7, that $\mathcal{S}_\delta(E,\gamma)$ is stationary, so we can find a countable elementary submodel M^* of V_{θ^*} containing all the relevant objects such that $M \in \mathcal{S}_\delta(E,\gamma)$, where $M = M^* \cap V_\theta$. Let q be the condition p^M as in Lemma 2.5 (or rather its version for $\mathbb{Q}_\delta$). Since $\dot{C} \in M^*$ and q is $(M^*, \mathbb{Q}_\delta)$-generic, it follows that q forces that $M \cap \omega_1$ belongs to $\dot{C}$. Also, note that the top model of $\mathcal{M}_q$ is M. Hence $h_q(\delta, M \cap \omega_1) = \text{o.t.}(M \cap \theta)$. Since $M \in \mathcal{S}_\delta(E,\gamma)$ we have that $\text{o.t.}(M \cap \theta) < h_{G_\delta}(\gamma, M \cap \omega_1)$. It follows that q forces that $M \cap \omega_1$ belongs to the intersection of $\dot{\varphi}(E,\gamma,\delta)$ and $\dot{C}$, as required. □

We now have the following conclusion.

Theorem 3.9. *Let G be V-generic over $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$. Then, in $V[G]$, $\theta = \omega_2$ and NS_{ω_1} is not precipitous.*

Proof. We already know that $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$ is $\mathcal{S}^* \cup \mathcal{T}^*$-strongly proper. This implies that ω_1 and θ are preserved. Moreover, by Lemma 2.11, we know that all cardinals between ω_1 and θ are collapsed to $\aleph_1$. Therefore, θ becomes ω_2 in $V[G]$. In order to show that NS_{ω_1} is not precipitous we describe a winning strategy τ for Player I in $\mathcal{G}_{\text{NS}_{\omega_1}}$. On the side, Player I will pick a sequence $(\gamma_n)_n$ of ordinals $< \theta$. So, Player I starts by playing $E_0 = \omega_1$ and letting $\gamma_0 = 0$. Suppose, in the n-th inning, Player II has played a stationary set E_{2n+1}. Player I applies Lemma 3.8 to find $\delta < \theta$ such that $\varphi(E_{2n+1}, \gamma_n, \delta)$ is stationary. He then lets $\gamma_{n+1} = \delta$ and plays $E_{2n+2} = \varphi(E_{2n+1}, \gamma_n, \gamma_{n+1})$. Suppose the game continues ω moves and II respects the rules. We need to show that $\bigcap_n E_n$ is empty. Indeed, if $\xi \in \bigcap_n E_n$, then $\xi \in D_{\gamma_n}$, for all n, and $h_{G,\gamma_0}(\xi) > h_{G,\gamma_1}(\xi) > \ldots$ is an infinite decreasing sequence of ordinals, a contradiction. □

4. The working parts

In this section we show how to add the working part to the side condition poset described in §2, which allows us to define a Neeman style iteration. As in [7], if at each stage we choose a proper forcing, the resulting forcing notion will be proper as well. By a standard argument, if we use a Laver function to guide our choices, we obtain PFA in the final model. The point is that the relevant lemmas from §2 and §3 go through almost verbatim and hence we obtain, as before, that NS_{ω_1} will be nonprecipitous in the final model.

Let us now recall the iteration technique from [7]. We fix an inaccessible cardinal θ and a function $F : \theta \to V_\theta$. Let $\mathcal{K}$ be the structure $(V_\theta, \in, F)$. Let $\mathcal{S}$ be the set of all countable elementary submodels of $\mathcal{K}$ and T the set of all $\alpha < \theta$ of uncountable cofinality such that V_α is an elementary submodel of $\mathcal{K}$. Let $\mathcal{T} = \{V_\alpha : \alpha \in T\}$. Define $\mathcal{S}^*$, T^* and $\mathcal{T}^*$ as before. Note that if $\alpha \in T^*$, then $T^* \cap \alpha$ is definable in $\mathcal{K}$ from parameter α. Hence, if $\mathcal{M} \in \mathcal{S}$ and $\alpha \in T^*$, then $M \cap V_\alpha \in \mathcal{S}^*$. We will define, by induction on $\alpha \in T^* \cup \{\theta\}$, a forcing notion $\mathbb{P}_\alpha$. In general, $\mathbb{P}_\alpha$ consists of triples p of the form $(\mathcal{M}_p, d_p, w_p)$ such that $(\mathcal{M}_p, d_p)$ is a reflecting condition in $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$ and w_p is a finite partial function from $T^* \cap \alpha$ to V_α with some properties. If $\alpha < \beta$ are in $T^* \cup \{\theta\}$ and $p \in \mathbb{P}_\beta$ we let $p \restriction \alpha$ denote $(\mathcal{M}_p \cap V_\alpha, d_p \restriction (\mathcal{M}_p \cap V_\alpha), w_p \restriction V_\alpha)$. It will be immediate from the definition that $p \restriction \alpha \in \mathbb{P}_\alpha$. Moreover, since $(\mathcal{M}_p, d_p)$ is reflecting, it will be an extension of $(\mathcal{M}_p \cap V_\alpha, d_p \restriction (\mathcal{M}_p \cap V_\alpha))$. For $\alpha \in T^*$ we will also be interested in the partial order $\mathbb{P}_\alpha \cap V_\alpha$. We let $\dot{G}_\alpha$ denote the canonical $\mathbb{P}_\alpha \cap V_\alpha$-name for the generic filter. If $M \in \mathcal{S} \cup \mathcal{T}$ and $\alpha \in M$ we let $M[\dot{G}_\alpha]$ be the canonical $\mathbb{P}_\alpha \cap V_\alpha$-name for the model $M[G_\alpha]$, where G_α is the generic filter. If $F(\alpha)$ is a $\mathbb{P}_\alpha \cap V_\alpha$-name which is forced by the maximal condition to be a proper forcing notion we let $\dot{\mathbb{F}}_\alpha$ denote $F(\alpha)$; otherwise let $\dot{\mathbb{F}}_\alpha$ denote the $\mathbb{P}_\alpha \cap V_\alpha$-name for the trivial forcing. Let $\leq_{\mathbb{F}_\alpha}$ be the name for the ordering on $\dot{\mathbb{F}}_\alpha$.

We are now ready for the main definition.

Definition 4.1. Suppose $\alpha \in T^* \cup \{\theta\}$. Conditions in $\mathbb{P}_\alpha$ are triples p of the form $(\mathcal{M}_p, d_p, w_p)$ such that:

(1) $(\mathcal{M}_p, d_p)$ is a reflecting condition in $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$,
(2) w_p is a finite function with domain contained in the set $\{\gamma \in T^* \cap \alpha : V_\gamma \in \mathcal{M}_p\}$,
(3) if $\gamma \in \mathrm{dom}(w_p)$, then:
  (a) $w_p(\gamma)$ is a canonical $\mathbb{P}_\gamma \cap V_\gamma$-name for an element of $\dot{\mathbb{F}}_\gamma$,
  (b) if $M \in \mathcal{S} \cap \mathcal{M}_p$ and $\gamma \in M$, then

$$p \restriction \gamma \Vdash_{\mathbb{P}_\gamma \cap V_\gamma} w_p(\gamma) \text{ is } (M[\dot{G}_\alpha], \dot{\mathbb{F}}_\alpha)\text{-generic.}$$

We let $q \leq p$ if $(\mathcal{M}_q, d_q)$ extends $(\mathcal{M}_p, d_p)$ in $\mathbb{M}^*_{\mathcal{S},\mathcal{T}}$, $\mathrm{dom}(w_p) \subseteq \mathrm{dom}(w_q)$ and, for all $\gamma \in \mathrm{dom}(p)$,

$$q \restriction \gamma \Vdash_{\mathbb{P}_\gamma \cap V_\gamma} w_q(\gamma) \leq_{\mathbb{F}_\gamma} w_p(\gamma).$$

Our poset $\mathbb{P}_\alpha$ is almost identical to the poset $\mathbb{A}_\alpha$ from [7]. The difference is that we have a requirement that the height function ht_p of a condition p is preserved when going to a stronger condition and we also added the decoration d_p. We are restricting ourselves to reflecting conditions p since we then know that p is an extension of $p \restriction \alpha$, for any $\alpha \in T^*$ such that V_α is a node in $\mathcal{M}_p$. Of course, the working part w_p is defined only for such α. These modifications do not affect the relevant arguments from [7]. We state the main properties of our posets and refer to [7] for the proofs.

Lemma 4.2. *Suppose β belongs to $T^* \cup \{\theta\}$.*

(1) *Let $p \in \mathbb{P}_\beta$ and let $V_\alpha \in \mathcal{M}_p \cap \mathcal{T}^*$. Then p is $(V_\alpha, \mathbb{P}_\beta)$-strongly generic.*
(2) *Let $p \in \mathbb{P}_\beta$, let $V_\alpha \in \mathcal{T}$ and suppose $p \in V_\alpha$. Then $(\mathcal{M}_p \cup \{V_\alpha\}, d_p, w_p)$ is a condition in $\mathbb{P}_\beta$.*
(3) *$\mathbb{P}_\beta$ is $\mathcal{T}^*$-strongly proper.*

Proof. This is essentially the same as Lemma 6.7 from [7]. □

Lemma 4.3. *Suppose $\beta \in T^* \cup \{\theta\}$ and $p \in \mathbb{P}_\beta$. Let $M \in \mathcal{S}$ be such that $p \in M$. Then there is a condition $q \in \mathbb{P}_\beta$ extending p such that M is the top model of q.*

Proof. First, let $\mathcal{M}$ be closure of $\mathcal{M}_p \cup \{M\}$ under intersections and let d be the extension of d_p to $\mathcal{M}$ defined by letting $d(N) = \emptyset$, for all $N \in \mathcal{M} \setminus \mathcal{M}_p$. Then $(\mathcal{M}, d) \in \mathbb{M}^*_{\mathcal{S},\mathcal{T}}$. By Corollary 2.9 we can find a reflecting condition $(\mathcal{M}_q, d_q) \leq (\mathcal{M}, d)$ such that the top model of $\mathcal{M}_q$ is M. Now, we need to define w_q. If $\alpha \in \mathrm{dom}(w_p)$, then $\mathbb{P}_\alpha \cap V_\alpha, \dot{\mathbb{F}}_\alpha \in M$. Since $\dot{\mathbb{F}}_\alpha$ is forced by the maximal condition in $\mathbb{P}_\alpha \cap V_\alpha$ to be proper and $w_p(\alpha) \in M$ is a canonical name for a member of $\dot{\mathbb{F}}_\alpha$, we can fix a canonical $\mathbb{P}_\alpha \cap V_\alpha$-name $w_q(\alpha)$ for a member of $\dot{\mathbb{F}}_\alpha$ such that $p \restriction \alpha$ forces in $\mathbb{P}_\alpha \cap V_\alpha$ that $w_q(\alpha)$ extends $w_p(\alpha)$ and is $(M[\dot{G}_\alpha], \dot{\mathbb{F}}_\alpha)$-generic. Then the condition $q = (\mathcal{M}_q, d_q, w_q)$ is as required. □

Lemma 4.4. *Suppose* $\beta \in T^* \cup \{\theta\}$ *and* $p \in \mathbb{P}_\beta$. *Let* $\theta^* > \theta$ *be such that* $(V_{\theta^*}, \in)$ *satisfies a sufficient fragment of* ZFC. *Let* M^* *be a countable elementary submodel of* V_{θ^*} *containing all the relevant parameters. Let* $M = M^* \cap V_\theta$ *and suppose* $M \in \mathcal{M}_p$. *Then* p *is* $(M^*, \mathbb{P}_\beta)$*-generic.*

Proof. This is essentially the same as Lemma 6.11 from [7]. □

Then, as in [7], we have the following.

Proposition 4.5. *Suppose that* θ *is supercompact and* F *is a Laver function on* θ. *Let* G_θ *be a* V*-generic filter over* $\mathbb{P}_\theta$. *Then* $V[G_\theta]$ *satisfies* PFA.

Now, if $\delta \in T^*$ and G_δ is a V-generic filter over $\mathbb{P}_\delta \cap V_\delta$, we can define the function h_{G_δ} and the factor forcing $\mathbb{Q}_\delta$ as in §3. Further, we define the set $\mathcal{S}_\delta$ in an analogous way to Definition 3.1 and show that it is stationary as in Lemma 3.4. We show, as in Lemma 4.2 that $\mathbb{Q}_\delta$ is $\mathcal{T}_\delta^*$-strongly proper. By Lemma 4.3 and Lemma 4.4 we also get that, in $V[G_\delta]$, $\mathbb{Q}_\delta$ is $\mathcal{S}_\delta^*$-proper. For every subset E of ω_1 which belongs to $V[G_\delta]$, we define the set $\mathcal{S}_\delta(E)$ as in §3 and prove a version of Lemma 3.5. Then, proceeding in the same way, for every $\gamma < \delta$ we define $\mathcal{S}_\delta(E, \gamma)$ and prove an analog of Lemma 3.7. Then, turning to the final model $V[G_\theta]$, we prove an analog of Lemma 3.8. Finally, combining the arguments of Theorem 3.9 and Proposition 4.5 we get the following conclusion.

Theorem 4.6. *Suppose* θ *is supercompact and* F *is a Laver function on* θ. *Let* G_θ *be* V*-generic over* $\mathbb{P}_\theta$. *Then, in* $V[G_\theta]$, PFA *holds and* NS_{ω_1} *is not precipitous.*

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