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GLOBAL VISCOSITY SOLUTIONS OF GENERALIZED KÄHLER-RICCI FLOW

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ABSTRACT. We apply ideas from viscosity theory to establish the existence of a unique global weak solution to the generalized Kähler-Ricci flow in the setting of commuting complex structures. Our results are restricted to the case of a smooth manifold with smooth background data. We discuss the possibility of extending these results to more singular settings, pointing out a key error in the existing literature on viscosity solutions to complex Monge-Ampère equations/Kähler-Ricci flow.

1. Introduction

Generalized Kähler geometry and generalized Calabi-Yau structures arose from research on supersymmetric sigma models [11]. They were rediscovered by Hitchin [13], growing out of investigations into natural volume functionals on differential forms. These points of view were connected in the thesis of Gualtieri [12]. These structures have recently attracted enormous interest in both the physics and mathematical communities as natural generalizations of Kähler Calabi-Yau structures, inheriting a rich physical and geometric theory. The author and Tian [18] developed a natural notion of Ricci flow in generalized Kähler geometry, and we will call this flow *generalized Kähler-Ricci flow* (GKRF). Explicitly it takes the form

$$\begin{aligned} \frac{\partial}{\partial t} g = -2 \operatorname{Rc}^g + \frac{1}{2}\mathcal{H}, \qquad \frac{\partial}{\partial t} H = \Delta_d H, \\ \frac{\partial}{\partial t} I = L_{\theta_I^\sharp} I, \qquad \frac{\partial}{\partial t} J = L_{\theta_J^\sharp} J, \end{aligned} \tag{1.1}$$

where $\mathcal{H}_{ij} = H_{ipq} H_j^{pq}$, and θ_I, θ_J are the Lee forms of the corresponding Hermitian structures.

A special case of this flow arises when $[J_A, J_B] = 0$, a condition preserved by the flow [16], and moreover causes the complex structures to be fixed along the flow. As shown in [16], the GKRF reduces to a single parabolic scalar PDE in this setting. We recall that, suppressing all background geometry terms, the Kähler-Ricci flow is known to reduce locally to the parabolic complex Monge-Ampère equation. In the present setting, the local reduction is to the parabolic complex "twisted" Monge-Ampère equation. Namely, one has a splitting $\mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^l$, and we denote

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$z = (z_1, \ldots, z_n) = (z_+, z_-)$ where $z_+ \in \mathbb{C}^k, z_- \in \mathbb{C}^l$. Then the "twisted"[1] equation is

$$\frac{\partial}{\partial t} u = \log \frac{\det \sqrt{-1}\partial_+ \overline{\partial}_+ u}{\det(-\sqrt{-1}\partial_- \overline{\partial}_- u)}.$$

As observed in [19], this equation is formally related to the parabolic complex Monge-Ampère equation via partial Legendre transformation in the z_- variables. This observation was exploited to establish a $C^{2,\alpha}$ estimate of Evans-Krylov type for this equation, overcoming the nonconvexity which prevents applying standard machinery. This estimate can be combined with further global a priori estimates which hold in specific geometric/topological situations [16, 17] to establish global existence and convergence results for the GKRF.

Despite the partial results and natural estimates which have been established for this flow, a full regularity theory is lacking due to the lack of general a priori estimates on the parabolicity of the equation. Here again the nonconvexity of the equation causes difficulty as the potential function alone cannot be added to test functions to apply the maximum principle as in the traditional Monge-Ampère theory [21]. For this reason it is natural to pursue alternative methods for establishing low order estimates, and here we look to viscosity theory. Our main result establishes the existence of such solutions. In the statement below τ^* is the maximal possible smooth existence time based on cohomological obstructions (see Definition 2.5).

Theorem 1.1. *Let (M^{2n}, g, J_A, J_B) be a generalized Kähler manifold with $[J_A, J_B] = 0$. There exists a unique maximal viscosity solution to GKRF on $[0, \tau^*)$, realized as the supremum of all subsolutions.*

Remark 1.2.

(1) In the work [2] a general theory of viscosity solutions is developed for equations on Riemannian manifolds. They require adapting the "variable-doubling method" *globally* on M, which forces the use of the global distance function. The analytic details require some convexity properties for the distance function, which are only satisfied under strong curvature hypotheses such as nonnegative sectional curvature.

(2) A remarkable feature of the viscosity theory for the complex Monge-Ampère equation is that the traditional definition of viscosity subsolution naturally picks out *elliptic* subsolutions (cf. [7, Proposition 1.3]). For instance, the function $|z_1|^2 - |z_2|^2 - |z_3|^2$ is *not* a viscosity solution of $\det \sqrt{-1}\partial\overline{\partial} u = 1$ on $\mathbb{C}^3$. This is related to the simple but important observation that the supremum of two subsolutions is again a subsolution. This presents an extra challenge due to the natural mixed plurisub/superharmonic condition needed for ellipticity of the twisted equation, which for instance is not preserved under taking supremums. These issues are overcome by making careful definitions of sub/supersolutions which naturally split up the two pieces of the ellipticity condition so that part is satisfied by subsolutions, part by supersolutions.

[1] The terminology 'twisted Monge-Ampère' appears in other places in the literature often referring to a usual Monge-Ampère equation modified by some further terms involving specialized background geometry. Despite this clash we will use this terminology as it seems to economically capture the notion that the equation exploits a nonstandard combination of Monge-Ampère operators.

(3) While it is satisfying to construct a global solution with some (very weak) regularity, it is of course unsatisfying because ultimately we expect the solution to be smooth, and it is unclear if the viscosity approach can eventually lead to the full regularity. Viscosity theory holds the promise to understand generalized Kähler-Ricci flow, perhaps for instance flowing through singularities. This is the approach taken in a series of works based on [7, 8] in efforts to better understand the complex Monge-Ampère equation/Kähler-Ricci flow in singular settings. In the course of the author's investigations into these works a crucial error was discovered which renders those works and many subsequent works incomplete. After the appearance of the first version of this article, an erratum appeared [6] which allows for a comparison principle in the ample locus when the background cohomology class is big, with the background form sufficiently regular. It seems likely that the methods employed in [6] can apply in the setting considered here to show uniqueness and continuity of the solution we construct. We leave the investigation of this and the further regularity of the solution we construct for future work.

2. Smooth twisted Monge-Ampère flows

In this section we recall and refine the discussion in [16] wherein the pluriclosed flow in the setting of generalized Kähler geometry with commuting complex structures is reduced to a fully nonlinear parabolic PDE. First we recall the fundamental aspects of the relevant differential geometry.

2.1. **Tangent bundle splitting.** Let (M^{2n}, g, J_A, J_B) be a generalized Kähler manifold satisfying $[J_A, J_B] = 0$. Define

$$\Pi := J_A J_B \in \mathrm{End}(TM).$$

It follows that $\Pi^2 = \mathrm{Id}$, and Π is g-orthogonal, hence Π defines a g-orthogonal decomposition into its ± 1 eigenspaces, which we denote

$$TM = T_+M \oplus T_-M.$$

Moreover, on the complex manifold (M^{2n}, J_A) we can similarly decompose the complexified tangent bundle $T^{1,0}_{\mathbb{C}}$. For notational simplicity we denote

$$T^{1,0}_{\pm} := \ker\,(\Pi \mp I) : T^{1,0}_{\mathbb{C}}(M, J_A) \to T^{1,0}_{\mathbb{C}}(M, J_A).$$

We use similar notation to denote the pieces of the complex cotangent bundle. Other tensor bundles inherit similar decompositions. The one of most importance to us is

$$\begin{aligned}\Lambda^{1,1}_{\mathbb{C}}(M, J_A) = &\left(\Lambda^{1,0}_+ \oplus \Lambda^{1,0}_-\right) \wedge \left(\Lambda^{0,1}_+ \oplus \Lambda^{0,1}_-\right)\\ = &\left[\Lambda^{1,0}_+ \wedge \Lambda^{0,1}_+\right] \oplus \left[\Lambda^{1,0}_+ \wedge \Lambda^{0,1}_-\right] \oplus \left[\Lambda^{1,0}_- \wedge \Lambda^{0,1}_+\right] \oplus \left[\Lambda^{1,0}_- \wedge \Lambda^{0,1}_-\right].\end{aligned}$$

Given $\mu \in \Lambda^{1,1}_{\mathbb{C}}(M, J_A)$ we will denote this decomposition as

$$\mu := \mu^+ + \mu^{\pm} + \mu^{\mp} + \mu^-. \tag{2.1}$$

These decompositions allow us to decompose differential operators as well. In particular we can express

$$d = d_+ + d_-, \qquad \partial = \partial_+ + \partial_-, \qquad \overline{\partial} = \overline{\partial}_+ + \overline{\partial}_-.$$

The crucial differential operator governing the local generality of generalized Kähler metrics in this setting is

$$\square := \sqrt{-1}\left(\partial_+\overline{\partial}_+ - \partial_-\overline{\partial}_-\right).$$

2.2. **A characteristic class.**

Definition 2.1. Let (M^{2n}, J_A, J_B) be a bicomplex manifold such that $[J_A, J_B] = 0$. Let

$$\chi(J_A, J_B) = c_1^+(T_+^{1,0}) - c_1^-(T_+^{1,0}) + c_1^-(T_-^{1,0}) - c_1^+(T_-^{1,0}).$$

The meaning of this formula is the following: fix Hermitian metrics $h_\pm$ on the holomorphic line bundles $\det T_\pm^{1,0}$, and use these to define elements of $c_1(T_\pm^{1,0})$, and then project according to the decomposition (2.1). In particular, given such metrics $h_\pm$ we let $\rho(h_\pm)$ denote the associated representatives of $c_1(T_\pm^{1,0})$, and then let

$$\chi(h_\pm) = \rho^+(h_+) - \rho^-(h_+) + \rho^-(h_-) - \rho^+(h_-).$$

This definition yields a well-defined class in a certain cohomology group, defined in [16], which we now describe.

Definition 2.2. Let (M^{2n}, J_A, J_B) be a bicomplex manifold with $[J_A, J_B] = 0$. Given $\zeta_A \in \Lambda^{1,1}_{J_A,\mathbb{R}}$, let $\zeta_B = -\zeta_A(\Pi\cdot,\cdot) \in \Lambda^{1,1}_{J_B,\mathbb{R}}$. We say that ζ_A is *formally generalized Kähler* if

$$\begin{aligned} d^c_{J_A}\zeta_A &= -\, d^c_{J_B}\zeta_B, \\ dd^c_{J_A}\zeta_A &= 0. \end{aligned} \tag{2.2}$$

This definition captures every aspect of a generalized Kähler metric compatible with J_A, J_B, except for being positive definite. As we will show in Lemma 2.8 below, such forms are locally expressed as $\square f$. It is therefore natural to define the following cohomology space.

Definition 2.3. Let (M^{2n}, J_A, J_B) denote a bicomplex manifold such that $[J_A, J_B] = 0$. Let

$$H^{1,1}_{GK} := \frac{\left\{\zeta_A \in \Lambda^{1,1}_{J_A,\mathbb{R}} \mid \zeta_A \text{ satisfies (2.2)}\right\}}{\{\square f \mid f \in C^\infty(M)\}}.$$

It follows from direct calculations using the transgression formula for c_1 (cf. [16]) that χ yields a well-defined class in $H^{1,1}_{GK}$.

2.3. **Pluriclosed flow in commuting generalized Kähler geometry.** With this setup we describe how to reduce pluriclosed flow to a scalar PDE in the setting of commuting generalized Kähler manifolds. First we recall that it follows from ([16, Proposition 3.2, Lemma 3.4]) that the pluriclosed flow in this setting reduces to

$$\frac{\partial}{\partial t}\omega = -\,\chi(\omega_\pm). \tag{2.3}$$

To capture the idea of the formal maximal existence time, we first define the analogous notion to the "Kähler cone", which we refer to as $\mathcal{P}$, the "positive cone":

Definition 2.4. Let (M^{2n}, g, J_A, J_B) denote a bicomplex manifold such that $[J_A, J_B] = 0$. Let

$$\mathcal{P} := \left\{[\zeta] \in H^{1,1}_{GK} \mid \exists\, \omega \in [\zeta], \omega > 0\right\}.$$

From the discussion above, we thus see that a solution to (2.3) induces a solution to an ODE in $\mathcal{P}$, namely

$$[\omega_t] = [\omega_0] - t\chi.$$

It is clear now that there is a formal obstruction to the maximal smooth existence time of the flow.

Definition 2.5. Given (M^{2n}, g, J_A, J_B) a generalized Kähler manifold with $[J_A, J_B] = 0$, let

$$\tau^*(g) := \sup \{t \geq 0 \mid [\omega] - t\chi \in \mathcal{P}\}.$$

Now fix $\tau < \tau^*$, so that by hypothesis if we fix arbitrary metrics $\widetilde{h}_\pm$ on $T^{1,0}_\pm$, there exists $a \in C^\infty(M)$ such that

$$\omega_0 - \tau\chi(\widetilde{h}_\pm) + \Box a > 0.$$

Now set $h_\pm = e^{\pm\frac{a}{2\tau}}\widetilde{h}_\pm$. Thus $\omega_0 - \tau\chi(h_\pm) > 0$, and by convexity it follows that

$$\hat{\omega}_t := \omega_0 - t\chi(h_\pm) > 0$$

is a smooth one-parameter family of generalized Kähler metrics. Furthermore, given a function $f \in C^\infty(M)$, let

$$\omega^f := \hat{\omega} + \Box f,$$

with g^f the associated Hermitian metric. Now suppose that u satisfies

$$\frac{\partial}{\partial t}u = \log \frac{(\omega^u_+)^k \wedge (\zeta_-)^l}{(\zeta_+)^l \wedge (\omega^u_-)^l}, \tag{2.4}$$

where ζ denotes the Kähler form of the Hermitian metric h. An elementary calculation using the transgression formula for the first Chern class ([16, Lemma 3.4]) yields that ω^u solves (2.3).

2.4. **Twisted Monge-Ampère flows.** We now codify the discussion of the previous subsection by making some general definitions, and then use these to define our notion of viscosity sub/supersolutions.

Definition 2.6. Let (M^{2n}, g, J_A, J_B) be a generalized Kähler manifold with $[J_A, J_B] = 0$. Fix

(1) ω a continuous family of formally generalized Kähler forms.
(2) $0 \leq \mu_+(z,t) \in C^0(M, \Lambda^{k,k}_+), 0 \leq \mu_-(z,t) \in C^0(M, \Lambda^{l,l}_-)$ continuous families of partial volume forms.
(3) $F : M \times [0,T)\times \to \mathbb{R}$ a continuous function.

A function $u \in C^2(M \times [0,T))$ is a solution of $(\omega, \mu_\pm, F)$-*twisted Monge-Ampère flow* if

(1) $\omega^{u_t} > 0$ for all $t \in [0,T)$.
(2) $(\omega_+ + \sqrt{-1}\partial_+\overline{\partial}_+ u)^k \wedge \mu_- = e^{u_t + F(x,t)}(\omega_- - \sqrt{-1}\partial_-\overline{\partial}_- u)^l \wedge \mu_+$.

Definition 2.7. Let (M^{2n}, g, J_A, J_B) be a generalized Kähler manifold with $[J_A, J_B] = 0$. Fix data $(\omega, \mu_\pm, F)$ as in Definition 2.6. A function $u \in \mathrm{USC}(M \times [0,T))$ is a *viscosity subsolution of* $(\omega, \mu_\pm, F)$-*twisted Monge-Ampère flow* if for all $\phi \in$

$C^\infty(M \times [0,T))$ such that $u - \phi$ has a local maximum at $(z,t) \in M \times (0,T)$, one has that, at (z,t),

$$(\omega_+ + \sqrt{-1}\partial_+\overline{\partial}_+\phi)^k \wedge \mu_- \geq e^{\phi_t + F(x,t)} \left[(\omega_- - \sqrt{-1}\partial_-\overline{\partial}_-\phi)_+\right]^l \wedge \mu_+,$$

where for a section $\zeta \in \Lambda^{1,1}_-$ the notation ζ^k_+ means ζ^l if $\eta > 0$ and zero otherwise.

Likewise, a function $v \in \mathrm{LSC}(M \times [0,T))$ is a *viscosity supersolution of* $(\omega, \mu_\pm, F)$*-twisted Monge-Ampère flow* if for all $\phi \in C^\infty(M \times [0,T))$ such that $v - \phi$ has a local minimum at $(z,t) \in M \times (0,T)$, one has that, at (z,t),

$$\left[(\omega_+ + \sqrt{-1}\partial_+\overline{\partial}_+\phi)_+\right]^k \wedge \mu_- \leq e^{\phi_t + F(x,t)} \left(\omega_- - \sqrt{-1}\partial_-\overline{\partial}_-\phi\right)^l \wedge \mu_+,$$

where for a section $\eta \in \Lambda^{1,1}_+$ the notation η^k_+ means η^k if $\eta > 0$ and zero otherwise.

A remarkable feature of the viscosity theory for complex Monge-Ampère equations is that it naturally selects elliptic solutions to the problem. In a sense it is forced upon the solutions through the use of the projection operators onto the positive part of the complex Hessian of the test functions, and the fact that the inequality must hold for arbitrary test functions, as explained in ([7, Proposition 1.3]). In our case the notion of ellipticity is more delicate, and yet the viscosity theory still allows us to set up our definitions so as to ensure we obtain elliptic solutions to the problem. This is surprising due to the nonconvexity of the equation at hand.

Even further, the Perron process, which involves taking supremums of subsolutions, natually preserves the plurisubharmonicity of subsolutions in the z_+ directions, but would not preserve the plurisuperharmonicity in the z_- directions if we attempted to impose this by hand. Only a fortiori, having constructed a sub/supersolution at the end of the Perron process, do we ensure that our final solution is parabolic. We clarify this in the rest of the subsection. The first step is to exhibit a local version of the $\partial\overline{\partial}$-lemma adapted to this setting. This result is stated in [11] without proof, which is however elementary.

Lemma 2.8. *Let $\omega = \omega_+ + \omega_-$ be formally generalized Kähler on $U \subset \mathbb{C}^k \times \mathbb{C}^l$. There exists $f \in C^\infty(U)$ such that $\omega = \Box f$.*

Proof. First observe that since $d_+\omega_+ = 0$, on each $w \equiv$ const complex k-plane we can apply the $\partial\overline{\partial}$-lemma to obtain a function $\psi_+(z)$ such that $\sqrt{-1}\partial_+\overline{\partial}_+\psi_+ = \omega_+$ on that plane. Since ω_+ is smooth, we can moreover choose these on each slice so that the resulting function $\psi_+(z,w)$ is smooth, and satisfies $\sqrt{-1}\partial_+\overline{\partial}_+\psi_+ = \omega_+$ on U. Arguing similarly we obtain a function ψ_- such that $\sqrt{-1}\partial_-\overline{\partial}_-\psi_- = \omega_-$ everywhere on U. We note now that the fact that ω is pluriclosed implies that

$$0 = \sqrt{-1}\partial_+\overline{\partial}_+\omega_- + \sqrt{-1}\partial_-\overline{\partial}_-\omega_+ = -\partial_+\overline{\partial}_+\partial_-\overline{\partial}_-\left(\psi_+ + \psi_-\right).$$

We next claim that any element in the kernel of the operator $\partial_+\overline{\partial}_+\partial_-\overline{\partial}_-$, in particular $\psi_+ + \psi_-$, can be expressed as

$$\psi_+ + \psi_- = \lambda_1(z,\overline{z},w) + \overline{\lambda}_1(z,\overline{z},\overline{w}) + \lambda_2(w,\overline{w},z) + \overline{\lambda}_2(w,\overline{w},\overline{z}). \tag{2.5}$$

To see this we first note that if $\phi := \psi_+ + \psi_-$ satisfies $\partial_+\overline{\partial}_+\partial_-\overline{\partial}_-\phi = 0$, then $\partial_-\overline{\partial}_-\phi$ can be expressed as the real part of a $\overline{\partial}_+$-holomorphic function, so $(\partial_-\overline{\partial}_-\phi)_{\omega_i\overline{w}_j} = \mu_1^{i\overline{j}}(w,\overline{w},z) + \overline{\mu}_1^{i\overline{j}}(w,\overline{w},\overline{z})$, where the indices on the μ refer to the fact that each component of the $\partial_-\overline{\partial}_-$-Hessian can be expressed this way. It follows that $\Delta_-\phi := \sqrt{-1}\phi_{,w_i\overline{w}_i}$ is the real part of a $\overline{\partial}_+$-holomorphic function. Applying the Green's function on each z-slice it follows that ϕ can be expressed as the real part of a $\overline{\partial}_+$-holomorphic function, up to the addition of an arbitrary $\overline{\partial}_-$-holomorphic function. Thus (2.5) follows.

We claim that $f = \psi_+ - \lambda_2 - \overline{\lambda}_2$ is the required potential function. In particular, since $\sqrt{-1}\partial_+\overline{\partial}_+\left(\lambda_2 + \overline{\lambda}_2\right) = 0$ it follows that $\sqrt{-1}\partial_+\overline{\partial}_+ f = \omega_+$. Also, we compute using (2.5),

$$\begin{aligned} -\sqrt{-1}\partial_-\overline{\partial}_- f =& -\sqrt{-1}\partial_-\overline{\partial}_-\left(-\psi_- + \lambda_1 + \overline{\lambda}_1\right) \\ &= \sqrt{-1}\partial_-\overline{\partial}_-\psi_- \\ &= \omega_-. \end{aligned}$$

The lemma follows. □

Lemma 2.9. *Let (M^{2n}, g, J_A, J_B) be a generalized Kähler manifold with $[J_A, J_B] = 0$. Suppose $\omega_t, t \in [0,T]$ is a one-parameter family of smooth generalized Kähler metrics on M. There exists a locally finite open cover $\mathcal{U} = \{U_\beta\}$ of M such that*

1. *Each U_β is the domain of a bicomplex coordinate chart.*
2. *For each β there is a smooth function $f_\beta : U_\beta \times [0,T] \to \mathbb{R}$ such that $\omega = \Box f$.*

Proof. The existence of local bicomplex coordinates around each point follows from ([1, Theorem 4]), and then the existence of a locally finite cover follows from standard arguments. At any time t we can construct a local potential f by Lemma 2.8, and it is clear by the proof of that lemma that f can be chosen to depend smoothly on ω, and so the lemma follows. □

Lemma 2.10. *Let (M^{2n}, g, J_A, J_B) be a generalized Kähler manifold with $[J_A, J_B] = 0$. Fix data $(\omega, \mu_\pm, F)$ as in Definition 2.6, and fix a cover $\mathcal{U}$ as in Lemma 2.9. Suppose that on $U_\beta \in \mathcal{U}$ there are continuous density functions $\zeta_\pm$ satisfying*

$$\begin{aligned} \mu_+ &= e^{\zeta_+}(\sqrt{-1}dz_+^1 \wedge d\overline{z}_+^1) \wedge \cdots \wedge (\sqrt{-1}dz_+^k \wedge d\overline{z}_+^k), \\ \mu_- &= e^{\zeta_-}(\sqrt{-1}dz_-^1 \wedge d\overline{z}_-^1) \wedge \cdots \wedge (\sqrt{-1}dz_-^k \wedge d\overline{z}_-^k). \end{aligned}$$

If u is a subsolution of $(\omega, \mu_\pm, F)$-twisted Monge-Ampère flow, then $u_\beta := u + f_\beta$ is a subsolution of

$$(\sqrt{-1}\partial_+\overline{\partial}_+ w)^k \geq e^{w_t - f_t + F(x,t) + \zeta_+ - \zeta_-}(-\sqrt{-1}\partial_-\overline{\partial}_- w)_+^l.$$

Likewise, if v is a viscosity supersolution of $(\omega, \mu_\pm, F)$-twisted Monge-Ampère flow, then $v_\beta := v + f_\beta$ is a supersolution of

$$(\sqrt{-1}\partial_+\overline{\partial}_+ w)_+^k \leq e^{w_t - f_t + F(x,t) + \zeta_+ - \zeta_-}(-\sqrt{-1}\partial_-\overline{\partial}_- w)^l.$$

Proof. This is an immediate consequence of unraveling the definitions. □

Observe that the inequalities defining sub/supersolutions in Lemma 2.10 are expressed as inequalities of scalars in the chosen coordinates, whereas the original inequalities of Definition 2.7 are expressed in terms of sections of $\Lambda^{n,n}$. Moreover, the meaning of viscosity sub/supersolution in this context is the classic one. As the key arguments in the proofs of the comparison theorems are local in nature, it suffices to consider this localized version of the flow, which has the advantage of stripping away much notation and making things more concrete in coordinates.

We will refer to this setup informally as a *localized flow.* Now we are ready to state our ellipticity claim.

Lemma 2.11. *Local viscosity subsolutions of twisted Monge-Ampère flow as in Lemma* 2.10 *are plurisubharmonic in the* z_+*-variables, and viscosity supersolutions of twisted Monge-Ampère flow as in Lemma* 2.10 *are plurisuperharmonic in the* z_-*-variables.*

Proof. Let u be a local viscosity subsolution of twisted Monge-Ampère flow. Without loss of generality we assume the domain is $B_1(0) \times [0,T)$. Fix $(z_0,t_0) \in B_1(0)\times[0,T)$ such that $u(z_0) \neq -\infty$. Choose a function $\phi \in C^2(B_1(0)\times[0,T))$ such that $u-\phi$ has a local maximum at (z_0,t_0). It follows directly from the definition of subsolution that

$$(\sqrt{-1}\partial_+\overline{\partial}_+\phi)^k \geq e^{\phi_t + F(z_0,t_0)}(-\sqrt{-1}\partial_-\overline{\partial}_-\phi)^l_+ \geq 0.$$

We claim that $\sqrt{-1}\partial_+\overline{\partial}_+\phi \geq 0$. First note that, if we fix a $k\times k$ Hermitian positive semidefinite matrix H_+, and set

$$\phi_{H_+}(z,t) := \phi(z,t) + H_+(z_+ - (z_0)_+)(\overline{z}_+ - (\overline{z}_0)_+).$$

The function ϕ_{H_+} has a local maximum at (z_0,t_0) as well. Hence arguing as above we have

$$(\sqrt{-1}\partial_+\overline{\partial}_+\phi_{H_+})^k = (\sqrt{-1}\partial_+\overline{\partial}_+\phi + H_+)^k \geq 0.$$

Since H_+ is arbitrary, by an elementary linear algebra argument this implies $\sqrt{-1}\partial_+\overline{\partial}_+\phi \geq 0$. It then follows that for any positive definite matrix H_+ one has

$$H_+^{\overline{j}_+ i_+}\frac{\partial^2\phi}{\partial z^i \partial \overline{z}^j} \geq 0.$$

This implies that u is a viscosity subsolution of $\Delta_H \phi \geq 0$. Since H_+ is arbitrary, using results from linear elliptic PDE theory ([14]) as in ([7, Proposition 1.3]) it follows that u is plurisubharmonic in the z_+-variables. The argument for u being plurisuperharmonic in the z_--variables is directly analogous. ☐

3. Proof of Theorem 1.1

Lemma 3.1. *Let (M^{2n}, g, J_A, J_B) be a generalized Kähler manifold with $[J_A,J_B] = 0$. Fix data $(\omega,\mu_\pm,F)$ as in Definition* 2.6. *Suppose $\underline{u}$ is a bounded viscosity subsolution of $(\omega,\mu_\pm,F)$-twisted Monge-Ampère flow, and suppose $\overline{u}$ is a smooth supersolution of $(\omega,\mu_\pm,F)$-twisted Monge-Ampère flow. If $\overline{u}(x,0) \geq \underline{u}(x,0)$ for all $x \in M$, then $\overline{u}(x,t) \geq \underline{u}(x,t)$ for all $(x,t) \in M\times[0,T)$.*

Proof. Suppose there exists $(x_0,t_0) \in M\times[0,T)$ such that $\overline{u}(x_0,t_0) < \underline{u}(x_0,t_0)$. It follows directly from the definitions that for $\delta > 0$, the function

$$\overline{u}_\delta(x,t) := \overline{u}(x,t) + \frac{\delta}{T-t}$$

is also a smooth supersolution. Moreover, for δ chosen sufficiently small it follows that $\overline{u}_\delta(x_0,t_0) < \underline{u}(x_0,t_0)$. Since $\underline{u}$ is bounded and $\lim_{t\to T}\overline{u}_\delta(x,t) = \infty$ for all $x \in M$, it follows that $\underline{u} - \overline{u}_\delta$ attains a positive maximum at some point (x_0',t_0'), $0 < t_0' < T$.

The function $\overline{u}_\delta$ is smooth, and so can be used in the definition of $\underline{u}$ being a subsolution to yield, at the point (x_0', t_0'), the inequality

$$(\omega_+ + \sqrt{-1}\partial_+\overline{\partial}_+\overline{u}_\delta)^k \wedge \mu_- \geq e^{(\overline{u}_\delta)_t + F(x_0',t_0')}(\omega_- - \sqrt{-1}\partial_-\overline{\partial}_-\overline{u}_\delta)_+^l \wedge \mu_+.$$

Since $\overline{u}(\cdot,t) \in \mathcal{P}_{\omega_t}$ for all t, we can ignore the projection operator on the right hand side and apply elementary identities to obtain

$$(\omega_+ + \sqrt{-1}\partial_+\overline{\partial}_+\overline{u})^k \wedge \mu_- \geq e^{\overline{u}_t + \delta/(T-t_0')^2 + F(x_0',t_0')}(\omega_- - \sqrt{-1}\partial_-\overline{\partial}_-\overline{u})^l \wedge \mu_+.$$

On the other hand, since $\overline{u}$ is already a supersolution and $\overline{u}(\cdot,t) \in \mathcal{P}_{\delta_t}$ for all t we have

$$(\omega_+ + \sqrt{-1}\partial_+\overline{\partial}_+\overline{u})^k \wedge \mu_- \leq e^{\overline{u}_t + F(x_0',t_0')}(\omega_- - \sqrt{-1}\partial_-\overline{\partial}_-\overline{u})^l \wedge \mu_+.$$

Putting the previous two inequalities together yields

$$e^{\delta/(T-t_0')^2} \leq e^{-\overline{u}_t - F(x_0',t_0')}\frac{(\omega_+ + \sqrt{-1}\partial_+\overline{\partial}_+\overline{u})^k \wedge \mu_-}{(\omega_- - \sqrt{-1}\partial_-\overline{\partial}_-\overline{u})_+^l \wedge \mu_+} \leq 1,$$

a contradiction. □

Proof of Theorem 1.1. We first observe the existence of smooth, bounded sub/ super solutions. In particular, since g^{u_0}, h are smooth metrics, one has that

$$\sup_M \left|\log \frac{\det g_+^{u_0} \det h_-}{\det h_+ \det g_-^{u_0}}\right| \leq A.$$

It follows immediately that the smooth functions

$$\overline{u} := u_0 + tA, \qquad \underline{u} := u_0 - tA,$$

are smooth sub/supersolutions to the problem.

Now let

$$u = \sup\{w \mid \underline{u} \leq w \leq \overline{u},\ w \text{ is a subsolution to } (\omega, \mu_\pm, F) \text{ twisted MA flow}\}.$$

We claim that u is a viscosity solution in the sense that the usc regularization u^* is a subsolution, whereas the lower semicontinuous regularization u_* is a supersolution. It is a standard argument (cf. [3]) to show that, as a supremum of subsolutions, u itself is a subsolution to $(\omega, \mu_\pm, F)$-twisted Monge-Ampère flow. It follows that in fact $u^* = u$ is a subsolution. Next we show that u_* is a supersolution. If not, there exists $(z_0, t_0) \in M \times [0, T)$ and ϕ a C^2 function such that $u_* - \phi$ has a local minimum of zero at (z_0, t_0) and, at that point,

$$\left[(\omega_+ + \sqrt{-1}\partial_+\overline{\partial}_+\phi)_+\right]^k \wedge \mu_- > e^{\phi_t + F(x,t)}\left(\omega_- - \sqrt{-1}\partial_-\overline{\partial}_-\phi\right)^l \wedge \mu_+.$$

Choose coordinates around z_0, fix constants $\gamma, \delta > 0$ and consider

$$\phi_{\gamma,\delta} = \phi + \delta - \gamma\left|z\right|^2.$$

It follows that

$$\left[(\omega_+ + \sqrt{-1}\partial_+\overline{\partial}_+\phi_{\gamma,\delta})_+\right]^k \wedge \mu_- > e^{\phi_t + F(x,t)}\left(\omega_- - \sqrt{-1}\partial_-\overline{\partial}_-\phi_{\gamma,\delta}\right)^l \wedge \mu_+$$

on $P_r(z_0)$, for sufficiently small $r > 0$. If we choose $\delta = (\gamma r^2)/8$, then it follows that $u_* > \phi_{\gamma,\delta}$ for $r/2 \leq ||z|| \leq r$, whereas $\phi_{\gamma,\delta}(z_0, t_0) > u_*(z_0, t_0) + \delta$. We now define, supressing the identification with the given coordinate chart,

$$\Phi(z,t) = \begin{cases} \max\{u_*(z,t), \phi_{\gamma,\delta}(z,t)\} & z \in B_r(z_0), \\ u_*(z,t) & \text{otherwise.} \end{cases}$$

As the supremum of subsolutions, Φ is a subsolution to $(\omega, \mu_{\pm}, F)$-twisted Monge-Ampère flow. Now choose a sequence $(z_n, t_n) \to (z_0, t_0)$ such that $u(z_n, t_n) \to u_*(z_n, t_n)$. For sufficiently large n it follows that $\Phi(z_n, t_n) = \phi_{\gamma,\delta}(z_n, t_n) > u(z_n, t_n)$, contradicting the definition of u. $\square$

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