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A NON-PI MINIMAL SYSTEM IS LI-YORKE SENSITIVE

SONG SHAO AND XIANGDONG YE

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Abstract. It is shown that any non-PI minimal system is Li-Yorke sensitive. Consequently, any minimal system with non-trivial weakly mixing factor (such a system is non-PI) is Li-Yorke sensitive, which answers affirmatively an open question by Akin and Kolyada in [Nonlinearity, 16 (2003) pp. 1421–1433].

1. Introduction

1.1. **Li-Yorke sensitivity.** *A topological dynamical system* (X,T) is a compact metric space (X,ρ) endowed with a continuous surjective map $T: X \to X$. For simplicity, we only consider homeomorphisms in this paper.

(X,T) is *Li-Yorke sensitive*, briefly LYS or LYS_ϵ, if there is an $\epsilon > 0$ with the property that every $x \in X$ is a limit of points $y \in X$ such that the pair (x,y) is proximal but not ϵ-asymptotic, i.e., if

$$\liminf_{n\to\infty} \rho(T^n x, T^n y) = 0, \text{and} \quad \limsup_{n\to\infty} \rho(T^n x, T^n y) > \epsilon.$$

Each pair satisfying the above condition is called an *ϵ-Li-Yorke pair*. A subset $S \subseteq X$ is called an *ϵ-scrambled set* if each pair with distinct elements is an ϵ-Li-Yorke pair. (X,T) is *ϵ-Li-Yorke chaotic*, briefly LYC_ϵ if it has an uncountable ϵ-scrambled subset, for some $\epsilon > 0$.

Li-Yorke chaos [LY75] and sensitivity [Gu79, GW93] are two basic notions to describe the complexity of a topological dynamical system. The notion of Li-Yorke sensitivity, combining the above two notions together, was introduced and studied by Akin and Kolyada [AK03]. For surveys on Li-Yorke sensitivity and other concepts of chaos, see [K04, LiY16]. For more details related to Li-Yorke sensitivity, see [AK03, CM06, CM09, CM16].

In [AK03] the authors showed that every non-trivial weak mixing system is LYS, and they stated five conjectures concerning LYS. Three of them were disproved in [CM06] and [CM09]. In particular, it was proved that a minimal LYS system needs not to have a non-trivial weak mixing factor [CM06], and that a minimal system with a non-trivial LYS factor needs not to be LYS [CM09]. The remaining two open problems are the following:

Question 1.1 ([AK03, Question 4.]). Is every minimal system with a non-trivial weak mixing factor LYS?

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Question 1.2 ([AK03, Question 2.]). Does Li-Yorke sensitivity imply Li-Yorke chaos?

In this paper we give an affirmative answer to Question 1.1. Before stating our main result, we need some basic notions on topological dynamics. In the article, integers, non-negative integers and natural numbers are denoted by $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ respectively.

1.2. **Topological dynamics.** A topological dynamical system (X,T) is *transitive* if for each pair of opene (i.e. open and non-empty) subsets U and V, $N(U,V) = \{n \in \mathbb{Z} : U \cap T^{-n}V \neq \emptyset\}$ is non-empty. (X,T) is (topologically) *weakly mixing* if the product system $(X \times X, T \times T)$ is transitive. A point $x \in X$ is a *transitive point* if its orbit $Orb(x,T) = \{T^n x : n \in \mathbb{Z}\}$ is dense in X. If every point of X is transitive, we say that (X,T) is *minimal*. A point is *minimal* if its orbit closure is a minimal subsystem. It is well known that for a compact metric space X without isolated points, a system (X,T) is transitive if and only if there exists a transitive point if and only if there is a point $x \in X$ with forward orbit $\{T^n x : n \in \mathbb{Z}_+\}$ dense in X [AC12].

A pair $(x,y) \in X \times X$ is said to be *proximal* if $\liminf_{n\to\infty} \rho(T^n x, T^n y) = 0$, and it is called *asymptotic* when $\lim_{n\to\infty} \rho(T^n x, T^n y) = 0$. The set of proximal pairs is denoted by $P(X,T)$ or P when the system is clear. P is a reflexive, symmetric, T-invariant relation but in general is neither transitive nor closed. For $x \in X$, the set $P[x] = \{y \in X : (x,y) \in P\}$ is called the *proximal cell* of x. An important result concerning the proximal cell is that for any $x \in X$, $P[x]$ contains a minimal point; more precisely, every minimal subset of $\overline{Orb(x,T)}$ meets $P[x]$ [Au88, Theorem 5.3.]. A point $x \in X$ is called a *distal* point if $P[x]$ is a singleton. Since $P[x]$ contains a minimal point, it follows immediately that a distal point x is minimal. A system is called a *distal system* when every point is distal. A system (X,T) with some distal point x whose orbit $Orb(x,T)$ is dense in X is called a *point-distal system.* For a weakly mixing system, the proximal cell $P[x]$ is "big", in the sense that it is a residual subset of X for all $x \in X$ [AK03]. In fact, for mixing systems, proximal cells can be very complicated [HSY].

For a topological dynamical system (X,T), a pair is said to be a *Li-Yorke pair* if it is proximal but not asymptotic. A point $x \in X$ is *recurrent* if there is a subsequence $\{n_i\}$ of $\mathbb{N}$ with $n_i \to +\infty$ such that $T^{n_i}x \to x$. A pair $(x,y) \in X \times X \setminus \Delta_X$ is said to be a *strong Li-Yorke pair* if it is proximal and is also a recurrent point of $(X \times X, T \times T)$, where $\Delta_X = \{(x,x) \in X \times X : x \in X\}$. A subset $S \subset X$ is called *scrambled* (*resp. strongly scrambled*) if every pair of distinct points in S is Li-Yorke (resp. strong Li-Yorke). A system (X,T) is said to be *Li-Yorke chaotic* (*resp. strong Li-Yorke* chaotic) if it contains an uncountable scrambled (resp. strongly scrambled) subset.

Let (X,T) be a dynamical system. A subset K of X is *uniformly recurrent* if for every $\epsilon > 0$ there is an $n \in \mathbb{N}$ with $d(T^n x, x) < \epsilon$ for all x in K. K is *recurrent* if every finite subset of K is uniformly recurrent. The subset K is called *uniformly proximal* if for every $\epsilon > 0$ there is $n \in \mathbb{N}$ with $\operatorname{diam} T^n K < \epsilon$. A subset K of X is called *proximal* if every finite subset of K is uniformly proximal.

An *extension* $\pi : (X,T) \to (Y,S)$ is a continuous surjective map from X to Y such that $S \circ \pi = \pi \circ T$. In this case we also say that (X,T) is an *extension* of (Y,S) and that (Y,S) is a *factor* of (X,T). An extension π is determined by the corresponding closed invariant equivalence relation $R_\pi = \{(x_1, x_2) : \pi x_1 = \pi x_2\} = (\pi \times \pi)^{-1}\Delta_Y \subset X \times X$.

1.3. **Main result of the paper.** Here is our main result of the paper (for the definition of PI, see the next subsection):

Main Theorem. *Let (X, T) be a minimal system. If (X, T) is not PI, then there is some $\epsilon > 0$ such that for any $x \in X$ and any neighbourhood U of x, there is a subset $S \subseteq U$ such that*

(1) *S is uncountable and $x \in S \subseteq P[x] \cap U$;*
(2) *S is ϵ-scrambled;*
(3) *S is proximal;*
(4) *S is recurrent.*

In particular, (X, T) is ϵ-Li-Yorke chaotic, strongly Li-Yorke chaotic and Li-Yorke sensitive.

It is known that a factor of a minimal PI system is PI (see Theorem 2.4). Using this fact it is easy to show that every minimal system with a non-trivial weak mixing factor is not a PI system. So a minimal system with a non-trivial weakly mixing factor is LYS, answering Question 1.1 affirmatively. Moreover, by the Main Theorem, we have that any non-PI system is (strongly) Li-Yorke chaotic, which was proved first in [AGHSY]. Note that here we offer a different approach.

1.4. **On the structure of minimal systems.** Our main tool to prove the Main Theorem is the structure theorem of minimal systems. In this subsection we state the structure theorem for minimal systems and give the definition of a PI system. For other undefined notions, see [Au88, G76, V77].

We first recall definitions of extensions. An extension $\pi : (X, T) \to (Y, S)$ is called *proximal* if $R_\pi \subset P(X, T)$. An extension π is an *equicontinuous* or *almost periodic* extension if for every $\epsilon > 0$ there is $\delta > 0$ such that $(x, y) \in R_\pi$ and $\rho(x, y) < \delta$ imply $\rho(T^n x, T^n y) < \epsilon$, for every $n \in \mathbb{Z}$. In the metric case an equicontinuous extension is also called an *isometric extension*. An extension π is a *weakly mixing extension* if $(R_\pi, T \times T)$ as a subsystem of the product system $(X \times X, T \times T)$ is transitive. An extension π is called a *relatively incontractible (RIC) extension* if it is open and for every $n \geq 1$ the minimal points are dense in the relation

$$R_\pi^n = \{(x_1, \ldots, x_n) \in X^n : \pi(x_i) = \pi(x_j),\ \forall\ 1 \leq i \leq j \leq n\}.$$

Note that $R_\pi^1 = X$ and $R_\pi^2 = R_\pi$. We say that a minimal system (X, T) is a *strictly PI system* (PI means proximal-isometric) if there is an ordinal η (which is countable when X is metrizable) and a family of systems $\{(W_\iota, w_\iota)\}_{\iota \leq \eta}$ such that

(1) W_0 is the trivial system,
(2) for every $\iota < \eta$ there exists an extension $\phi_\iota : W_{\iota+1} \to W_\iota$ which is either proximal or equicontinuous (isometric when X is metrizable),
(3) for a limit ordinal $\nu \leq \eta$ the system W_ν is the inverse limit of the systems $\{W_\iota\}_{\iota < \nu}$,
(4) $W_\eta = X$.

We say that (X, T) is a *PI system* if there exists a strictly PI system $\tilde{X}$ and a proximal extension $\theta : \tilde{X} \to X$.

Finally, we have the structure theorem for minimal systems (see Ellis-Glasner-Shapiro [EGS75], McMahon [Mc76], Veech [V77], and Glasner [G76]).

Theorem 1.3 (Structure theorem for minimal systems). *Let (X,T) be a minimal system. Then we have the following diagram:*

$$\begin{array}{ccc} X & \xleftarrow[\theta]{} & X_\infty \\ & & \big\downarrow {\scriptstyle \pi_\infty} \\ & & Y_\infty \end{array}$$

where X_∞ is a proximal extension of X and an RIC weakly mixing extension of the strictly PI system Y_∞.

The extension π_∞ is an isomorphism (so that $X_\infty = Y_\infty$) if and only if X is a PI system.

Remark 1.4. If (X,T) is a weakly mixing minimal system, then in the structure theorem, $X = X_\infty$ and Y_∞ is trivial. Hence if a minimal system is both PI and weakly mixing, then it is trivial.

2. Proof of the Main Theorem

In this section we will give the proofs of the main results of the paper. To this aim, we need some basic results from the theory of minimal flows.

First recall some basic notions related to the Ellis semigroup. For more details on Ellis semigroups, please refer to Chapter 3 and Chapter 6 of [Au88]. Given a system (X,T) its *enveloping semigroup* or *Ellis semigroup* $E(X,T)$ is defined as the closure of the set $\{T^n : n \in \mathbb{Z}\}$ in X^X (with its compact, usually non-metrizable, pointwise convergence topology). Let $(X,T),(Y,S)$ be systems and $\pi : X \to Y$ be an extension. Then there is a unique continuous semigroup homomorphism $\pi^* : E(X,T) \to E(Y,S)$ such that $\pi(px) = \pi^*(p)\pi(x)$ for all $x \in X, p \in E(X,T)$. When there is no confusion, we usually regard the fact that the enveloping semigroup of X acts on Y: $p\pi(x) = \pi(px)$ for $x \in X$ and $p \in E(X,T)$.

For a semigroup an element u with $u^2 = u$ is called an *idempotent*. The well-known Ellis-Numakura theorem states that for any enveloping semigroup E the set $J(E)$ of idempotents of E is not empty. An idempotent $u \in J(E)$ is *minimal* if $v \in J(E)$ and $vu = v$ implies $uv = u$. A point $x \in X$ is minimal if and only if $ux = x$ for some minimal idempotent $u \in E(X,T)$.

For $n \in \mathbb{N}$, let $T^{(n)} = T \times T \times \ldots \times T$ (n times). For a subset $A \subseteq X^n$, let

$$Orb(A, T^{(n)}) = \bigcup \left\{ \left(T^{(n)}\right)^k A : k \in \mathbb{Z} \right\}.$$

The following theorem is crucial to prove the Main Theorem. Note that we assume that X and Y are metrizable.

Theorem 2.1 ([SY, Lemma B.2 and Theorem B.3]). *Let (X,T) and (Y,S) be minimal systems and let $\pi : X \to Y$ be an RIC weakly mixing extension. Let $y \in Y$ with $uy = y$, where $u \in E(Y,S)$ is a minimal idempotent. Then for all $n \ge 2$, any non-empty open subset U of $\overline{u\pi^{-1}(y)}$ and any transitive point $x' = (x'_1, \cdots, x'_{n-1}) \in R_\pi^{n-1}$ with $\pi(x'_j) = y, j = 1, \cdots, n-1$, one has that*

$$\overline{Orb(\{x'\} \times U, T^{(n)})} = R_\pi^n.$$

Moreover, for each transitive point $x' = (x'_1, \cdots, x'_{n-1}) \in R_\pi^{n-1}$ with $\pi(x'_j) = y, j = 1, \cdots, n-1$, there is some residual subset D of $\overline{u\pi^{-1}(y)}$ such that if $x'' \in D$, then $\overline{Orb((x', x''), T^{(n)})} = R_\pi^n$.

Remark 2.2. We have the following:

(1) To prove Theorem 2.1 one needs the so-called Ellis trick by Glasner in [G76]. We refer to [G05] for more discussions about weakly mixing extensions.
(2) In Theorem 2.1, when $n = 2$, $R_\pi^{n-1} = R_\pi^1 = X$, i.e. for each $x \in X$ with $\pi(x) = y$ there is some residual subset D of $\overline{u\pi^{-1}(y)}$ such that if $x' \in D$, $\overline{Orb((x, x'), T \times T)} = R_\pi$. This result is Theorems 14.27 and 14.28 in [Au88].

Corollary 2.3. *Let (X, T) and (Y, S) be minimal systems and let $\pi : X \to Y$ be an RIC weakly mixing extension. Let $y \in Y$ with $uy = y$, where u is a minimal idempotent. If π is non-trivial, then $\overline{u\pi^{-1}(y)}$ is perfect, and hence it has the cardinality of the continuum.*

Proof. If $\overline{u\pi^{-1}(y)}$ is not perfect, then there is some $x \in \overline{u\pi^{-1}(y)}$ such that $\{x\}$ is relatively open in $\overline{u\pi^{-1}(y)}$. Take $n = 2$ and $U = \{x\}$ in Theorem 2.1; then we have

$$\Delta_X = \overline{Orb(\{x\} \times \{x\}, T \times T)} = R_\pi.$$

Thus π is trivial, a contradiction! The proof is completed. □

The following theorem will be used in the proof of the Main Theorem.

Theorem 2.4 ([EGS75, Corollary 7.4])**.** *A factor of a PI flow is PI.*

Now we are ready to give the proof of the Main Theorem.

Proof of the Main Theorem. By the structure theorem for minimal systems we have the following diagram:

$$\begin{array}{ccc} X & \xleftarrow[\theta]{} & X_\infty \\ & & \Big\downarrow{\scriptstyle \pi_\infty} \\ & & Y_\infty \end{array}$$

where θ is a proximal extension, π_∞ is a non-trivial weakly mixing RIC extension and Y_∞ is a strictly PI system. Let

$$d = \max\{\rho(\theta(x_1), \theta(x_2)) : (x_1, x_2) \in R_{\pi_\infty}\}.$$

Then $d > 0$, since if $d = 0$, then $R_{\pi_\infty} \subseteq R_\theta$, which implies that X is a factor of Y_∞, a contradiction by Theorem 2.4. Put $\epsilon = \frac{1}{2}d > 0$.

Fix $x \in X$, an open neighbourhood U of x and choose $x_1' \in \theta^{-1}(x)$. Let $ux_1' = x_1'$ for some minimal idempotent u, and let $y = \pi_\infty(x_1')$. Let $U' = \theta^{-1}(U)$; then U' is a neighbourhood of x_1'. We are going to construct increasing subsets $X_\alpha \subset \overline{u\pi_\infty^{-1}(y)} \cap U'$, $\alpha < \Omega$ such that each non-empty finite subset F of X_α is a transitive point of $R_{\pi_\infty}^{|F|}$, where Ω is the first uncountable ordinal number and $|F|$ means the cardinality of F.

Put $X_1 = \{x_1'\}$. By Theorem 2.1 and Corollary 2.3, there is some $x_2' \in \overline{u\pi_\infty^{-1}(y)} \cap U'$ such that (x_1', x_2') is a transitive point of $R_{\pi_\infty}^2$. Let $X_2 = \{x_1', x_2'\}$.

Now assume that $\alpha < \Omega$ is an ordinal number and X_β has been constructed for any $\beta < \alpha$ such that each non-empty finite subset F of X_β is a transitive point of $R_{\pi_\infty}^{|F|}$. If α is a limit ordinal number, then we put $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$. It is clear that each finite non-empty subset F of X_α is a transitive point of $R_{\pi_\infty}^{|F|}$. Assume now α is not a limit ordinal number. For each finite non-empty subset F of $X_{\alpha-1}$, by

Theorem 2.1 and Corollary 2.3, the set Z_F of points $z \in \overline{u\pi_\infty^{-1}(y)}$ such that (F, z) is transitive in $R_{\pi_\infty}^{|F|+1}$ is residual in $\overline{u\pi_\infty^{-1}(y)}$. Since $X_{\alpha-1}$ is countable, the number of finite non-empty subsets of $X_{\alpha-1}$ is also countable. Hence $Z = \bigcap_F Z_F$ is also residual in $\overline{u\pi_\infty^{-1}(y)}$, where F runs over all finite non-empty subsets of $X_{\alpha-1}$. Choose $x'_\alpha \in (Z \setminus X_{\alpha-1}) \cap U'$ and put $X_\alpha = X_{\alpha-1} \cup \{x'_\alpha\}$. It is clear that $X_\alpha \subset \overline{u\pi_\infty^{-1}(y)} \cap U'$ and each finite non-empty subset F of X_α is a transitive point of $R_{\pi_\infty}^{|F|}$.

Let $X_\Omega = \bigcup_{\alpha<\Omega} X_\alpha$ and $S = \bigcup_{\alpha<\Omega} \theta(X_\alpha)$. Then $S \subseteq U$ and $x = \theta(x'_1) \in S$. First we show that $\theta|_{X_\Omega} : X_\Omega \to S$ is injective, and hence S is uncountable. Assume to the contrary that there are some $\alpha < \beta < \Omega$ such that $x_\alpha = \theta(x'_\alpha) = \theta(x'_\beta) = x_\beta$. In particular,

$$(x'_\alpha, x'_\beta) \in R_\theta.$$

As (x'_α, x'_β) is a transitive point of $(R_{\pi_\infty}, T \times T)$, it follows that

$$R_{\pi_\infty} = \overline{Orb((x'_\alpha, x'_\beta), T \times T)} \subseteq R_\theta.$$

Hence X is a factor of Y_∞, and X is a PI flow by Theorem 2.4, a contradiction.

Now we show that S is proximal and recurrent. Let $n \in \mathbb{N}$ and $\alpha_1 < \alpha_2 < \ldots < \alpha_n < \Omega$. Let $\mathbf{x}' = (x'_{\alpha_1}, x'_{\alpha_2}, \ldots, x'_{\alpha_n})$ and $\mathbf{x} = (x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n})$, where $x_{\alpha_i} = \theta(x'_{\alpha_i}), 1 \le i \le n$. Since the diagonal of ${X_\infty}^n$ is contained in $R_{\pi_\infty}^n$ and $\mathbf{x}'$ is the transitive point of $R_{\pi_\infty}^n$, there are a sequence $\{n_i\} \subseteq \mathbb{N}$ and some point $z' \in X_\infty$ such that $(T^{n_i} x'_{\alpha_1}, \ldots, T^{n_i} x'_{\alpha_n}) \to (z', \ldots, z')$ as $n_i \to \infty$. Since θ is continuous,

$$(T^{n_i} x_{\alpha_1}, \ldots, T^{n_i} x_{\alpha_n}) = (\theta(T^{n_i} x'_{\alpha_1}), \ldots, \theta(T^{n_i} x'_{\alpha_n})) \to (\theta(z'), \ldots, \theta(z')),\ n_i \to \infty.$$

Thus $\{x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}\}$ is uniformly proximal.

Since $\mathbf{x}'$ is the transitive point of $R_{\pi_\infty}^n$, $\mathbf{x}'$ is recurrent in the product system $(X_\infty^n, T^{(n)})$ and hence its $\theta \times \ldots, \times\theta$-image $\mathbf{x}$ is also recurrent. So S is recurrent.

It is left to show that for all $\alpha < \beta < \Omega$, $(x_\alpha, x_\beta) = (\theta(x'_\alpha), \theta(x'_\beta))$ is LYS$_\epsilon$. By the definition of d, there is some pair $(p, q) \in R_{\pi_\infty}$ such that $\rho(\theta(p), \theta(q)) > \frac{1}{2} d = \epsilon$. Since (x'_α, x'_β) is the transitive point of R_{π_∞}, there is a sequence $\{m_i\} \subseteq \mathbb{N}$ such that $(T^{m_i} x'_\alpha, T^{m_i} x'_\beta) \to (p, q)$ as $m_i \to \infty$. As θ is continuous,

$$\lim_i \rho(T^{m_i} x_\alpha, T^{m_i} x_\beta) = \lim_i \rho(\theta(T^{m_i} x'_\alpha), \theta(T^{m_i} x'_\beta)) = \rho(\theta(p), \theta(q)) > \epsilon.$$

Hence (x_α, x_β) is LYS$_\epsilon$. The proof is completed. □

Corollary 2.5. *Any minimal system with a non-trivial weakly mixing factor is Li-Yorke sensitive.*

Proof. Let (X, T) be a minimal system with a non-trivial weakly mixing factor (Y, T). By the Main Theorem, it remains to show that (X, T) is not PI. If not, then as a factor of (X, T), (Y, T) is PI by Theorem 2.4. Hence (Y, T) is both weakly mixing and PI, which means that (Y, T) is trivial (Remark 1.4). A contradiction! □

Remark 2.6. In this paper, we consider $T : X \to X$ as a homeomorphism for simplicity. When T is a continuous surjective map, one can use the natural extension to get the corresponding results. Recall that for a system (X, T), let $(\widetilde{X}, \widetilde{T})$ be the *natural extension* of (X, T), i.e. $\widetilde{X} = \{(x_1, x_2, \ldots) \in \prod_{i=1}^\infty X : T(x_{i+1}) = x_i,\ i \in \mathbb{N}\}$ (as the subspace of the product space) and $\widetilde{T}(x_1, x_2, \ldots) = (T(x_1), x_1, x_2, \ldots)$. It is clear that $\widetilde{T} : \widetilde{X} \to \widetilde{X}$ is a homeomorphism and $p_1 : \widetilde{X} \to X$ is an asymptotic extension, where p_1 is the projection to the first coordinate. When (X, T) is minimal, $(\widetilde{X}, \widetilde{T})$ is minimal and p_1 is almost one-to-one [AGHSY, Corollary 5.18]. We say that a topological system (X, T) is PI, if its natural extension $(\widetilde{X}, \widetilde{T})$ is PI. Then the Main Theorem and Corollary 2.5 remain true.

Remark 2.7. We say that a subset of A of X is a *Mycielski set* if it can be expressed as a union of countably many Cantor sets. Using the methods in the proof of Theorem 5.10 in [AGHSY] and some more efforts, our Main Theorem can be stated replacing "S is uncountable" by "S is Mycielski".

3. A question

Finally we ask a question related to Question 1.2.

Question 3.1. Is a non-point-distal minimal system Li-Yorke chaotic?

Since each LYS minimal system is not point-distal, an affirmative answer to Question 3.1 will also give an affirmative answer to Question 1.2 for the minimal case.

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Wu Wen-Tsun Key Laboratory of Mathematics, USTC, Chinese Academy of Sciences and Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, 230026, People's Republic of China

E-mail address: songshao@ustc.edu.cn

Wu Wen-Tsun Key Laboratory of Mathematics, USTC, Chinese Academy of Sciences and Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, 230026, People's Republic of China

E-mail address: yexd@ustc.edu.cn