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THE DEGREE OF A TROPICAL BASIS

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ABSTRACT. We give an explicit upper bound for the degree of a tropical basis of a homogeneous polynomial ideal. As an application f-vectors of tropical varieties are discussed. Various examples illustrate differences between Gröbner and tropical bases.

1. INTRODUCTION

Computations with ideals in polynomial rings require an explicit representation in terms of a finite set of polynomials which generate that ideal. The size, i.e., the amount of memory required to store this data, depends on four parameters: the number of variables, the number of generators, their degrees and the sizes of their coefficients. For purposes of computational complexity it is of major interest to obtain explicit bounds for these parameters. An early step in this direction is Hermann's degree bound [Her26] on solutions of linear equations over $\mathbb{Q}[x_1, \ldots, x_n]$. In practice, however, not all generating sets are equally useful, and so it is important to seek complexity results for generating sets which have additional desirable properties. A landmark result here is the worst case space complexity estimate for Gröbner bases by Mayr and Meyer [MM82].

Tropical geometry associates with an algebraic variety a piecewise linear object in the following way. Let $\mathbb{K}$ be a field with a real-valued valuation, which we denote as val. We consider an ideal I in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ and its vanishing locus $\mathrm{V}(I)$, which is an affine variety. The *tropical variety* $\mathcal{T}(I)$ is defined as the topological closure of the set

$$(1) \qquad \mathrm{val}\big(\mathrm{V}(I)\big) \;=\; \big\{ \big(\mathrm{val}(z_1), \ldots, \mathrm{val}(z_n)\big) \;\big|\; z \in \mathrm{V}(I) \cap (\mathbb{K} \setminus \{0\})^n \big\} \quad \subset \; \mathbb{R}^n \;.$$

In general, $\mathcal{T}(I)$ is a polyhedral complex whose dimension agrees with the Krull dimension of I; see Bieri and Groves [BG84]. If, however, the ideal I has a generating system of polynomials whose coefficients are mapped to zero by val that polyhedral complex is a fan. This is the *constant coefficient case*. A major technical challenge in tropical geometry is the fact that, in general, intersections of tropical varieties do not need to be tropical varieties. Therefore, the following concept is crucial for an approach via computational commutative algebra. A finite generating subset T of I is a *tropical basis* if the tropical variety $\mathcal{T}(I)$ coincides with the intersection of

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the finitely many tropical hypersurfaces $\mathcal{T}(f)$ for $f \in T$; see [MS15, §2.6] for the details.

Our main result states that each such ideal has a tropical basis whose degree does not exceed a certain bound which is given explicitly. While the bound which we are currently able to achieve is horrendous, to the best of our knowledge this is the first result of this kind. The main result comes in two versions: Theorem 5 covers the case of constant coefficients, while Theorem 10 deals with the general case. Moreover, we present examples of tropical bases which exhibit several interesting features. We close this paper with an application to f-vectors of tropical varieties and two open questions.

2. Degree bounds

In this section we will assume that the valuation on the field $\mathbb{K}$ is trivial, i.e., we are in the constant coefficient case. Throughout the following let I be a homogeneous ideal in the polynomial ring $R := \mathbb{K}[x_1, \ldots, x_n]$. Bogart et al. were the first to describe an algorithm for computing a tropical basis [BJS+07, Thm. 11]. This algorithm is implemented in `Gfan`, a software package for computing Gröbner fans and tropical varieties [Jen]. Since our proof rests on the method of Bogart et al. we need to give a few more details. Every weight vector $w \in \mathbb{R}^n$ gives rise to a generalized term order on R. The generalization lies in the fact that this order may only be partial, which is why the initial form $\mathrm{in}_w(f)$ of a polynomial f does not need to be a monomial. Now the tropical variety of I can be described as the set

$$\mathcal{T}(I) \;=\; \{ w \in \mathbb{R}^n \mid \mathrm{in}_w(I) \text{ does not contain any monomial}\} \;,$$

where the *initial ideal* $\mathrm{in}_w(I)$ is generated from all initial forms of polynomials in I. Declaring two weight vectors equivalent whenever their initial ideals agree yields a stratification of $\mathbb{R}^n$ into relatively open polyhedral cones; this is the *Gröbner fan* of I. Each maximal cone of the Gröbner fan corresponds to a proper term order or, equivalently, to a monomial initial ideal and a reduced Gröbner basis. A Gröbner basis is *universal* if it is a Gröbner basis for each term order. By construction $\mathcal{T}(I)$ is a subfan of the Gröbner fan. A polynomial $f \in I$ is a *witness* for a weight vector $w \in \mathbb{R}^n$ if its initial form $\mathrm{in}_w(f)$ is a monomial. Such a polynomial f certifies that the Gröbner cone containing w is not contained in $\mathcal{T}(I)$. The algorithm in [BJS+07] now checks each Gröbner cone and adds witnesses to a universal Gröbner basis to obtain a tropical basis.

An ideal I contains the monomial $x^m = x_1^{m_1} \cdots x_n^{m_n}$ if and only if the quotient

$$I : x^m \;=\; \{ f \in R \mid \; x^m f \in I\}$$

contains a unit. The ideal

$$(I : x^m)^\infty \;=\; \bigcup_{k \in \mathbb{N}} (I : x^{km})$$

is called the *saturation* of I with respect to x^m. Since the ring R is Noetherian there exists a smallest number k such that $I : x^{km} = (I : x)^\infty$. That number k is the *saturation exponent*. Hence the total degree of any witness does not exceed αn, where α is the maximal saturation exponent of all initial ideals of I with respect to $x_1 \cdots x_n$. We need to get a grip on that parameter α. The *degree* of a finite set of polynomials is the maximal total degree which occurs.

Proposition 1. *Let I be a homogeneous ideal. The saturation exponent α of I with respect to $x_1 \cdots x_n$ is bounded by*

$$\alpha \;\leq\; \deg H,$$

where H is a universal Gröbner basis for I.

Proof. Since H is universal it contains a Gröbner basis $\{f_1, \ldots, f_s\}$ for the reverse lexicographic order. By [Eis95, Prop. 15.12] the set

$$\left\{ \frac{f_1}{\gcd(x_n, f_1)}, \frac{f_2}{\gcd(x_n, f_2)}, \ldots, \frac{f_s}{\gcd(x_n, f_s)} \right\}$$

is a Gröbner basis for $I : x_n$. Thus the saturation exponent of I with respect to x_n is bounded by the degree $\deg_{x_n}(H)$ of H in the variable x_n. Permuting the variables implies a similar statement for x_i. It follows that $\alpha = \max_{1 \leq i \leq n} \deg_{x_i} H \;\leq\; \deg H$. □

Notice that the tropical variety of a homogeneous ideal I coincides with the tropical variety of the saturated ideal $I : (x_1 \cdots x_n)^\infty$. For the next step we need to determine the degree of a universal Gröbner basis. The key ingredient is a result of Mayr and Ritscher [MR10]. Here and below d is the *degree* of I, i.e., the minimum of the degrees of all generating sets, and r is the Krull dimension.

Proposition 2 (Mayr and Ritscher)**.** *Assume that $r \geq 1$. Each reduced Gröbner basis G of the ideal I satisfies*

$$\deg G \;\leq\; 2 \left(\frac{d^{n-r} + d}{2} \right)^{2^{r-1}} . \tag{2}$$

Lakshman and Lazard [LL91] give an asymptotic bound of the degree on zero-dimensional ideals, that is, for $r = 0$. For Gröbner bases one could argue that the degree is more interesting than the number of polynomials. This is due to the following simple observation.

Remark 3. A reduced Gröbner basis of degree e (of any ideal in R) can contain at most $\binom{e+n-1}{e} = \binom{e+n-1}{n-1}$ polynomials. The reason is that no two leading monomials can divide one another.

We are ready to bound the degree of a universal Gröbner basis. In view of the previous remark this also entails a bound on the number of polynomials. Since we will use Proposition 2, throughout this section we will assume that $r \geq 1$.

Corollary 4. *There is a universal Gröbner basis for I whose degree is bounded by* (2)*.*

Proof. The union of the reduced Gröbner bases for all term orders is universal. The claim follows since the bound in Proposition 2 is uniform. □

For our main result we apply the bounds which we just obtained to the output of the algorithm in [BJS+07].

Theorem 5. *Suppose that the valuation* val *on the coefficients is trivial. There is a universal Gröbner basis U and a tropical basis T of the homogenous ideal I with*

$$\deg T \;\leq\; \max\{\deg U, \alpha n\} \;\leq\; n \deg U \;\leq\; 2n \left(\frac{d^{n-r} + d}{2} \right)^{2^{r-1}} . \tag{3}$$

Proof. The number αn bounds the degree of a witness, and so the first inequality follows from the correctness of the algorithm [BJS+07, Thm. 11]. For a weight vector w we abbreviate $J := \mathrm{in}_w(I)$. From U we can obtain a universal Gröbner basis H for J, and this satisfies $\deg H \leq \deg U$. The initial ideal J coincides with the initial ideal of I with respect to a perturbation of the term order that yields J in direction w. From Proposition 1 we thus get the second inequality. Finally, the third inequality follows from (2) and Corollary 4. $\square$

Replacing (2) by other estimates gives variations of the last inequality in (3). For example, the bound

$$\deg G \;\leq\; 2\left(\frac{d^2}{2}+d\right)^{2^{n-1}} \tag{4}$$

of Dubé [Dub90] does not rely on the dimension r. Multiplying that bound by n also yields an upper bound on the degree of a tropical basis. Note that the results of this section hold for arbitrary characteristic of $\mathbb{K}$.

3. Examples

Throughout this section, we will be looking at the case $\mathbb{K} = \mathbb{C}$, and val sends each non-zero complex number to zero. In particular, as above, we are considering constant coefficients.

It is known that, in general, a universal Gröbner basis does not need to be a tropical basis; see [BJS+07, Ex. 10] or [MS15, Ex. 2.6.7]. That is, it cannot be avoided to compute witness polynomials. In fact, the following example, which is a simple modification of [BJS+07, Ex. 10], shows that adding witnesses may even increase the degree.

Example 6. Let $I \subset \mathbb{C}[x,y,z]$ be the ideal generated by the six degree 3 polynomials

$$\begin{array}{ccc} x^2y+xy^2\,, & x^2z+xz^2\,, & y^2z+yz^2\,, \\ x^3+x^2y+x^2z\,, & xy^2+y^3+y^2z\,, & xz^2+yz^2+z^3. \end{array}$$

These six generators together with the ten polynomials of degree 3 below form a universal Gröbner basis for I

$$\begin{array}{ccc} x^3-xy^2-xz^2\,, & x^2y-y^3+yz^2\,, & x^2z+y^2z-z^3\,, \\ x^3-xy^2+x^2z\,, & xy^2+y^3-yz^2\,, & xz^2-y^2z+z^3\,, \\ x^3+x^2y-xz^2\,, & x^2y-y^3-y^2z\,, & x^2z-yz^2-z^3\,, \\ & x^3+y^3+z^3. & \end{array}$$

The monomial x^2yz of degree 4 is contained in I. This is a witness to the fact that the tropical variety $\mathcal{T}(I)$ is empty. Since, however, there is no monomial of degree 3 contained in I, any tropical basis must have degree at least 4. One such tropical basis, T, is given by the six generators and the monomial x^2yz. This also shows that a tropical basis does not need to contain a universal Gröbner basis.

A tropical basis does not even need to be any Gröbner basis, as the next example shows.

Example 7. Consider the three polynomials

$$x^5\,, \quad x^4+x^2y^2+y^4\,, \quad y^5,$$

in $\mathbb{C}[x, y]$. They form a tropical basis for the ideal they generate. However each Gröbner basis has to include at least one of the S-polynomials $x^3y^2 + xy^4$ or $x^4y + x^2y^3$.

For conciseness the Examples 6 and 7 address tropical varieties which are empty. One can modify the above to obtain ideals and systems of generators with similar properties for tropical varieties of arbitrarily high dimension. We leave the details to the reader.

It is obvious that the final upper bound in (3) is an extremely coarse estimate. However, better bounds on the degree of the universal Gröbner basis can clearly be exploited. The following example may serve as an illustration.

Example 8. Let $I = \langle xy - zw + uv\rangle \subset \mathbb{C}[x, y, z, u, v, w]$. In this case we have $d = 2$, $n = 6$ and $r = 5$. Since I is a principal ideal the single generator forms a Gröbner basis, which is even universal and also a tropical basis. The degree of that universal Gröbner basis is $d = 2$, which needs to be compared with the upper bound of 2^{17} from (2). For the saturation exponent we have $\alpha = 1 \leq 2$, and the degree of the tropical basis equals $d = 2$. This is rather close to the bound $\alpha n = 6$, whereas the final upper bound in (3) is as much as $3 \cdot 2^{18}$.

Our final example generalizes the previous. In fact, Example 8 re-appears below for $D = 2$ and $N = 4$.

Example 9. The Plücker ideal $I_{D,N}$ captures the algebraic relations among the $D\times D$-minors of a generic $D\times N$-matrix with coefficients in the field $\mathbb{K}$. This is a homogeneous prime ideal in the polynomial ring over $\mathbb{K}$ with $n = \binom{N}{D}$ variables. The variety $\mathrm{V}(I_{D,N})$ is the *Grassmannian* of D-planes in $\mathbb{K}^N$. Its tropicalization $\mathcal{T}(I_{D,N})$ is the *tropical Grassmannian* of Speyer and Sturmfels [SS04]; see also [MS15, §4.3].

The Plücker ideal is generated by quadratic relations; see [Stu08, Thm 3.1.7]. Its dimension equals $r = (N - D)D + 1$; see [SS04, Cor 3.1]. From this data we derive that there is a tropical basis $T_{D,N}$ of degree

$$\deg T_{D,N} \;\leq\; 2 \cdot \tbinom{N}{D} \cdot \left(2^{\binom{N}{D} - ND + D^2 - 2} + 1\right)^{2^{ND - D^2}} .$$

To the best of our knowledge explicit tropical bases for $I_{D,N}$ are known only for $D = 2$ and $(D, N) \in \{(3, 6), (3, 7)\}$; see [SS04] and [HJJS09]. Note that for $D = 2$ the degree of a universal Gröbner basis grows with n while the quadratic 3-term Plücker relations form a tropical basis.

4. Non-constant coefficients

Recently, Markwig and Ren [MR16] presented a new algorithm which extends [BJS+07] to the case of non-constant coefficients. We will use their method to generalize Theorem 5 accordingly. To this end we will browse through our exposition in Section 2 and indicate the necessary changes to the arguments.

Let $\mathbb{K}$ be a field equipped with a non-trivial discrete valuation val. The *valuation ring* $\mathfrak{r} := \{ a \in \mathbb{K} \mid \mathrm{val}(a) \geq 0\}$ has a unique maximal ideal $\mathfrak{m} := \{ a \in \mathbb{K} \mid \mathrm{val}(a) > 0\}$. The ideal $\mathfrak{m}$ is generated by a prime element, which we denote as t. The *residue field* of $\mathbb{K}$ is the quotient $\Bbbk := \mathfrak{r}/\mathfrak{m}$. The *initial form* of a homogeneous polynomial

$f = \sum_{u \in \mathbb{N}^n} c_u x^u$ in $\mathbb{K}[x_1, \ldots, x_n]$ with respect to a weight vector $w \in \mathbb{R}^n$ is

$$\mathrm{in}_w(f) \;=\; \sum_{\substack{w \cdot u - \mathrm{val}(c_u) \\ \text{maximal}}} \overline{t^{-\mathrm{val}(c_u)} c_u}\, x^u \;\in\; \Bbbk[x_1, \ldots, x_n] \ ,$$

where $\overline{\cdot}$ describes the canonical projection from $\mathfrak{r}$ to $\Bbbk$. The initial ideal $J = \mathrm{in}_w(I)$, the tropical variety $\mathcal{T}(I)$ of an ideal I, witnesses and tropical bases are defined as in the constant coefficient case. The key difference to the classical case is that the stratification of $\mathbb{R}^n$ by initial ideals yields a polyhedral complex, the *Gröbner complex* $\Gamma(I)$, which does not need to be a fan; see [MS15, Section 2.5]. The tropical variety $\mathcal{T}(I)$ is a subcomplex of $\Gamma(I)$.

Let $w \in \mathbb{R}^n$ be a generic vector, i.e., it is an interior point of some maximal cell of $\Gamma(I)$. Like in the classical case, a *Gröbner basis* of I with respect to w is a set of generators such that their initial forms with respect to w generate the entire initial ideal $\mathrm{in}_w(I)$. Further, a Gröbner basis is *universal* if it works for all weight vectors. Again, a universal Gröbner basis enhanced with a witness for each cell in $\Gamma(I) \setminus \mathcal{T}(I)$ forms a tropical basis. As before, the degree of a witness with respect to w is bounded by the saturation exponent of the saturation exponent of $(\mathrm{in}_w(I) : x)^\infty$.

We are ready to state and prove the following generalization of Theorem 5, which was suggested to us by Yue Ren. We are grateful for this hint.

Theorem 10. *Suppose that* val *is a non-constant discrete valuation on* $\mathbb{K}$*. There is a universal Gröbner basis* U *and a tropical basis* T *of the homogeneous ideal* I *with*

$$\deg T \;\le\; \max\{\deg U, \alpha n\} \;\le\; n \deg U \;\le\; 2n\left(\frac{d^2}{2} + d\right)^{2^{n-1}} . \tag{5}$$

Proof. Our proof is based on the algorithm of Markwig and Ren [MR16], which is a direct generalization of [BJS+07]. Let U be a universal Gröbner basis of the ideal I. For $w \in \mathbb{R}^n$ the set $\{\mathrm{in}_w(f) \mid f \in U\}$ is a Gröbner basis of J. By Proposition 1, the saturation exponent of the saturation $(J : x)^\infty$ is bounded by the degree $\deg U$. This establishes the first two inequalities in (5). The final inequality follows from Dubé's bound (4). That result was extended to non-constant coefficients by Chan and Maclagan; see [CM13, Theorem 3.1]. $\square$

Remark 11. The canonical valuation on the field of Puiseux series $\mathbb{C}\{\!\{t\}\!\}$ is not discrete, and the valuation ring is not Noetherian; see [MS15, Remark 2.4.13]. However, the computation of a tropical basis for any finitely generated ideal can be restricted to a polynomial ring over an appropriate discretely valuated subfield. The degree bound in Theorem 10 does not depend on the choice of that subfield. Thus Theorem 10 also holds for $\mathbb{K} = \mathbb{C}\{\!\{t\}\!\}$, provided that I is finitely generated.

5. The f-vector of a tropical variety

The *f-vector* of a polyhedral complex, which counts the number of cells by dimension, is a fundamental combinatorial complexity measure. In this section we will give an explicit bound on the f-vector of a tropical variety $\mathcal{T}(I)$, with arbitrary valuation on the field $\mathbb{K}$, in terms of the number s of polynomials in a tropical basis T and the degree d of a tropical basis, T. Notice that in the previous sections 'd' was the degree of I.

First we discuss the case of a tropical hypersurface, that is, $s = 1$, as in Example 8. Let $g \in R$ be an arbitrary homogeneous polynomial of degree d. As in Section 4, here we are admitting non-constant coefficients. A tropical hypersurface $\mathcal{T}(g)$ is dual to the regular subdivision of the Newton polytope $N(g)$ of g, which is gotten from lifting the lattice points in $N(g)$, which correspond to the monomials in g, to the valuation of their coefficients [MS15, Prop. 3.1.6]. See the monograph [DLRS10] for details on polytopal subdivisions of finite point sets. The polynomial g has at most $\binom{d+n-1}{n-1}$ monomials, which correspond to the lattice points in the dth dilation of the $(n-1)$-dimensional simplex $d \cdot \Delta_{n-1}$. The *standard simplex* Δ_{n-1} is the $(n-1)$-dimensional convex hull of the n standard basis vectors $e_1, \ldots, e_n$. The maximal f-vector of a polytopal subdivision of $d \cdot \Delta_{n-1}$ by lattice points is (simultaneously for all dimensions) attained for a unimodular triangulation [BM85, Thm. 2]. If Δ is such a unimodular triangulation, then its vertices use all lattice points in $d \cdot \Delta_{n-1}$. The converse does not hold if $n \geq 4$. The f-vector of Δ equals

$$f_j^{\Delta} \ = \ \sum_{i=0}^{j} (-1)^{i+j} \binom{j}{i} \binom{di+d+n-1}{n-1} \ ; \tag{6}$$

see [DLRS10, Thm. 9.3.25]. By duality the bound in (6) translates into a bound on the f-vector for the tropical hypersurface $\mathcal{T}(g)$:

$$f_j^{\mathcal{T}(g)} \ \leq \ f_{n-j-1}^{\Delta} \ \leq \ \sum_{i=1}^{n-j} (-1)^{n+i-j} \binom{n-j-1}{i-1} \binom{di+n-1}{n-1} \ . \tag{7}$$

From the above computation we can derive the following general result.

Proposition 12. *Let I be a homogeneous ideal in R. Then the f-vector of the tropical variety $\mathcal{T}(I)$ with a tropical basis T, consisting of s polynomials of degree at most d, is bounded by*

$$f_j \ \leq \ \sum_{i=1}^{n-j} (-1)^{n+i-j} \binom{n-j-1}{i-1} \binom{sdi+n-1}{n-1} \ .$$

Proof. Let g denote the product $h_1 \cdots h_s$ of all polynomials in the tropical basis T. The tropical hypersurface of g is the support of the $(n-1)$-skeleton of the polyhedral complex dual to a regular subdivision of the Newton polytope $N(g)$; see [MS15, Prop. 3.1.6]. This polytope is the Minkowski sum of all Newton polytopes $N(h)$ for $h \in T$. Moreover, the polyhedral subdivision of $N(g)$ dual to $\mathcal{T}(g)$ is the common refinement of the subdivisions of the Newton polytopes for the polynomials in T. The tropical variety $\mathcal{T}(I)$ is a subcomplex of this refinement since, by the definition of T, we have

$$\mathcal{T}(I) \ = \ \bigcap_{f \in T} \mathcal{T}(f) \ .$$

The polynomial g is of degree at most sd. From the inequality (7) we get the claim. $\square$

Let us now discuss the special case of a tropical hypersurface $\mathcal{T}(g)$ with constant coefficients. That is, we assume that the valuation map applied to each coefficient of the homogeneous polynomial g yields zero. In this case the lifting is trivial

TABLE 1. The vectors $\lambda(d,n)$ for small values of d and n. A star indicates only a lower bound, which is due to the fact that we could not complete our ad hoc computation with the given resources.

$n\backslash d$	1	2	3	4	5
2	(2)	(2)	(2)	(2)	(2)
3	$(3,3)$	$(4,4)$	$(6,6)$	$(6,6)$	$(8,8)$
4	$(4,6,4)$	$(7,12,8)$	$(12,18,10)$	$(15,24,16)$	$(20,36,22)$
5	$(5,10,10,5)$	$(11,30,30,10)$	$(20,48,50,20)$	$(28,83,86,33)^*$	$(33,96,101,36)^*$

and thus $\mathcal{T}(g)$ is dual to a lattice polytope contained in the simplex $d \cdot \Delta_{n-1}$; see [MS15, Prop. 3.1.10]. We introduce the parameter

$$\lambda_j(d,n) \;=\; \max\left\{ f_j^P \;\middle|\; P \text{ is a lattice polytope in } d\cdot\Delta_{n-1} \right\} \quad ,$$

which measures how combinatorially complex tropical hypersurfaces (with constant coefficients) can be. We arrive at the following conclusion.

Corollary 13. *Let I be a homogeneous ideal in R which is generated by polynomials with constant coefficients. Then the f-vector of the tropical variety $\mathcal{T}(I)$ a tropical basis T, consisting of s polynomials of degree at most d, is bounded by*

$$f_j \;\leq\; \lambda_{n-j-1}(sd,n) \quad . \tag{8}$$

Notice that the $(n-1)$-simplex has $\lambda_0(1,n) = n$ vertices and an interval has $\lambda_0(d,2) = 2$ vertices. The number of vertices $\lambda_0(d,n)$ does not exceed the sum of the number of vertices in $(d-1)\cdot\Delta_{n-1}$ and $d\cdot\Delta_{n-2}$. Hence, e.g., the number of $(n-1)$-cells of $\mathcal{T}(I)$ in (8) is bounded by

$$f_{n-1} \;\leq\; \lambda_0(sd,n) \;\leq\; \sum_{i=0}^{sd-2} 2\binom{i+n-3}{i} + \sum_{i=0}^{n-3}(n-i)\binom{i+sd-2}{i} \quad .$$

We calculated the numbers $\lambda_j(d,n)$ for small values of d and n with `polymake` [GJ00]. The result is summarized in Table 1. Note that, e.g., for $d=2$ and $n=4$ there is no polytope that maximizes f_j simultaneously for all j. We expect that it is difficult to explicitly determine the values for $\lambda_j(d,n)$. The somewhat related question of determining the (maximal) f-vectors of 0/1-polytopes is a challenging open problem; see [Zie00].

6. Open questions

For constant coefficients, Hept and Theobald [HT09] developed an algorithm for computing tropical bases, which is based on projections.

Question A. Can their approach be used to obtain better degree bounds?

Our current techniques employ the Gröbner complex of an ideal, i.e., a universal Gröbner basis. Yet, as Example 7 shows tropical bases and Gröbner bases are not related in a straightforward way.

Question B. Is it possible to directly obtain a tropical basis from the generators of an ideal, i.e., without the need to compute any Gröbner basis?

Notice the algorithm of Hept and Theobald [HT09] uses elimination (Gröbner bases). However, one may ask if techniques from polyhedral geometry can further be exploited to obtain yet another method for computing tropical bases and tropical varieties.

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