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VANISHING SIMPLICIAL VOLUME FOR CERTAIN AFFINE MANIFOLDS

MICHELLE BUCHER, CHRIS CONNELL, AND JEAN-FRANÇOIS LAFONT

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Abstract. We show that closed aspherical manifolds supporting an affine structure, whose holonomy map is injective and contains a pure translation, must have vanishing simplicial volume. As a consequence, these manifolds have zero Euler characteristic, satisfying the Chern Conjecture. Along the way, we provide a simple cohomological criterion for aspherical manifolds with normal amenable subgroups of π_1 to have vanishing simplicial volume. This answers a special case of a question due to Lück.

1. Introduction

The topology of affine manifolds remains quite poorly understood. In this short note, we consider the simplicial volume of affine manifolds. We show:

Main Theorem. *Let M be a closed aspherical manifold. Suppose that M admits an affine structure for which the holonomy representation $\rho : \pi_1(M) \to \mathrm{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes GL_n(\mathbb{R})$ is injective, and has non-trivial translational subgroup $\rho(\pi_1(M)) \cap \mathbb{R}^n$. Then the simplicial volume of M vanishes.*

Recall that, for a closed oriented manifold M, the simplicial volume $||M||$ is a topological invariant which measures how efficiently the fundamental class of M can be represented as a real singular chain. This non-negative real valued invariant was introduced by Gromov [Gr82] and Thurston [Th80, Chapter 6]. More precisely, for a topological space X, the real vector spaces $C_*(X;\mathbb{R})$ appearing in the chain complex for the singular homology $H_*(X;\mathbb{R})$ come equipped with a canonical basis, consisting of the set of all continuous maps from an appropriate dimensional simplex into the space X. There is thus an associated l^1-norm on $C_*(X;\mathbb{R})$, which descends to a semi-norm $||\cdot||_1$ on the homology $H_*(X;\mathbb{R})$. The *simplicial volume* $||M||$ of a closed oriented manifold is then defined to be $||[M]||_1$, where $[M]$ is the fundamental class in $H_n(M;\mathbb{R})$. This invariant is multiplicative under covers, so the definition can be extended to closed non-orientable manifolds. Now dual to this l^1-norm on $C_*(X;\mathbb{R})$, we also have an l^∞-norm $||\cdot||_\infty$ on the real vector spaces $C^*(X;\mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(C_*(X;\mathbb{R});\mathbb{R}) = [C_*(X;\mathbb{R})]^\vee$ arising in the cochain complex for singular cohomology $H^*(X;\mathbb{R})$. By considering the bounded elements, one obtains a subcomplex of the cochain complex, whose homology yields the *bounded cohomology* $H_b^*(X;\mathbb{R})$. The natural inclusion of cochain complexes induces a comparison

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map $c : H_b^*(X;\mathbb{R}) \to H^*(X;\mathbb{R})$ from the bounded cohomology to the ordinary cohomology. Elements in the image are cohomology classes which have bounded representatives, and hence have a well-defined l^∞-norm. The simplicial volume of M vanishes if and only if the comparison map in top dimension is the zero map.

A smooth manifold M supports an affine structure if one can choose charts for M so that all transition maps are locally constant affine maps. If M has an affine structure, then there is an associated *holonomy representation* $\rho : \pi_1(M) \to \mathrm{Aff}(\mathbb{R}^n)$, and a ρ-equivariant *developing map* $D : \tilde{M} \to \mathbb{R}^n$ (unique up to affine transformations). If the developing map D is a homeomorphism, then the affine structure is called *complete*.

If M has a complete affine structure, then it follows that the holonomy representation is injective, and that M is aspherical. Thus from our **Main Theorem** we obtain (see also the discussion in Section 4):

Corollary 1.1. *If M is a closed manifold with a complete affine structure, and the holonomy contains a pure translation, then $||M|| = 0$.*

Remark 1.2. There exist examples of complete affine manifolds whose (necessarily injective) holonomy representation contains no pure translations – and hence are not covered by our **Main Theorem**. A concrete example can be obtained as follows: consider the simply transitive affine action $\phi : \mathbb{R}^2 \to \mathrm{Aff}(\mathbb{R}^2)$, given by

$$\phi(s,t) \cdot (x,y) = (x + ty + (s + t^2/2), y + t).$$

Taking any lattice in $\Lambda \leq \mathbb{R}^2$, the quotient $M := \mathbb{R}^2/\phi(\Lambda)$ is a complete affine 2-torus. On the other hand, the only elements which act via a translation are those lying in the 1-dimensional subgroup $K := \mathbb{R} \times \{0\}$. Picking a lattice Λ with $\Lambda \cap K = \{0\}$ gives the desired example. We learned of this example from Bill Goldman, who attributed it to N. Kuiper.

We have established a special case of the following natural:

Conjecture. *If M is a closed affine manifold, then $||M|| = 0$.*

Remark 1.3. In the context of closed complete affine manifolds, the Auslander Conjecture predicts that the fundamental group of such a manifold is virtually polycyclic. Since manifolds whose fundamental group are virtually polycyclic have vanishing simplicial volume, for this class of manifolds our conjecture would follow immediately from the Auslander Conjecture. In particular, from the work of Abels-Margulis-Soifer [AMS02], we see that for *complete* affine manifolds, our conjecture holds in dimensions ≤ 6.

Remark 1.4. Another famous problem is the Chern Conjecture, which asserts that affine manifolds have zero Euler characteristic. The Chern Conjecture has so far only been established for particular families of affine structures or of manifolds. For example, it is known to hold for complete affine manifolds [KoSu75], for affine manifolds with linear holonomy in $\mathrm{SL}(n,\mathbb{R})$ [Kl15] (a conjecture of Markus predicts this is equivalent to being complete), for surfaces [Be60], for higher rank irreducible locally symmetric manifolds [GoHi84], for manifolds which are locally a product of hyperbolic planes [BuGe11], and for complex hyperbolic surfaces [Pi16].

Note that the tangent bundle of an affine manifold is flat, so the Euler class can be represented by a bounded cocycle (see [Gr82]), i.e. $||e(TM)||_\infty < \infty$. Since one has the inequality $|\chi(M)| \leq ||e(TM)||_\infty \cdot ||M||$, our conjecture immediately implies the Chern Conjecture.

In view of the previous remark, our **Main Theorem** also yields some additional cases of the Chern Conjecture:

Corollary 1.5. *If M is a closed aspherical affine manifold, with injective holonomy representation, whose holonomy contains a non-trivial translation, then $\chi(M) = 0$.*

Note that in this corollary, we are not assuming that the affine structure on M is complete.

2. Normal amenable subgroups and a question of Lück

In Lück's book on L^2-invariants, the following question is raised – see [L02, Question 14.39]:

Question. Let M be a closed aspherical manifold whose fundamental group contains a non-trivial amenable normal subgroup $A \triangleleft \pi_1(M)$. Does the simplicial volume of M vanish?

Affirmative answers to this question are only known in some special cases. It is easy to check for fibrations for which the fiber has amenable fundamental group [L02, Exercise 14.15]. Furthermore, Neofytidis proves it for aspherical manifolds whose fundamental group is infinite index presentable by products while the quotient by the center of the fundamental group is not presentable by products [Ne15, Corollary 1.2]. While we will not require it for the proof of our **Main Theorem**, we first establish a special case of the question. Both of these results will rely on the following elementary lemma.

Lemma 2.1. *Let M be a closed connected n-manifold, and $\Gamma = \pi_1(M)$. Assume $A \triangleleft \Gamma$ is an amenable normal subgroup, and $q : \Gamma \to \Lambda := \Gamma/A$ is the quotient map. If the induced map $q^* : H^n(\Lambda; \mathbb{R}) \to H^n(\Gamma; \mathbb{R})$ is the zero map, then $\|M\| = 0$.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccccc}
H_b^n(\Lambda;\mathbb{R}) & \xrightarrow{q_b^*} & H_b^n(\Gamma;\mathbb{R}) & \xrightarrow{\phi_b^*} & H_b^n(M;\mathbb{R}) \\
\downarrow{\scriptstyle c_\Lambda} & & \downarrow{\scriptstyle c_\Gamma} & & \downarrow{\scriptstyle c_M} \\
H^n(\Lambda;\mathbb{R}) & \xrightarrow{q^*} & H^n(\Gamma;\mathbb{R}) & \xrightarrow{\phi^*} & H^n(M;\mathbb{R}),
\end{array}$$

where the vertical arrows are the comparison maps from bounded cohomology to ordinary cohomology, the horizontal arrows in the first block are the morphisms induced by the surjection $q : \Gamma \twoheadrightarrow \Lambda$, and the horizontal arrows in the second block are induced by the classifying map $\phi : M \to B\Gamma$.

Let us focus on the first commutative square. From the hypothesis, the bottom map q^* is the zero map. On the other hand, since A is amenable, we have that the top map q_b^* is an isomorphism (see [Gr82, Section 3.1], or take $E = \mathbb{R}$ in [M01, Remark 8.5.4]). Commutativity of the diagram forces c_Γ to also be the zero map. Now consider the second commutative square. The classifying map $\phi : M \to B\Gamma$ always induces an isomorphism on bounded cohomology. Since c_Γ is the zero map, so is c_M, which immediately implies $||M|| = 0$ by the "duality" of ℓ^1 and ℓ^∞-norms [Gr82]. ☐

Theorem 2.2. *Let M be a closed connected aspherical n-manifold whose fundamental group $\Gamma = \pi_1(M)$ contains a non-trivial amenable normal subgroup $A \lhd \pi_1(M)$, and let $\Lambda = \pi_1(M)/A$. Assume that the quotient group Λ has* finite *cohomological dimension* $\mathrm{cdim}_{\mathbb{R}}(\Lambda) = \ell < \infty$*, and that $H^\ell(\Lambda; H^k(A;\mathbb{R})) \neq 0$, where $k = \mathrm{cdim}_{\mathbb{R}}(A)$. Then $||M|| = 0$.*

Proof. We consider the cohomological Lyndon-Hochschild-Serre spectral sequence with real coefficients $\mathbb{R}$ (and trivial module structure) associated to the short exact sequence $1 \to A \to \Gamma \to \Lambda \to 1$. The E_2-page is given by

$$E_2^{p,q} := H^p(\Lambda; H^q(A;\mathbb{R})) \Rightarrow H^{p+q}(\Gamma;\mathbb{R}),$$

and the spectral sequence convergences to the cohomology $H^*(\Gamma;\mathbb{R}) \cong H^*(M;\mathbb{R})$. Note that $\tilde{M}/A$ is an n-dimensional model for a $K(A,1)$, and hence $\mathrm{cdim}_{\mathbb{R}}(A) = k \leq n$ is finite. From our hypotheses, we obtain the following observations:

(1) Since $\mathrm{cdim}_{\mathbb{R}}(A) = k$, it follows that $H^q(A;\mathbb{R}) = 0$ for all $q > k$. This forces $E_2^{p,q} = 0$ for all $q > k$.
(2) Since $\mathrm{cdim}_{\mathbb{R}}(\Lambda) = \ell$, we see that $H^p(\Lambda; -) = 0$ for all $p > \ell$, regardless of the coefficient $\mathbb{R}[\Lambda]$-module. In particular, this forces $E_2^{p,q} = 0$ for all $p > \ell$.

Whence we see that the E^2-page looks like

$$\begin{array}{c|ccccc}
 & q & & & & \\
\vdots & 0 & 0 & 0 & 0 & \\
k & H^0(\Lambda; H^k(A;\mathbb{R})) & \cdots & H^\ell(\Lambda; H^k(A;\mathbb{R})) & 0 & \\
\vdots & \vdots & \ddots & \vdots & 0 & \\
0 & H^0(\Lambda;\mathbb{R}) & \cdots & H^\ell(\Lambda;\mathbb{R}) & 0 & \\
\hline
 & & & & & p \\
 & 0 & \cdots & \ell & \cdots &
\end{array}$$

By hypothesis, we also have that the $E_2^{\ell,k}$ entry is non-zero. Thus the $E_2^{\ell,k}$ entry survives to the E_∞-page, establishing that $\mathrm{cdim}_{\mathbb{R}}(\Gamma) \geq \ell + k$.

Now the closed orientable aspherical manifold M^n is a model for $K(\Gamma, 1)$, so we obtain the lower bound $n \geq \ell + k$. Since A is non-trivial, we have that $k > 0$, and hence that $n > \ell = \mathrm{cdim}_{\mathbb{R}}(\Lambda)$. This forces $H^n(\Lambda;\mathbb{R}) = 0$, and Lemma 2.1 allows us to conclude $\|M\| = 0$. $\square$

Remark 2.3. As was pointed out to us by C. Löh, the proof of Theorem 2.2 still works if instead of M aspherical, we only have $\mathrm{cdim}_{\mathbb{R}}(\pi_1(M)) = n$.

Remark 2.4. It is tempting to use Lemma 2.1 to attack the general case of Lück's question. Notice that the induced homomorphism $q^* : H^n(\Lambda;\mathbb{R}) \to H^n(\Gamma;\mathbb{R})$

appears naturally inside the Lyndon-Hochschild-Serre spectral sequence. Indeed, $H^n(\Lambda;\mathbb{R})$ appears as the $E_2^{0,n}$-term in the spectral sequence. Thus, whether or not the induced homomorphism q^* is zero translates to whether or not the $E_2^{0,n}$ survives to the E_∞-page, i.e. whether or not $E_\infty^{0,n}=0$. In the proof of Theorem 2.2, our hypotheses already forced $E_2^{0,n}=0$.

3. Proof of Main Theorem

This section is devoted to the proof of the **Main Theorem**. So let us assume that M is a connected closed aspherical affine manifold, $\Gamma=\pi_1(M)$, and the holonomy representation $\rho:\Gamma\to \mathrm{Aff}(\mathbb{R}^n)=\mathbb{R}^n\rtimes GL_n(\mathbb{R})$ is injective with $\rho(\Gamma)\cap\mathbb{R}^n$ non-zero. Since the simplicial volume is multiplicative under finite covers, it is sufficient to show that a finite cover of M has vanishing simplicial volume. Since the hypotheses in our theorem are inherited by finite covers, we will from now on assume that the manifold M is orientable.

We have the following commutative diagram relating the various groups we are interested in:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, hook, "i"] & \mathrm{Aff}(\mathbb{R}^n) \arrow[r, "L"] & \mathrm{GL}_n(\mathbb{R}) \\ A \arrow[u, hook] \arrow[r, hook, "i"] & \Gamma \arrow[u, hook, "\rho"] \arrow[r, two heads, "q"] & \Lambda. \arrow[u, hook, "\bar{\rho}"] \end{tikzcd} \tag{3.1}$$

Here A is the purely translational part of $\pi_1(M)$ – and by hypothesis, A is non-trivial, so of rank ≥ 1. This forces $\mathrm{cdim}_{\mathbb{R}}(A)\geq 1$. Since $A\triangleleft\pi_1(M)$ and M is aspherical, $\pi_1(M)$ and A are torsion-free. Note that $\tilde{M}/A$ is an n-dimensional model for a $K(A,1)$, which immediately gives us:

Fact 1. $\mathrm{cdim}_{\mathbb{R}}(A)=k$, with $1\leq k\leq n$, and hence $A\cong\mathbb{Z}^k$.

Next we consider $\mathrm{cdim}_{\mathbb{R}}(\Lambda)$, where Λ is the linear part of the holonomy action. A special case of the main theorem of [AlSh81] states:

Theorem (Alperin-Shalen)**.** *If S is a finitely generated integral domain of characteristic zero, then $G<\mathrm{GL}(S)$ has finite virtual cohomological dimension* $\mathrm{vcdim}_{\mathbb{Z}}(G)$ $<\infty$ *if and only if there is an upper bound on the ranks of abelian subgroups of G.*

For a finite generating set $\{g_1,\ldots,g_r\}\subset\Lambda$, take $S\subset\mathbb{R}$ to be the subring of $\mathbb{R}$ generated (over $\mathbb{Z}$) by the finite collection of matrix entries of $\{\bar{\rho}(g_1),\ldots,\bar{\rho}(g_r)\}$. Then S is a (finitely generated) characteristic zero integral domain, since it is a subring of $\mathbb{R}$, and $\bar{\rho}(\Lambda)\subset\mathrm{GL}(S)\subset\mathrm{GL}_n(\mathbb{R})$. We now use the embedding $\bar{\rho}$ to identify Λ with its isomorphic copy in $\mathrm{GL}(S)$. Since Λ is a finitely generated linear group, it has a finite index torsion-free subgroup Λ'; we replace Λ,Γ by the finite index subgroups $\Lambda',\Gamma':=q^{-1}(\Lambda')$. This replaces M by a finite cover M', so we can now also assume that the quotient group Λ is torsion-free.

Taking a finitely generated abelian subgroup $H<\Lambda$ (necessarily torsion-free), we have a corresponding exact sequence:

$$\begin{tikzcd} A \arrow[r, hook, "i"] & \Gamma \arrow[r, "q"] & \Lambda \\ A \arrow[u, equal] \arrow[r, hook, "i"] & q^{-1}(H) \arrow[u, hook] \arrow[r, two heads, "q"] & H. \arrow[u, hook] \end{tikzcd} \tag{3.2}$$

Since A, H are finitely generated torsion-free abelian, we see that $\hat{H} := q^{-1}(H)$ is a finitely generated nilpotent group. Also, $\widetilde{M}/\hat{H}$ is an n-dimensional $K(\hat{H}, 1)$. Hence $h(\hat{H}) = \mathrm{cdim}_{\mathbb{Z}}(\hat{H}) \leq n$, where $h(\hat{H})$ is the Hirsch length of $\hat{H}$ (see Gruenberg [G70, Section 8.8]). But from the two step nilpotence sequence above, the Hirsch length of $\hat{H}$ is just $k + r$ where r is the rank of H. Hence $k + r \leq n$, and $n - k$ is the desired upper bound on the rank of the abelian subgroups of Λ. Applying Alperin and Shalen's result, we conclude that Λ has finite cohomological dimension $\mathrm{cdim}_{\mathbb{Z}}(\Lambda) < \infty$. Since any finite length free $\mathbb{Z}[\Lambda]$ resolution of $\mathbb{Z}$ can be tensored with $\mathbb{R}$ to obtain a same length free $\mathbb{R}[\Lambda]$ resolution of $\mathbb{R}$, this yields $\mathrm{cdim}_{\mathbb{R}}(\Lambda) \leq \mathrm{cdim}_{\mathbb{Z}}(\Lambda)$, which establishes

Fact 2. $\mathrm{cdim}_{\mathbb{R}}(\Lambda) = \ell$ for some finite ℓ.

Finally, we will use the following result of Fel'dman [F71, Theorem 2.4]:

Theorem (Fel'dman)**.** *For G a group, and $H \triangleleft G$ a normal subgroup, F a field. If H is of type FP (over the field F), and $\mathrm{cdim}_F(G/H) < \infty$, then*

$$\mathrm{cdim}_F(G) = \mathrm{cdim}_F(H) + \mathrm{cdim}_F(G/H).$$

From Fact 1 and Fact 2 we see that the hypotheses of Fel'dman's theorem hold for $A \triangleleft \Gamma$ (with $F = \mathbb{R}$). Since $\mathrm{cdim}_{\mathbb{R}}(A) > 0$, Fel'dman's theorem gives us the inequality $\mathrm{cdim}_{\mathbb{R}}(\Lambda) < \mathrm{cdim}_{\mathbb{R}}(\Gamma) = n$. Applying Lemma 2.1 concludes the proof of the **Main Theorem**.

Remark 3.1. As the reader can easily see, this same proof applies to the following more general setting. Let M be a closed connected n-manifold, and assume that $\Gamma = \pi_1(M)$ has $\mathrm{cdim}_{\mathbb{R}}(\Gamma) = n$. If $A \triangleleft \Gamma$ is a normal elementary amenable subgroup of type FP, and $\Lambda = \Gamma/A$ is a linear group, then $||M|| = 0$. For the portions of the proof relying on Hirsch length, one can use Hillman's extension of the Hirsch length to elementary amenable groups; see [H91, Theorem 1]. Note also that elementary amenable groups of type FP are automatically virtually solvable; see [KMPN09].

4. Concluding remarks.

As mentioned in the introduction, the case of closed *complete* affine manifolds provides a large class of manifolds satisfying the hypotheses of our **Main Theorem**. For these manifolds, it is tempting to try and give a more direct, geometrical proof that $||M|| = 0$. Indeed, one can consider the foliation of $\mathbb{R}^n$ given by affine subspaces in the directions spanned by the (non-trivial) translational subgroup. Since the developing map is a homeomorphism $D : \widetilde{M} \to \mathbb{R}^n$, the translational subgroup acts discretely (hence cocompactly) on the leaves of this foliation. Normality of the translational subgroup implies that this foliation of $\widetilde{M} \cong \mathbb{R}^n$ descends to a foliation of M by closed submanifolds, where each leaf is finitely covered by a torus. If this foliation were a fibration, then it would follow that $||M|| = 0$ [L02, Exercise 14.15]. More generally, $||M|| = 0$ if M admits a polarized $\mathcal{F}$-structure (see [CG86]). This geometric approach then motivates the following interesting

Question. If M is a closed aspherical manifold, with a foliation all of whose leaves are finitely covered by tori, does it follow that $||M|| = 0$?

In the non-complete case, one can still foliate the image of the development map D by affine subspaces in the directions spanned by the translational subgroup, and then pull back this foliation via D to a foliation on $\widetilde{M}$. The foliation on $\widetilde{M}$ will still descend to a foliation on M, but it is unclear whether the leaves of the resulting foliation on M are even *closed*. Indeed, if one takes a leaf of the foliation in $\mathbb{R}^n$, its pre-image in $\widetilde{M}$ could consist of countably infinitely many leaves for the induced foliation of $\widetilde{M}$. The pre-image of the translational subgroup could then act by permuting these individual leaves in $\widetilde{M}$, none of which would close up in M.

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Section de Mathematiques, Université de Genève, 2-4 rue du Livre, Case postale 64, 1211 Geneva, 4, Switzerland

E-mail address: Michelle.Bucher-Karlsson@unige.ch

Department of Mathematics, Indiana University, 115 Rawles Hall, Bloomington, Indiana 47405

E-mail address: connell@indiana.edu

Department of Mathematics, Ohio State University, 231 W. 18th Avenue, Columbus, Ohio 43210

E-mail address: jlafont@math.ohio-state.edu