PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 3, March 2018, Pages 1207–1216
http://dx.doi.org/10.1090/proc/13815
Article electronically published on October 6, 2017

BOUNDS ON THE GREEN FUNCTION FOR INTEGRAL OPERATORS AND FRACTIONAL HARMONIC MEASURE WITH APPLICATIONS TO BOUNDARY HARNACK

LUIS A. CAFFARELLI AND YANNICK SIRE

(Communicated by Joachim Krieger)

Abstract. We prove a priori bounds on the Green function for general integral operators in divergence form in the spirit of Littman, Stampacchia and Weinberger's result. For general linear integral operators with bounded measurable coefficients, we introduce the so-called fractional harmonic measure and prove several estimates on it. As an application, we prove a new boundary Harnack principle for these operators. Once the bounds on the Green function are known, the proof follows the approach of Caffarelli-Fabes-Mortola-Salsa and K. Bogdan.

1. Introduction

The aim of the present note is three-fold: first we derive bounds on the Green function for integral operators in the spirit of [LSW63] (see also [GW82]). The operators in consideration can be linear or nonlinear, modeled in the latter case on the fractional $p-$Laplacian. Second, for general integral operators in divergence form with bounded measurable coefficients we introduce the fractional harmonic measure and derive, thanks to the previous bounds, some estimates. Finally, we state, as an application, a boundary Harnack principle in Non-Tangentially Accessible (NTA) domains. The purpose of the note is to collect some known results for the fractional Laplacian and generalize them, thanks to our bounds on the Green function, to *any* type of integral operator in divergence form. The proof follows closely known ideas, but our contribution is to put such a theory in a unified way.

The work of Littman, Stampacchia and Weinberger [LSW63] establishes that any second-order differential operator with bounded measurable coefficients of the form

$$\mathcal{L} := \sum_{i,j=1}^n \partial_i(a_{ij}(x)\partial_j) + \sum_{i=1}^n b_i\partial_i + c \ \text{ in } \Omega \subset \mathbb{R}^n, \ n \geq 3,$$

admits a Green function in Ω with pole at $y \in \Omega$ denoted $G_\Omega(.,y)$ comparable with the one of the Laplacian; i.e. there exist constants $c_1, c_2 > 0$ such that

$$\frac{c_1}{|x-y|^{n-2}} \leq G_\Omega(x,y) \leq \frac{c_2}{|x-y|^{n-2}}.$$

Received by the editors April 26, 2017 and, in revised form, May 4, 2017.
2010 *Mathematics Subject Classification.* Primary 35B45.
The first author was supported by NSF DMS-1540162.

The main goal of this note is to obtain such a result for general nonlocal operators and give some applications to fractional harmonic measure and boundary Harnack principle in nonsmooth domains.

Results on the boundary Harnack principle for nonlocal equations. The first boundary Harnack principle was obtained by Bogdan for the fractional Laplacian $(-\Delta)^\alpha$ with $\alpha \in (0,1)$ in [Bog97] for Lipschitz domains. The approach he followed is in the seminal paper by Caffarelli, Fabes, Mortola and Salsa [CFMS81] dealing with second-order equations in divergence form with bounded measurable coefficients. Later, his result was generalized to any open set by Song and Wu in [SW99]. We refer the reader also to [BKK08]. More generally, the paper [BKK15] deals with a wide class of Markov processes with jumps. Their theory particularly applies to operators of the type (see Example 5.6 in [BKK15])

$$\mathcal{L}u(x) = \int_{\mathbb{R}^n} \Big(\frac{u(x+y)+u(x-y)-2u(x)}{2}\Big) K(y)\,dy$$

for kernels $K(y) = K(-y)$ and

$$\frac{\lambda}{|y|^{n+2\alpha}} \le K(y) \le \frac{\Lambda}{|y|^{n+2\alpha}}, \quad y \in \mathbb{R}^n.$$

Finally, recently, Ros-Oton and Serra [ROS17] derived a boundary Harnack principle in open domains for nonlocal operators in nondivergence form (i.e. of the previous form) with bounded measurable kernels $K(x,y)$ comparable to the one of the fractional Laplacian.

Setting of the problem and main results. We will mainly consider in this paper operators of the form

$$\mathcal{L}u(x) = P.V. \int_{\mathbb{R}^n} (u(x)-u(y))K(x,y)\,dy \tag{1.1}$$

where $P.V.$ stands for the principal value in the Cauchy sense, $n \ge 2$ and the kernel K satisfies the bounds

$$\frac{1}{\Lambda|x-y|^{n+2\alpha}} \le K(x,y) \le \frac{\Lambda}{|x-y|^{n+2\alpha}}, \qquad 0<\alpha<1\,. \tag{1.2}$$

Written this way, the operator $\mathcal{L}$ is of divergence form with bounded measurable coefficients, with an associated Dirichlet form given by

$$\mathcal{E}_K(u,\eta) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (u(x)-u(y))[\eta(x)-\eta(y)]K(x,y)\,dx\,dy\,.$$

The previous assumptions (1.2) make the form coercive in $W^{\alpha,2}(\mathbb{R}^n)$ and the related nonlocal equations elliptic. More generally, we will also consider operators defined by the energy

$$\mathcal{E}_K^\Phi(u,\eta) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(u(x)-u(y))(\eta(x)-\eta(y))K(x,y)\,dx\,dy$$

with associated operator $\mathcal{L}_\Phi$ and where the function $\Phi : \mathbb{R} \mapsto \mathbb{R}$ is assumed to be continuous, satisfying $\Phi(0) = 0$ together with the monotonicity property

$$\Lambda^{-1}|t|^p \le \Phi(t)t \le \Lambda|t|^p\,, \qquad \forall\, t \in \mathbb{R}\,. \tag{1.3}$$

Finally, the kernel $K\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is assumed to be measurable, satisfying the following ellipticity/coercivity properties:

$$\frac{1}{\Lambda|x-y|^{n+\alpha p}} \leq K(x,y) \leq \frac{\Lambda}{|x-y|^{n+\alpha p}} \qquad \forall\, x,y \in \mathbb{R}^n,\ x \neq y, \tag{1.4}$$

where $\Lambda \geq 1$ and

$$\alpha \in (0,1)\,, \qquad p \geq 2. \tag{1.5}$$

The canonical example for Φ is $\Phi(t) = |t|^p$, yielding the fractional $p-$Laplacian.

We focus on the following measure data problem: Let Ω be a bounded domain in $\mathbb{R}^n$ and let δ_y be the Dirac mass at $y \in \mathbb{R}^n$. Consider

$$\begin{cases} -\mathcal{L}_\Phi G_\Omega = \delta_y \ \text{ in } \Omega, \\ \ G_\Omega = 0 \ \text{ in } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{1.6}$$

Then the function $G_\Omega(.,y)$ is the Green function of $\mathcal{L}_\Phi$ at point y in Ω. We prove the following theorem.

Theorem 1.1. *Let $\mathcal{L}_\Phi$ be defined as before and consider problem* (1.6)*. Then there exist two constants $c_1, c_2 > 0$ such that*

$$\frac{c_1}{|x-y|^{n-\alpha p}} \leq G_\Omega(x,y) \leq \frac{c_2}{|x-y|^{n-\alpha p}}.$$

Remark 1.1. We refer the reader to the proof for the precise definition in which sense one has to consider problem (1.6).

In Theorem 1.1, taking $\Phi(t) = |t|^2$ and $p = 2$, one gets for the operator $\mathcal{L}$ defined in (1.1) the following bounds for its Green function:

$$\frac{c_1}{|x-y|^{n-2\alpha}} \leq G_\Omega(x,y) \leq \frac{c_2}{|x-y|^{n-2\alpha}}.$$

The previous theorem has the following applications.

Theorem 1.2. *Let $\alpha \in (0,1)$ and Ω be an NTA domain such that $0 \in \partial\Omega$. Let $u, v \geq 0$ in $\mathbb{R}^n$, continuous in B_1 satisfying*

$$\mathcal{L}u = \mathcal{L}v, \ \ \textit{in } \Omega \cap B_1,$$

$$u = v = 0, \ \ \textit{in } B_1 \backslash \Omega,$$

and $u(x_0) = v(x_0) = 1$ at some interior point $x_0 \in \Omega$. Then there exists $C > 0$ such that

$$\frac{1}{C} u \leq v \leq Cu, \ \ \textit{in } B_{1/2}.$$

Here the constant C depends only on n, α, Ω and the ellipticity constants.

We also have

Theorem 1.3. *Let $\alpha \in (0,1)$ and Ω be an NTA domain such that $0 \in \partial\Omega$. Let $u, v \geq 0$ in $\mathbb{R}^n$, continuous in B_1 satisfying*

$$\mathcal{L}u = \mathcal{L}v, \ \ \textit{in } \Omega \cap B_1,$$

$$u = v = 0, \ \ \textit{in } B_1 \backslash \Omega.$$

Then there exists $C > 0$ such that

$$\left\| \frac{u}{v} \right\|_{C^\beta(\bar{\Omega} \cap B_{1/2})} \leq C.$$

Here the constants C, β depend only on n, α, Ω and the ellipticity constants.

We refer the reader to [JK82] for an extensive theory of nontangentially accessible (NTA) domains. Basically, an NTA domain is a domain such that the interior admits a corkscrew chain and the complement has positive density. As in [CFMS81], we will do the proof when Ω is a Lipschitz domain checking that what we need is just the previous characterization of NTA domains.

2. Bounds on the Green function: Proof of Theorem 1.1

As already mentioned, the Littman-Stampachia-Weinberger theory ensures bounds on the Green function in bounded domains for general bounded measurable second-order operators (see [LSW63] and also [GW82]). The goal of this section is to derive such bounds for the Green functions of our operators and prove our main theorem, Theorem 1.1. This relies on zero-order potential estimates obtained in [KMS15].

2.1. Existence of a Green function.

Definition 1. Let $\mu \in (W^{\alpha,2}(\Omega))'$ and $g \in W^{\alpha,2}(\mathbb{R}^n)$. A weak (energy) solution to the problem

$$\begin{cases} -\mathcal{L}_\Phi u = \mu & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{2.1}$$

is a function $u \in W^{\alpha,2}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(u(x)-u(y))(\psi(x)-\psi(y))K(x,y)\,dx\,dy = \langle \mu, \psi \rangle$$

holds for any $\psi \in C_0^\infty(\Omega)$ and such that $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.

First let us introduce some notation:

$$\mathrm{Tail}(v; x_0, r) := \left[r^{2\alpha} \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{|v(x)|}{|x-x_0|^{n+\alpha p}}\,dx \right] \tag{2.2}$$

and the (truncated) Wolff potential

$$\mathbf{W}^{\mu}_{\beta,p}(x_0,r) := \int_0^r \left(\frac{|\mu|(B_\varrho(x_0))}{\varrho^{n-\beta p}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho}, \qquad \beta > 0.$$

We are now ready for the following:

Definition 2 (Solutions obtained by Limiting Approximations for the Dirichlet problem)**.** Let $\mu \in \mathcal{M}(\mathbb{R}^n)$, $g \in W^{s,p}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^{p-1}_{sp}(\mathbb{R}^n)$ and let $-\mathcal{L}_\Phi$ be defined as above under assumptions (1.3)-(1.5). We say that a function $u \in W^{h,q}(\Omega)$ for

$$h \in (0,s), \qquad \max\{1, p-1\} =: q_* \le q < \bar{q} := \min\left\{ \frac{n(p-1)}{n-s}, p \right\} \tag{2.3}$$

is a SOLA to (2.1) if it is a distributional solution to $-\mathcal{L}_\Phi u = \mu$ in Ω; that is,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(u(x)-u(y))(\varphi(x)-\varphi(y))K(x,y)\,dx\,dy = \int_{\mathbb{R}^n} \varphi\,d\mu \tag{2.4}$$

holds whenever $\varphi \in C_0^\infty(\Omega)$, if $u = g$ a.e. in $\mathbb{R}^n \setminus \Omega$. Moreover it has to satisfy the following approximation property: There exists a sequence of functions $\{u_j\} \subset W^{s,p}(\mathbb{R}^n)$ weakly solving the approximate Dirichlet problems

$$\begin{cases} -\mathcal{L}_\Phi u_j = \mu_j & \text{in } \Omega, \\ u_j = g_j & \text{on } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{2.5}$$

in the sense of Definition 1, such that u_j converges to u a.e. in $\mathbb{R}^n$ and locally in $L^q(\mathbb{R}^n)$. Here the sequence $\{\mu_j\} \subset C_0^\infty(\mathbb{R}^n)$ converges to μ weakly in the sense of measures in Ω and moreover satisfies

$$\limsup_{j\to\infty} |\mu_j|(B) \leq |\mu|(\overline{B}) \tag{2.6}$$

whenever B is a ball. The sequence $\{g_j\} \subset C_0^\infty(\mathbb{R}^n)$ converges to g in the following sense: For all balls $B_r \equiv B_r(z)$ with center in z and radius $r > 0$, it holds that

$$g_j \to g \quad \text{in} \quad W^{s,p}(B_r) \qquad \text{and} \qquad \lim_j \operatorname{Tail}(g_j - g; z, r) = 0\,. \tag{2.7}$$

2.2. **Potential estimates.**

Theorem 2.1 (Kuusi, Mingione, Sire [KMS15])**.** *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ be a nonnegative measure, and let $-\mathcal{L}_\Phi$ be defined as before. Let u be a SOLA as in the previous definition which is nonnegative in the ball $B_r(x_0) \subset \Omega$ and such that the approximating sequence $\{\mu_j\}$ for μ is made of nonnegative functions. Then the estimate*

$$\mathbf{W}^{\mu}_{\alpha,p}(x_0, r/8) \leq cu(x_0) + c\operatorname{Tail}(u_-; x_0, r/2) \leq c\mathbf{W}^{\mu}_{\alpha,p}(x_0, r) + c \fint_{B_r(x_0)} |u|\, dx \tag{2.8}$$
$$+ c\operatorname{Tail}(u; x_0, r)$$

holds for a constant $c \equiv c(n, s, p, \Lambda)$, as soon as $\mathbf{W}^{\mu}_{\alpha,p}(x_0, r/8)$ is finite, where $u_- := \max\{-u, 0\}$.

2.3. **Proof of Theorem 1.1.** Assume $y = 0$. From this, we conclude directly. Indeed, since $G \geq 0$ everywhere by the maximum principle we have

$$\operatorname{Tail}(G_-; x_0, r/2) \equiv 0$$

for any $r > 0$.

Since $G = 0$ outside Ω, this gives

$$\mathbf{W}^{\delta_0}_{\alpha,p}(x_0, r/8) \leq cG(x_0) \leq c\mathbf{W}^{\delta_0}_{\alpha,p}(x_0, r) + c \fint_{B_r(x_0)} |G(x)|\, dx \tag{2.9}$$
$$+ r^{p\alpha} \int_{\Omega\setminus B_r(x_0)} \frac{|G(x)|}{|x - x_0|^{n+p\alpha}}\, dx.$$

Now by the properties of the Wolff potentials, one gets

$$\mathbf{W}^{\delta_0}_{\alpha,p}(x_0, r/8) \sim \frac{1}{|x_0|^{n-p\alpha}}.$$

Trivially, one has

$$\int_{\Omega\setminus B_r(x_0)} \frac{|G(x)|}{|x - x_0|^{n+p\alpha}}\, dx \leq \frac{1}{r^{n+p\alpha}} \int_{\Omega\setminus B_r(x_0)} |G|.$$

Hence

$$\frac{C_1}{|x_0|^{n-p\alpha}} \leq G(x_0) \leq \frac{C_2}{|x_0|^{n-p\alpha}} + \frac{C}{r^n} \int_\Omega |G(x)|\, dx. \tag{2.10}$$

Now using Lemma 3.6 in [KMS15] (for $v = 0$ and then $w = u$), we deduce that $G \in L^1(\Omega)$ with a bound depending only on Ω. From the previous estimate we deduce the result.

3. The fractional harmonic measure

We now introduce the fractional harmonic measure. Consider

$$\begin{cases} -\mathcal{L}u(x) = 0 \text{ in } \Omega, \\ u = f \text{ in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{3.1}$$

where $f \in C(\mathbb{R}^n \backslash \Omega) \cap \mathcal{U}$ with

$$\mathcal{U} = \left\{ u : \mathbb{R}^n \to \mathbb{R} \; : \; \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2\alpha}} < \infty \right\}.$$

Remark 3.1. The last integrability condition ensures the convergence of the integral in $\mathcal{L}$ (see [Sil07]).

Therefore, the map

$$f \to u(x)$$

defines a positive linear functional on $C(\mathbb{R}^n \backslash \Omega)$. Consequently there exists a Borel measure $W^x_{\mathcal{L}}(., \Omega)$ supported on $\mathbb{R}^n \backslash \Omega$ such that

$$u(x) = \int_{\mathbb{R}^n \backslash \Omega} f(y) dW^x_{\mathcal{L}}(y, \Omega).$$

Definition 3. We call the measure $W^x_{\mathcal{L}}(., \Omega)$ the fractional harmonic measure at point $x \in \Omega$ associated to the operator $\mathcal{L}$. For any $A \subset \Omega^c$, we denote it $W^x_{\mathcal{L}}(A, \Omega)$.

Following [CFMS81], the harmonic measure plays a major role in obtaining boundary Harnack principles in Lipschitz (or NTA) domains. In this section, we derive several properties for fractional harmonic measure.

The fractional harmonic measure for the fractional Laplacian, called $\alpha-$harmonic measure, was investigated for the first time in the work of M. Riesz, who proved that in the case of the ball $B(0, r)$, one has for $\alpha \in (0, 2)$,

$$dW^x_{(-\Delta)^{\alpha/2}}(y, B(0, r)) = k_B(x, y)\, dx$$

where

$$k_B(x, y) = \begin{cases} c(n, \alpha) \left(\frac{r^2 - |x|^2}{|y|^2 - r^2} \right)^{\alpha/2} |x - y|^{-n}, & |y| > r, \\ 0, & |y| \leq r. \end{cases}$$

In particular this proves that the $\alpha-$harmonic measure on the ball is mutually absolutely continuous with respect to Lebesgue measure.

3.1. The fractional harmonic measure (or $\alpha-$harmonic measure) for the fractional Laplacian.

In this section, we collect a few known results on the $\alpha-$harmonic measure. Most can be found in the work of Bogdan [Bog97]. The results we present here will be valid for our general fractional harmonic measure under appropriate changes.

Bogdan proved in [Bog97, Lemma 6]:

Lemma 3.1. *Let $V \subset \mathbb{R}^n$ be a bounded open set with the outer cone property. Let A_1 and A_2 be Borel sets in V^c. Assume that there exists a constant C such that for every ball $B = B(x, r) \subset V$ satisfying dist$(B, V^c) =$ diam(B) we have*

$$W^x_{(-\Delta)^{\alpha/2}}(A_1, B) \leq C W^x_{(-\Delta)^{\alpha/2}}(A_2, B).$$

Then

$$W^x_{(-\Delta)^{\alpha/2}}(A_1, V) \leq C W^x_{(-\Delta)^{\alpha/2}}(A_2, V).$$

Furthermore the $\alpha-$harmonic measure is concentrated on intV^c and is absolutely continuous with respect to Lebesgue measure in V^c.

Bogdan also proved the following bound (see [Bog97, Corollary 1]):

Lemma 3.2. *There exist $C, R_0 > 0$ such that for all $Q \in \partial D$ and $r \in (0, R_0/2)$, and all $x \in D \backslash B(Q, 2r)$,*

$$C^{-1} r^{n-\alpha} G_D(a, A_{\rho r/2}(Q)) \leq W^x_{(-\Delta)^{\alpha/2}}(B(Q,r), D) \leq C r^{n-\alpha} G_D(a, A_{\rho r/2}(Q)),$$

where $A_r(Q)$ denotes a point $A \in D \cap B(Q,r)$ such that $B(A, \kappa r) \subset D \cap B(Q,r)$ for some $\kappa > 0$ depending only on the Lipschitz constant of the domain D.

The following theorem is due to Wu [Wu02], which will be valid in our case with suitable changes.

Theorem 3.1. *Let $0 < \alpha < 2$ and D be an NTA set such that ∂D has zero volume. Then for any $x \in D$, $W^x_{(-\Delta)^{\alpha/2}}(\partial D, D) = 0$.*

From the last lemma, invoking the Harnack inequality (see [BK05, DCKP14]), one deduces that the $\alpha-$harmonic measure is doubling.

3.2. **The fractional harmonic measure for general integral operators $\mathcal{L}$.** We now investigate the fractional harmonic measure as defined before. The following lemma is just a direct consequence of an integration by parts formula (see [DPROV16]).

Lemma 3.3. *Let $D \subset \Omega^c$ be open. Then the fractional harmonic measure is given by (up to a multiplicative constant)*

$$W^x(D, \Omega) = \int_D \Big(\int_\Omega K(z,y) G_\Omega(x,y)\, dy \Big)\, dz$$

and

$$dW^x(z, \Omega) = \int_\Omega K(z,y) G_\Omega(x,y)\, dy.$$

In particular, the fractional harmonic measure is mutually absolutely continuous with respect to Lebesgue measure in Ω^c.

Notice that an important fact from the previous lemma is that even if one considers operators with bounded measurable coefficients the fractional harmonic measure is never singular. This is in great contrast with the local case (see [CFK81, MM81]).

Now using the proof of Bogdan in [Bog97], by replacing the estimates on the Green function of the fractional Laplacian by our estimates in Theorem 1.1, one gets

Lemma 3.4. *There exists a constant $C > 0$ universal such that for $r_0 \leq 1/2$ and for all $x \in (\Omega \cap B_1) \backslash B_{1/2}$,*

$$\frac{1}{C} r^{n-2\alpha} G_\Omega(x, x_0) \leq W^x_{\mathcal{L}}(B_r(x_0), \Omega) \leq C r^{n-2\alpha} G_\Omega(x, x_0)$$

where $x_0 \in B_{1/2} \cap \partial\Omega$. Hence by Harnack inequality, the fractional harmonic measure is doubling.

Similarly, using our bounds on the Green function, one gets verbatim the proof in [Wu02].

4. Proof of Theorem 1.2

We first normalize the situation. As already mentioned in the introduction, it is enough to consider Lipschitz domains as soon as we only need the requirements of NTA properties. We then assume our domain in

$$B_1^+ = B_1 \cap \{x_n > 0\}$$

by a standard bilipshitz transformation which preserves the bounds of the kernel of the new operator $\mathcal{L}$. We then consider the problem

$$\begin{cases} \mathcal{L}u(x) = 0 \text{ in } B_1^+, \\ u = 0 \text{ on } \{x_n = 0\} \cap B_1, \\ u = f \text{ on the complement.} \end{cases} \tag{4.1}$$

One can then reflect in an odd fashion and one obtains an equation in B_1 with a new kernel satisfying the desired bounds.

4.1. **The Carleson estimate.**

We first collect the following several results.

Theorem 4.1 ([Kas09], De Giorgi-Nash-Moser Theorem)**.** *Any solution of* $\mathcal{L}v = 0$ *in* B_1 *is Hölder continuous on any sub-domain of* $\omega \Subset B_1$.

Theorem 4.2 ([BK05, DCKP14], Moser's Harnack inequality)**.** *Any solution of* $\mathcal{L}v = 0$ *in* B_1, $v \geq 0$, *in* $\mathbb{R}^n$ *satisfies the interior Harnack inequality: for all* $\omega \Subset B_1$, *we have*

$$\sup_\omega v \leq C \inf_\omega v.$$

Theorem 4.3 ([CCV11], De Giorgi oscillation lemma)**.** *Let* v *be a (sub-)solution of* $\mathcal{L}v = 0$ *in* B_1 *satisfying*

- $v \leq 1$ *in* $\mathbb{R}^n$,
- $|\{v \leq 0\}| > 0$.

Then

$$\sup_{B_1} v \leq \mu < 1.$$

Combining Theorems 4.1, 4.2 and 4.3 as in [CS05, Theorem 11.5] (see also [CFMS81]), one proves the following Carleson estimate (note that once the previous theorems are known, the Carleson estimate does not require any equation).

Theorem 4.4. *Let* u *be a positive solution of* (4.1) *continuously vanishing on* $\{x_n = 0\}$. *Assume that*

$$u(\frac{1}{2}e_n) = 1.$$

Then in B_1^+,

$$u \leq M$$

where M *depends on the dimension,* α, *and the ellipticity constants.*

5. Proof of Theorem 1.2

The proof follows using the Carleson estimate and the doubling property of the fractional harmonic measure as in Wu [Wu02], which uses only the positive density of the complement. See also Lemma 13 in [Bog97].

6. Proof of Theorem 1.3

The proof is in Lemma 16 of [Bog97].

Acknowledgment

The second author would like to thank the hospitality of the Department of Mathematics at UT Austin, where part of this work was done.

References

[BK05] Richard F. Bass and Moritz Kassmann, *Harnack inequalities for non-local operators of variable order*, Trans. Amer. Math. Soc. **357** (2005), no. 2, 837–850, DOI 10.1090/S0002-9947-04-03549-4. MR2095633

[BKK08] Krzysztof Bogdan, Tadeusz Kulczycki, and Mateusz Kwaśnicki, *Estimates and structure of α-harmonic functions*, Probab. Theory Related Fields **140** (2008), no. 3-4, 345–381, DOI 10.1007/s00440-007-0067-0. MR2365478

[BKK15] Krzysztof Bogdan, Takashi Kumagai, and Mateusz Kwaśnicki, *Boundary Harnack inequality for Markov processes with jumps*, Trans. Amer. Math. Soc. **367** (2015), no. 1, 477–517, DOI 10.1090/S0002-9947-2014-06127-8. MR3271268

[Bog97] Krzysztof Bogdan, *The boundary Harnack principle for the fractional Laplacian*, Studia Math. **123** (1997), no. 1, 43–80. MR1438304

[CCV11] Luis Caffarelli, Chi Hin Chan, and Alexis Vasseur, *Regularity theory for parabolic nonlinear integral operators*, J. Amer. Math. Soc. **24** (2011), no. 3, 849–869, DOI 10.1090/S0894-0347-2011-00698-X. MR2784330

[CFK81] Luis A. Caffarelli, Eugene B. Fabes, and Carlos E. Kenig, *Completely singular elliptic-harmonic measures*, Indiana Univ. Math. J. **30** (1981), no. 6, 917–924, DOI 10.1512/iumj.1981.30.30067. MR632860

[CFMS81] L. Caffarelli, E. Fabes, S. Mortola, and S. Salsa, *Boundary behavior of nonnegative solutions of elliptic operators in divergence form*, Indiana Univ. Math. J. **30** (1981), no. 4, 621–640, DOI 10.1512/iumj.1981.30.30049. MR620271

[CS05] Luis Caffarelli and Sandro Salsa, *A geometric approach to free boundary problems*, Graduate Studies in Mathematics, vol. 68, American Mathematical Society, Providence, RI, 2005. MR2145284

[DCKP14] Agnese Di Castro, Tuomo Kuusi, and Giampiero Palatucci, *Nonlocal Harnack inequalities*, J. Funct. Anal. **267** (2014), no. 6, 1807–1836, DOI 10.1016/j.jfa.2014.05.023. MR3237774

[DPROV16] Serena Dipierro, Xavier Ros-Oton, and Enrico Valdinoci, *Nonlocal problems with Neumann boundary conditions*, Rev. Mat. Iberoam. **33** (2017), no. 2, 377–416, DOI 10.4171/RMI/942. MR3651008

[GW82] Michael Grüter and Kjell-Ove Widman, *The Green function for uniformly elliptic equations*, Manuscripta Math. **37** (1982), no. 3, 303–342, DOI 10.1007/BF01166225. MR657523

[JK82] David S. Jerison and Carlos E. Kenig, *Boundary behavior of harmonic functions in nontangentially accessible domains*, Adv. in Math. **46** (1982), no. 1, 80–147, DOI 10.1016/0001-8708(82)90055-X. MR676988

[Kas09] Moritz Kassmann, *A priori estimates for integro-differential operators with measurable kernels*, Calc. Var. Partial Differential Equations **34** (2009), no. 1, 1–21, DOI 10.1007/s00526-008-0173-6. MR2448308

[KMS15] Tuomo Kuusi, Giuseppe Mingione, and Yannick Sire, *Nonlocal equations with measure data*, Comm. Math. Phys. **337** (2015), no. 3, 1317–1368, DOI 10.1007/s00220-015-2356-2. MR3339179

[LSW63] W. Littman, G. Stampacchia, and H. F. Weinberger, *Regular points for elliptic equations with discontinuous coefficients*, Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 43–77. MR0161019

[MM81] Luciano Modica and Stefano Mortola, *Construction of a singular elliptic-harmonic measure*, Manuscripta Math. **33** (1980/81), no. 1, 81–98, DOI 10.1007/BF01298340. MR596380

[ROS17] X. Ros-Oton and J. Serra, *The boundary Harnack principle for nonlocal elliptic operators in non-divergence form*, preprint (2-17).

[Sil07] Luis Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math. **60** (2007), no. 1, 67–112, DOI 10.1002/cpa.20153. MR2270163

[SW99] Renming Song and Jang-Mei Wu, *Boundary Harnack principle for symmetric stable processes*, J. Funct. Anal. **168** (1999), no. 2, 403–427, DOI 10.1006/jfan.1999.3470. MR1719233

[Wu02] Jang-Mei Wu, *Harmonic measures for symmetric stable processes*, Studia Math. **149** (2002), no. 3, 281–293, DOI 10.4064/sm149-3-5. MR1893056

Department of Mathematics, University of Texas at Austin, 2515 Speedway Stop C1200, Austin, Texas 78712

E-mail address: caffarel@math.utexas.edu

Department of Mathematics, Johns Hopkins University, 3400 N. Charles Street, Baltimore, Maryland 21218

E-mail address: sire@math.jhu.edu