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ON LOGARITHMIC COEFFICIENTS OF SOME CLOSE-TO-CONVEX FUNCTIONS

MD FIROZ ALI AND A. VASUDEVARAO

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ABSTRACT. The logarithmic coefficients γ_n of an analytic and univalent function f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalization $f(0) = 0 = f'(0) - 1$ are defined by $\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n$. Recently, D. K. Thomas [Proc. Amer. Math. Soc. 144 (2016), 1681–1687] proved that $|\gamma_3| \le \frac{7}{12}$ for functions in a subclass of close-to-convex functions (with argument 0) and claimed that the estimate is sharp by providing a form of an extremal function. In the present paper, we point out that such extremal functions do not exist and the estimate is not sharp by providing a much more improved bound for the whole class of close-to-convex functions (with argument 0). We also determine a sharp upper bound of $|\gamma_3|$ for close-to-convex functions (with argument 0) with respect to the Koebe function.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f'(0) - 1$. If $f \in \mathcal{A}$, then $f(z)$ has the following representation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n. \tag{1.1}$$

We will simply write $a_n := a_n(f)$ when there is no confusion. Let $\mathcal{S}$ denote the class of all univalent (i.e., one-to-one) functions in $\mathcal{A}$. A function $f \in \mathcal{A}$ is called starlike (convex respectively) if $f(\mathbb{D})$ is starlike with respect to the origin (convex respectively). Let $\mathcal{S}^*$ and $\mathcal{C}$ denote the class of starlike and convex functions in $\mathcal{S}$ respectively. It is well known that a function $f \in \mathcal{A}$ is in $\mathcal{S}^*$ if and only if $\mathrm{Re}\,(zf'(z)/f(z)) > 0$ for $z \in \mathbb{D}$. Similarly, a function $f \in \mathcal{A}$ is in $\mathcal{C}$ if and only if $\mathrm{Re}\,(1 + (zf''(z)/f'(z))) > 0$ for $z \in \mathbb{D}$. From the above it is easy to see that $f \in \mathcal{C}$ if and only if $zf' \in \mathcal{S}^*$. Given $\alpha \in (-\pi/2, \pi/2)$ and $g \in \mathcal{S}^*$, a function $f \in \mathcal{A}$ is said to be close-to-convex with argument α and with respect to g if

$$\mathrm{Re}\left(e^{i\alpha}\frac{zf'(z)}{g(z)}\right) > 0 \quad z \in \mathbb{D}. \tag{1.2}$$

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Let $\mathcal{K}_\alpha(g)$ denote the class of all such functions. Let

$$\mathcal{K}(g) := \bigcup_{\alpha\in(-\pi/2,\pi/2)} \mathcal{K}_\alpha(g) \quad \text{and} \quad \mathcal{K}_\alpha := \bigcup_{g\in\mathcal{S}^*} \mathcal{K}_\alpha(g)$$

be the classes of functions called close-to-convex functions with respect to g and close-to-convex functions with argument α, respectively. The class

$$\mathcal{K} := \bigcup_{\alpha\in(-\pi/2,\pi/2)} \mathcal{K}_\alpha = \bigcup_{g\in\mathcal{S}^*} \mathcal{K}(g)$$

is the class of all close-to-convex functions. It is well known that every close-to-convex function is univalent in $\mathbb{D}$ (see [2]). Geometrically, $f \in \mathcal{K}$ means that the complement of the image-domain $f(\mathbb{D})$ is the union of non-intersecting half-lines.

The logarithmic coefficients of $f \in \mathcal{S}$ are defined by

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n \tag{1.3}$$

where γ_n are known as the logarithmic coefficients. The logarithmic coefficients γ_n play a central role in the theory of univalent functions. Very few exact upper bounds for γ_n seem to have been established. The significance of this problem in the context of the Bieberbach conjecture was pointed out by Milin in his conjecture. Milin conjectured that for $f \in \mathcal{S}$ and $n \geq 2$,

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\gamma_k|^2 - \frac{1}{k}\right) \leq 0,$$

which led de Branges, by proving this conjecture, to the proof of the Bieberbach conjecture [1]. More attention has been given to the results of an average sense (see [2, 3]) than the exact upper bounds for $|\gamma_n|$. For the Koebe function $k(z) = z/(1-z)^2$, the logarithmic coefficients are $\gamma_n = 1/n$. Since the Koebe function $k(z)$ plays the role of extremal function for most of the extremal problems in the class $\mathcal{S}$, it is expected that $|\gamma_n| \leq \frac{1}{n}$ holds for functions in $\mathcal{S}$. But this is not true in general, even in order of magnitude. Indeed, there exists a bounded function f in the class $\mathcal{S}$ with logarithmic coefficients $\gamma_n \neq O(n^{-0.83})$ (see [2, Theorem 8.4]).

By differentiating (1.3) and equating coefficients we obtain

$$\gamma_1 = \frac{1}{2}a_2, \tag{1.4}$$

$$\gamma_2 = \frac{1}{2}(a_3 - \frac{1}{2}a_2^2), \tag{1.5}$$

$$\gamma_3 = \frac{1}{2}(a_4 - a_2a_3 + \frac{1}{3}a_2^3). \tag{1.6}$$

If $f \in \mathcal{S}$, then $|\gamma_1| \leq 1$ follows at once from (1.4). Using the Fekete-Szegö inequality [2, Theorem 3.8] for functions in $\mathcal{S}$ in (1.5), we can obtain the sharp estimate

$$|\gamma_2| \leq \frac{1}{2}(1 + 2e^{-2}) = 0.635\ldots.$$

For $n \geq 3$, the problem seems much harder, and no significant upper bound for $|\gamma_n|$ when $f \in \mathcal{S}$ appear to be known.

If $f \in \mathcal{S}^*$, then it is not very difficult to prove that $|\gamma_n| \leq \frac{1}{n}$ for $n \geq 1$ and the equality holds for the Koebe function $k(z) = z/(1-z)^2$. The inequality $|\gamma_n| \leq \frac{1}{n}$ for $n \geq 2$ extends to the class $\mathcal{K}$ was claimed in a paper of Elhosh [4]. However, Girela

[6] pointed out some error in the proof of Elhosh [4] and, hence, the result is not substantiated. Indeed, Girela proved that for each $n \geq 2$, there exists a function $f \in \mathcal{K}$ such that $|\gamma_n| > \frac{1}{n}$. In the same paper, it has been shown that $|\gamma_n| \leq \frac{3}{2n}$ holds for $n \geq 1$ whenever f belongs to the set of extreme points of the closed convex hull of the class $\mathcal{K}$. Recently, Thomas [12] proved that $|\gamma_3| \leq \frac{7}{12}$ for functions in $\mathcal{K}_0$ (close-to-convex functions with argument 0) with the additional assumption that the second coefficient of the corresponding starlike function g is real. Thomas claimed that this estimate is sharp and has given a form of the extremal function. But after rigorous reading of the paper [12], we observed that such functions do not belong to the class $\mathcal{K}_0$ (more details will be given in Section 2).

By fixing a starlike function g in the class $\mathcal{S}^*$, the inequality (1.2) gives a specific subclass of close-to-convex functions. One such important subclass is the class of close-to-convex functions with respect to the Koebe function $k(z) = z/(1-z)^2$. In this case, the inequality (1.2) becomes

$$\text{Re}\,\left(e^{i\alpha}(1-z)^2 f'(z)\right) > 0, \quad z \in \mathbb{D} \tag{1.7}$$

and defines the subclass $\mathcal{K}_\alpha(k)$. Several authors have extensively studied the class of functions $f \in \mathcal{S}$ that satisfies the condition (1.7) (see [5, 7, 9, 11]). Geometrically (1.7) says that the function $h := e^{i\alpha} f$ has the boundary normalization

$$\lim_{t \to \infty} h^{-1}(h(z) + t) = 1$$

and $h(\mathbb{D})$ is a domain such that $\{w + t : t \geq 0\} \subseteq h(\mathbb{D})$ for every $w \in h(\mathbb{D})$. Clearly, the image domain $h(\mathbb{D})$ is convex in the positive direction of the real axis. Denote by $\mathcal{CR}^+ := \mathcal{K}_0(k)$ the class of close-to-convex functions with argument 0 and with respect to Koebe function $k(z)$. That is,

$$\mathcal{CR}^+ = \left\{ f \in \mathcal{A} : \text{Re}\,(1-z)^2 f'(z) > 0, \ z \in \mathbb{D} \right\}.$$

Then clearly functions in $\mathcal{CR}^+$ are convex in the positive direction of the real axis. In the present article, we determine the upper bound of $|\gamma_3|$ for functions in $\mathcal{K}_0$ and $\mathcal{CR}^+$.

2. Main results

Let $\mathcal{P}$ denote the class of analytic functions P with positive real part on $\mathbb{D}$ which has the form

$$P(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{2.1}$$

Functions in $\mathcal{P}$ are sometimes called Carathéodory functions. To prove our main results, we need some preliminary lemmas. The first one is known as Carathéodory's lemma (see [2, p. 41] for example) and the second one is due to Libera and Złotkiewicz [10].

Lemma 2.1 ([2, p. 41])**.** *For a function $P \in \mathcal{P}$ of the form (2.1), the sharp inequality $|c_n| \leq 2$ holds for each $n \geq 1$. Equality holds for the function $P(z) = (1+z)/(1-z)$.*

Lemma 2.2 ([10])**.** *Let $P \in \mathcal{P}$ be of the form (2.1). Then there exist $x, t \in \mathbb{C}$ with $|x| \leq 1$ and $|t| \leq 1$ such that*

$$2c_2 = c_1^2 + x(4 - c_1^2)$$

and

$$4c_3 = c_1^3 + 2(4-c_1^2)c_1x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)t.$$

In [12], Thomas claimed that his result (i.e. $|\gamma_3| \le 7/12$) is sharp for functions in the class $\mathcal{K}_0$ by ascertaining the equality holds for a function f defined by $zf'(z) = g(z)P(z)$ where $g \in \mathcal{S}^*$ with $b_2(g) = 2, b_3(g) = 3$ and $P \in \mathcal{P}$ with $c_1(P) = 0$, $c_2(P) = c_3(P) = 2$. But in view of Lemma 2.2, it is easy to see that there does not exist a function $P \in \mathcal{P}$ with the property $c_1(P) = 0$, $c_2(P) = c_3(P) = 2$. Thus we can conclude that the result obtained by Thomas is not sharp. The main aim of the present paper is to obtain a better upper bound for $|\gamma_3|$ for functions in the class $\mathcal{K}_0$ than that obtained by Thomas [12]. To prove our main results we also need the following Fekete-Szegö inequality for functions in the class $\mathcal{S}^*$.

Lemma 2.3 ([8, Lemma 3]). *Let $g \in \mathcal{S}^*$ be of the form $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Then for any $\lambda \in \mathbb{C}$,*

$$|b_3 - \lambda b_2^2| \le \max\{1, |3-4\lambda|\}.$$

The inequality is sharp for $k(z) = z/(1-z)^2$ if $|3-4\lambda| \ge 1$ and for $(k(z^2))^{1/2}$ if $|3-4\lambda| < 1$.

For $f \in \mathcal{K}_0$ (close-to-convex functions with argument 0), we obtained the following improved result for $|\gamma_3|$ (compare [12]).

Theorem 2.1. *If $f \in \mathcal{K}_0$, then $|\gamma_3| \le \frac{1}{18}(3+4\sqrt{2}) = 0.4809$.*

Proof. Let $f \in \mathcal{K}_0$ be of the form (1.1). Then there exists a starlike function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ and a Carathéodory function $P \in \mathcal{P}$ of the form (2.1) such that

$$zf'(z) = g(z)P(z). \tag{2.2}$$

A comparison of the coefficients on the both sides of (2.2) yields

$$\begin{aligned}
a_2 &= \frac{1}{2}(b_2 + c_1),\\
a_3 &= \frac{1}{3}(b_3 + b_2c_1 + c_2),\\
a_4 &= \frac{1}{4}(b_4 + b_3c_1 + b_2c_2 + c_3).
\end{aligned}$$

By substituting the above expression for a_2, a_3 and a_4 in (1.6) and then further simplification gives

(2.3)

$$\begin{aligned}
2\gamma_3 &= a_4 - a_2a_3 + \frac{1}{3}a_2^3\\
&= \frac{1}{24}\left((6b_4 - 4b_2b_3 + b_2^3) + 2c_1\left(b_3 - \frac{1}{2}b_2^2\right) + b_2(2c_2 - c_1^2) + c_1^3 - 4c_1c_2 + 6c_3\right).
\end{aligned}$$

In view of Lemma 2.2 and writing c_2 and c_3 in terms of c_1 we obtain

$$\begin{aligned}
48\gamma_3 &= (6b_4 - 4b_2b_3 + b_2^3) + 2c_1\left(b_3 - \frac{1}{2}b_2^2\right) + b_2x(4-c_1^2)\\
&\quad + \frac{1}{2}c_1^3 + c_1x(4-c_1^2) - \frac{3}{2}c_1x^2(4-c_1^2) + 3(4-c_1^2)(1-|x|^2)t,
\end{aligned} \tag{2.4}$$

where $|x| \leq 1$ and $|t| \leq 1$. Note that if $\gamma_3(g)$ denotes the third logarithmic coefficient of $g \in \mathcal{S}^*$, then $|\gamma_3(g)| = \frac{1}{2}|b_4 - b_2b_3 + \frac{1}{3}b_2^3| \leq \frac{1}{3}$. Since $g \in \mathcal{S}^*$, in view of Lemma 2.3 we obtain

$$|6b_4 - 4b_2b_3 + b_2^3| \leq 6|b_4 - b_2b_3 + \frac{1}{3}b_2^3| + 2|b_2||b_3 - \frac{1}{2}b_2^2| \leq 8. \tag{2.5}$$

Since the class $\mathcal{K}_0$ is invariant under rotation, without loss of generality we can assume that $c_1 = c$, where $0 \leq c \leq 2$. Taking modulus on both sides of (2.4) and then applying triangle inequality and further using the inequality (2.5) and Lemma 2.3, it follows that

$$48|\gamma_3| \leq 8+2c+2|x|(4-c^2)+\left|\frac{1}{2}c^3 + cx(4-c^2) - \frac{3}{2}cx^2(4-c^2)\right|+3(4-c^2)(1-|x|^2),$$

where we have also used the fact $|t| \leq 1$. Let $x = re^{i\theta}$ where $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. For simplicity, by writing $\cos\theta = p$ we obtain

$$48|\gamma_3| \leq \psi(c,r) + |\phi(c,r,p)| =: F(c,r,p) \tag{2.6}$$

where $\psi(c,r) = 8 + 2c + 2r(4-c^2) + 3(4-c^2)(1-r^2)$ and

$$\phi(c,r,p) = \left(\frac{1}{4}c^6 + c^2r^2(4-c^2)^2 + \frac{9}{4}c^2r^4(4-c^2)^2 + c^4(4-c^2)rp \right.$$
$$\left. -\frac{3}{2}c^4r^2(4-c^2)(2p^2-1) - 3c^2(4-c^2)r^3p\right)^{1/2}.$$

Thus we need to find the maximum value of $F(c,r,p)$ over the rectangular cube $R := [0,2] \times [0,1] \times [-1,1]$.

By elementary calculus one can verify the following:

$$\max_{0\leq r\leq 1} \psi(0,r) = \psi\left(0,\frac{1}{3}\right) = \frac{64}{3}, \quad \max_{0\leq r\leq 1} \psi(2,r) = 12,$$

$$\max_{0\leq c\leq 2} \psi(c,0) = \psi\left(\frac{1}{3},0\right) = \frac{61}{3}, \quad \max_{0\leq c\leq 2} \psi(c,1) = \psi(0,1) = 16 \quad \text{and}$$

$$\max_{(c,r)\in[0,2]\times[0,1]} \psi(c,r) = \psi\left(\frac{3}{10},\frac{1}{3}\right) = \frac{649}{30} = 21.6333.$$

We first find the maximum value of $F(c,r,p)$ on the boundary of R, i.e., on the six faces of the rectangular cube R.

On the face $c = 0$, we have $F(0,r,p) = \psi(0,r)$, where $(r,p) \in R_1 := [0,1] \times [-1,1]$. Thus

$$\max_{(r,p)\in R_1} F(0,r,p) = \max_{0\leq r\leq 1} \psi(0,r) = \psi\left(0,\frac{1}{3}\right) = \frac{64}{3} = 21.33.$$

On the face $c = 2$, we have $F(2,r,p) = 16$, where $(r,p) \in R_1$.

On the face $r = 0$, we have $F(c,0,p) = 8 + 2c + 3(4-c^2) + \frac{1}{2}c^3$, where $(c,p) \in R_2 := [0,2] \times [-1,1]$. By using elementary calculus it is easy to see that

$$\max_{(c,p)\in R_2} F(c,0,p) = F\left(\frac{2}{3}(3-\sqrt{6}),0,p\right) = \frac{16}{9}\left(9+\sqrt{6}\right) = 20.3546.$$

On the face $r = 1$, we have $F(c,1,p) = \psi(c,1)+|\phi(c,1,p)|$, where $(c,p) \in R_2$. We first prove that $\phi(c,1,p) \neq 0$ in the interior of R_2. On the contrary, if $\phi(c,1,p) = 0$

in the interior of R_2, then

$$|\phi(c,1,p)|^2 = \left|\frac{1}{2}c^3 + ce^{i\theta}(4-c^2) - \frac{3}{2}ce^{2i\theta}(4-c^2)\right|^2 = 0$$

and hence,

$$\frac{1}{2}c^3 + cp(4-c^2) - \frac{3}{2}c(4-c^2)(2p^2-1) = 0 \text{ and } c(4-c^2)\sin\theta - \frac{3}{2}c(4-c^2)\sin 2\theta = 0. \tag{2.7}$$

On further simplification, (2.7) reduces to

$$\frac{1}{2}c^2 + p(4-c^2) - \frac{3}{2}(4-c^2)(2p^2-1) = 0 \quad \text{and} \quad 1-3p = 0,$$

which is equivalent to $p = 1/3$ and $c^2 = 6$. This contradicts the range of $c \in (0,2)$. Thus $\phi(c,1,p) \neq 0$ in the interior of R_2.

Next, we prove that $F(c,1,p)$ has no maximum at any interior point of R_2. Suppose that $F(c,1,p)$ has a maximum at an interior point of R_2. Then at such point $\frac{\partial F(c,1,p)}{\partial c} = 0$ and $\frac{\partial F(c,1,p)}{\partial p} = 0$. From $\frac{\partial F(c,1,p)}{\partial p} = 0$, (for points in the interior of R_2), a straightforward calculation gives

$$p = \frac{2\left(c^2-3\right)}{3c^2}. \tag{2.8}$$

Substituting the value of p as given in (2.8) in the relation $\frac{\partial F(c,1,p)}{\partial c} = 0$ and further simplification gives

$$3c^3 - 2c + (2c-1)\sqrt{6(c^2+2)} = 0. \tag{2.9}$$

It is easy to show that the function $\rho(c) = 3c^3 - 2c + (2c-1)\sqrt{6(c^2+2)}$ is strictly increasing in $(0,2)$. Since $\rho(0) < 0$ and $\rho(2) > 0$, the equation (2.9) has exactly one solution in $(0,2)$. By solving the equation (2.9) numerically, we obtain the approximate root in $(0,2)$ as 0.5772. But the corresponding value of p obtained by (2.8) is -5.3365 which does not belong to $(-1,1)$. Thus $F(c,1,p)$ has no maximum at any interior point of R_2.

Thus we find the maximum value of $F(c,1,p)$ on the boundary of R_2. Clearly, $F(0,1,p) = F(2,1,p) = 16$,

$$F(c,1,-1) = \begin{cases} 8 + 2c + 2(4-c^2) + c(10-3c^2) & \text{for} \quad 0 \le c \le \sqrt{\frac{10}{3}}, \\ 8 + 2c + 2(4-c^2) - c(10-3c^2) & \text{for} \quad \sqrt{\frac{10}{3}} < c \le 2, \end{cases}$$

and

$$F(c,1,1) = \begin{cases} 8 + 2c + 2(4-c^2) + c(2-c^2) & \text{for} \quad 0 \le c \le \sqrt{2}, \\ 8 + 2c + 2(4-c^2) - c(2-c^2) & \text{for} \quad \sqrt{2} < c \le 2. \end{cases}$$

By using elementary calculus we find that

$$\max_{0\le c\le 2} F(c,1,-1) = F\left(\frac{2}{9}(2\sqrt{7}-1),1,-1\right) = \frac{8}{243}\left(403 + 112\sqrt{7}\right) = 23.023 \quad \text{and}$$

$$\max_{0\le c\le 2} F(c,1,1) = F\left(\frac{2}{3},1,1\right) = \frac{427}{27} = 17.48.$$

Hence,

$$\max_{(c,p)\in R_2} F(c,1,p) = F\left(\frac{2}{9}(2\sqrt{7}-1),1,-1\right) = \frac{8}{243}\left(403+112\sqrt{7}\right) = 23.023.$$

On the face $p=-1$,

$$F(c,r,-1) = \begin{cases} \psi(c,r)+\eta_1(c,r) & \text{for} \quad \eta_1(c,r)\geq 0, \\ \psi(c,r)-\eta_1(c,r) & \text{for} \quad \eta_1(c,r)<0, \end{cases}$$

where $\eta_1(c,r) = c^3(3r^2+2r+1)-4cr(3r+2)$ and $(c,r)\in R_3 := [0,2]\times[0,1]$. Differentiating partially $F(c,r,-1)$ with respect to c and r and a routine calculation shows that

$$\max_{(c,r)\in \operatorname{int} R_3\setminus S_1} F(c,r,-1) = F\left(2(\sqrt{2}-1),\frac{1}{3}(1+\sqrt{2}),-1\right) = \frac{8}{3}(3+4\sqrt{2}) = 23.0849,$$

where $S_1 = \{(c,r)\in R_3 : \eta_1(c,r)=0\}$. Now we find the maximum value of $F(c,r,-1)$ on the boundary of R_3 and on the set S_1. Note that

$$\max_{(c,r)\in S_1} F(c,r,-1) \leq \max_{(c,r)\in R_3} \psi(c,r) = \frac{649}{30} = 21.6333.$$

On the other hand by using elementary calculus, as before, we find that

$$\max_{(c,r)\in \partial R_3} F(c,r,-1) = F\left(\frac{2}{9}(2\sqrt{7}-1),1,-1\right) = \frac{8}{243}\left(403+112\sqrt{7}\right) = 23.023,$$

where ∂R_3 denotes the boundary of R_3. Hence, by combining the above cases we obtain

$$\max_{(c,r)\in R_3} F(c,r,-1) = F\left(2(\sqrt{2}-1),\frac{1}{3}(1+\sqrt{2}),-1\right) = \frac{8}{3}(3+4\sqrt{2}) = 23.0849.$$

On the face $p=1$,

$$F(c,r,1) = \begin{cases} \psi(c,r)+\eta_2(c,r) & \text{for} \quad \eta_2(c,r)\geq 0, \\ \psi(c,r)-\eta_2(c,r) & \text{for} \quad \eta_2(c,r)<0, \end{cases}$$

where $\eta_2(c,r) = c^3(3r^2-2r+1)-4cr(3r-2)$ and $(c,r)\in R_3$. Differentiating partially $F(c,r,1)$ with respect to c and r and a routine calculation shows that

$$\max_{(c,r)\in \operatorname{int} R_3\setminus S_2} F(c,r,1) = F\left(\frac{1}{3}(10-2\sqrt{19}),\frac{1}{3},1\right) = \frac{16}{81}\left(28+19\sqrt{19}\right) = 21.89,$$

where $S_2 = \{(c,r)\in R_3 : \eta_2(c,r)=0\}$. Now, we find the maximum value of $F(c,r,1)$ on the boundary of R_3 and on the set S_2. By noting that

$$\max_{(c,r)\in S_2} F(c,r,1) \leq \max_{(c,r)\in R_3} \psi(c,r) = \frac{649}{30} = 21.6333$$

and proceeding similarly as in the previous case, we find that

$$\max_{(c,r)\in R_3} F(c,r,1) = F\left(\frac{1}{3}(10-2\sqrt{19}),\frac{1}{3},1\right) = \frac{16}{81}\left(28+19\sqrt{19}\right) = 21.89.$$

Let $S' = \{(c,r,p)\in R : \phi(c,r,p)=0\}$. Then

$$\max_{(c,r,p)\in S'} F(c,r,p) \leq \max_{(c,r)\in R_3} \psi(c,r) = \psi\left(\frac{3}{10},\frac{1}{3}\right) = \frac{649}{30} = 21.6333.$$

We prove that $F(c,r,p)$ has no maximum at any interior point of $R\setminus S'$. Suppose that $F(c,r,p)$ has a maximum at an interior point of $R\setminus S'$. Then at such point $\frac{\partial F}{\partial c}=0$, $\frac{\partial F}{\partial r}=0$ and $\frac{\partial F}{\partial p}=0$. Note that $\frac{\partial F}{\partial c}$, $\frac{\partial F}{\partial r}$ and $\frac{\partial F}{\partial p}$ may not exist at points in S'. In view of $\frac{\partial F}{\partial p}=0$ (for points in the interior of $R\setminus S'$), a straightforward but laborious calculation gives

$$p=\frac{3c^2r^2+c^2-12r^2}{6c^2r}. \tag{2.10}$$

Substituting the value of p as given in (2.10) in the relations $\frac{\partial F}{\partial c}=0$ and $\frac{\partial F}{\partial r}=0$ and simplifying (again, a long and laborious calculation), we obtain

$$\frac{3\sqrt{6}c^3(1-3r^2)+12(c(3r^2-2r-3)+1)\sqrt{c^2+2}+4\sqrt{6}c}{6\sqrt{c^2+2}}=0 \tag{2.11}$$

and

$$(4-c^2)\left((\sqrt{6(c^2+2)}-6)r+2\right)=0. \tag{2.12}$$

Since $0<c<2$, solving the equation (2.12) for r, we obtain

$$r=\frac{2}{6-\sqrt{6(c^2+2)}}. \tag{2.13}$$

Substituting the value of r in (2.11) and then further simplification gives

$$3c^3+6c-(6c-2)\sqrt{6\left(c^2+2\right)}=0.$$

Taking the last term on the right hand side and squaring on both sides yields

$$3\left(c^2+2\right)\left(3c^4-66c^2+48c-8\right)=0. \tag{2.14}$$

Clearly $c^2+2\neq 0$ in $0<c<2$. On the other hand the polynomial $q(c)=3c^4-66c^2+48c-8$ has exactly two roots in $(0,2)$, one lies in $(0,1/3)$ and another lies in $(1/3,1/2)$. This can be seen using the well-known Sturm theorem for isolating real roots and hence for the sake of brevity we omit the details. By solving the equation $q(c)=0$ numerically, we obtain two approximate roots 0.2577 and 0.4795 in $(0,2)$. But the corresponding value of p obtained from (2.13) and (2.10) are -23.6862 and -6.80595 which do not belong to $(-1,1)$. This proves that $F(c,r,p)$ has no maximum in the interior of $R\setminus S'$

Thus combining all the above cases we find that

$$\max_{(c,r,p)\in R}F(c,r,p)=F\left(2(\sqrt{2}-1),\frac{1}{3}(1+\sqrt{2}),-1\right)=\frac{8}{3}(3+4\sqrt{2})=23.0849,$$

and hence from (2.6) we obtain

$$|\gamma_3|\leq\frac{1}{18}(3+4\sqrt{2})=0.4809.$$

□

We obtained the following sharp upper bound for $|\gamma_3|$ for functions in the class $\mathcal{CR}^+$.

Theorem 2.2. *Let $f\in\mathcal{CR}^+$ be of the form* (1.1) *with $1\leq a_2\leq 2$. Then*

$$|\gamma_3|\leq\frac{1}{243}(28+19\sqrt{19})=0.4560. \tag{2.15}$$

The inequality is sharp.

Proof. If $f \in \mathcal{CR}^+$, then there exists a Carathéodory function $P \in \mathcal{P}$ of the form (2.1) such that $zf'(z) = g(z)P(z)$, where $g(z) := k(z) = z/(1-z)^2$. Following the same method as used in Theorem 2.1 and noting that $g(z) := k(z) = z + 2z^2 + 3z^3 + 4z^4 + \cdots$, a simple computation in (2.4) shows that

$$48\gamma_3 = 8 + 2c_1 + \frac{1}{2}c_1^3 + (4-c_1^2)(2x + c_1 x - \frac{3}{2}c_1 x^2) + 3(4-c_1^2)(1-|x|^2)t, \tag{2.16}$$

where $|x| \le 1$ and $|t| \le 1$. Since $1 \le a_2 \le 2$ and $2a_2 = 2 + c_1$, then $0 \le c_1 \le 2$. Taking modulus on both sides of (2.16) and then applying triangle inequality and writing $c = c_1$, it follows that

$$48|\gamma_3| \le \left| 8 + 2c_1 + \frac{1}{2}c_1^3 + (4-c_1^2)(2x + c_1 x - \frac{3}{2}c_1 x^2) \right| + 3(4-c^2)(1-|x|^2),$$

where we have also used the fact $|t| \le 1$. Let $x = re^{i\theta}$ where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. For simplicity, by writing $\cos\theta = p$ we obtain

$$48|\gamma_3| \le \psi(c,r) + |\phi(c,r,p)| =: F(c,r,p) \tag{2.17}$$

where $\psi(c,r) = 3(4-c^2)(1-r^2)$ and

$$\phi(c,r,p) = \Bigg((8 + 2c + \frac{1}{2}c^3)^2 + r^2(4-c^2)^2(4 + c^2 + \frac{9}{4}c^2r^2 + 4c - 6crp - 3c^2rp) \\ + 2(4-c^2)(8 + 2c + \frac{1}{2}c^3)(2rp + crp - \frac{3}{2}cr^2(2p^2-1)) \Bigg)^{1/2}.$$

Thus we need to find the maximum value of $F(c,r,p)$ over the rectangular cube $R = [0,2] \times [0,1] \times [-1,1]$.

We first find the maximum value of $F(c,r,p)$ on the boundary of R, i.e., on the six faces of the rectangular cube R. As before, let $R_1 = [0,1] \times [-1,1], R_2 = [0,2] \times [-1,1]$ and $R_3 = [0,2] \times [0,1]$. By elementary calculus it is not very difficult to prove that

$$\max_{(r,p)\in R_1} F(0,r,p) = F(0, \frac{1}{3}, 1) = \frac{64}{3} = 21.33,$$
$$\max_{(r,p)\in R_1} F(2,r,p) = F(2,r,p) = 16,$$
$$\max_{(c,p)\in R_2} F(c,0,p) = F\left(\frac{2}{3}(3-\sqrt{6}), 0, p\right) = \frac{16}{9}\left(9 + \sqrt{6}\right) = 20.3546.$$

On the face $r = 1$, we have $F(c,1,p) = |\phi(c,1,p)|$ where $(c,p) \in R_2$. As in the proof of Theorem 2.1, one can verify that $\phi(c,1,p) \ne 0$ in the interior of R_2 (otherwise, one can simply proceed to find maximum value $F(c,1,p)$ at an interior point of $R_2 \setminus T$, where $T = \{(c,p) \in R_2 : \phi_1(c,1,p) = 0\}$, as $F(c,1,p) = 0$ in T). Suppose that $F(c,1,p)$ has a maximum at an interior point of R_2. Then at such point $\frac{\partial F}{\partial c} = 0$ and $\frac{\partial F}{\partial p} = 0$. From $\frac{\partial F}{\partial p} = 0$ (for points in the interior of R_2), it follows that

$$p = \frac{2\left(c^3 - 2c + 4\right)}{3c\left(c^2 - 2c + 8\right)}. \tag{2.18}$$

By substituting the above value of p given in (2.18) in the relation $\frac{\partial F}{\partial c} = 0$ and further computation (a long and laborious calculation) gives

$$3c^8 - 17c^7 + 76c^6 - 136c^5 + 120c^4 + 640c^3 - 832c^2 - 192c + 128 = 0.$$

This equation has exactly two real roots in $(0,2)$, one lies in $(0,1)$ and another lies in $(1,2)$. This can be seen using the well-known Sturm theorem for isolating real roots therefore for the sake of brevity we omit the details. Solving this equation numerically we obtain two approximate roots 0.3261 and 1.2994 in $(0,2)$ and the corresponding values of p are 0.9274 and 0.2602 respectively. Thus the extremum points of $F(c,1,p)$ in the interior of R_2 lie in a small neighborhood of the points $A_1=(0.3261,1,0.9274)$ and $A_2=(1.2994,1,0.2602)$ (on the plane $r=1$). Now $F(A_1)=15.8329$ and $F(A_2)=18.6303$. Since the function $F(c,1,p)$ is uniformly continuous on R_2, the value of $F(c,1,p)$ would not vary too much in the neighborhood of the points A_1 and A_2. Again, proceeding similarly as in the proof of Theorem 2.1, we find that

$$\max_{(c,p)\in\partial R_2} F(c,1,p)=F(2,1,p)=16$$

and hence

$$\max_{(c,p)\in R_2} F(c,1,p)\approx 18.6306<\frac{64}{3}.$$

On the face $p=-1$,

$$F(c,r,-1)=\begin{cases}\psi(c,r)+\eta_1(c,r) & \text{for} \quad \eta_1(c,r)\geq 0,\\ \psi(c,r)-\eta_1(c,r) & \text{for} \quad \eta_1(c,r)\leq 0,\end{cases}$$

where $\eta_1(c,r)=c^3-3cr^2(4-c^2)+2(c-2)(c+2)^2r+4c+16$ and $(c,r)\in R_3$. Again, proceeding similarly as in the proof of Theorem 2.1, we can show that $F(c,r,-1)$ has no maximum in the interior of $R_3\setminus S_1$, where $S_1=\{(c,r)\in R_3:\eta_1(c,r)=0\}$. Computing the maximum value on the boundary of R_3 and on the set S_1 we conclude that

$$\max_{(c,r)\in R_3} F(c,r,-1)=F(0,0,-1)=20.$$

On the face $p=1$, we have $F(c,r,1)=\psi(c,r)+\eta_2(c,r)$, where

$$\begin{aligned}\eta_2(c,r)&=(c+2)(8-2c+c^2+8r-2c^2r-6cr^2+3c^2r^2)\\&\geq (c+2)\left(3+(1-c)^2+r(8-2c^2)+r^2(3c^2-6c+4)\right)\\&\geq 0\end{aligned}$$

for $(c,r)\in R_3$. Differentiating partially $F(c,r,1)$ with respect to c and r and a routine calculation shows that

$$\max_{(c,r)\in \operatorname{int} R_3} F(c,r,1)=F\left(\frac{1}{3}(10-2\sqrt{19}),\frac{1}{3},1\right)=\frac{16}{81}\left(28+19\sqrt{19}\right)=21.8902,$$

and on the boundary of R_3 we have

$$\max_{(c,r)\in\partial R_3} F(c,r,1)=F(0,\frac{1}{3},1)=\frac{64}{3}=21.33.$$

Thus,

$$\max_{(c,r)\in R_3} F(c,r,1)=F\left(\frac{1}{3}(10-2\sqrt{19}),\frac{1}{3},1\right)=\frac{16}{81}\left(28+19\sqrt{19}\right)=21.8902.$$

Let $S'=\{(c,r,p)\in R:\phi(c,r,p)=0\}$. Then

$$\max_{(c,r,p)\in S'} F(c,r,p)\leq \max_{(c,r)\in R_3}\psi(c,r)=12.$$

We now prove that $F(c,r,p)$ has no maximum at an interior point of $R\setminus S'$. Suppose that $F(c,r,p)$ has a maximum at an interior point of $R\setminus S'$. Then at such point $\frac{\partial F}{\partial c}=0$, $\frac{\partial F}{\partial r}=0$ and $\frac{\partial F}{\partial p}=0$. Note that $\frac{\partial F}{\partial c}$, $\frac{\partial F}{\partial r}$ and $\frac{\partial F}{\partial p}$ may not exist at points in S'. In view of $\frac{\partial F}{\partial p}=0$ (for points in the interior of $R\setminus S'$), a straightforward but laborious calculation gives

$$p=\frac{3c^3r^2+c^3-12cr^2+4c+16}{6cr(c^2-2c+8)}. \tag{2.19}$$

Substituting the value of p given in (2.19) in the relation $\frac{\partial F}{\partial r}=0$ and then further simplifying (again, a long and laborious calculation), we obtain

$$r(4-c^2)\left(c\sqrt{\frac{6(c^3-4c^2+14c+4)}{c(c^2-2c+8)}}-6\right)=0. \tag{2.20}$$

Since $0<c<2$ and $0<r<1$, we can divide by $r(4-c^2)$ on both sides of (2.20). Further, a simple computation shows that

$$\frac{6(4-c^2)(c^2-4c+12)}{c^2-2c+8}=0.$$

But this equation has no real roots in $(0,2)$. Therefore, $F(c,r,p)$ has no maximum at an interior point of $R\setminus S'$.

Thus combining all the cases we find that

$$\max_{(c,r,p)\in R}F(c,r,p)=F\left(\frac{1}{3}(10-2\sqrt{19}),\frac{1}{3},1\right)=\frac{16}{81}\left(28+19\sqrt{19}\right)=21.8902,$$

and hence, from (2.17) we obtain

$$|\gamma_3|\le\frac{1}{243}(28+19\sqrt{19})=0.4560.$$

We now show that the inequality (2.15) is sharp. It is pertinent to note that equality holds in (2.15) if we choose $c_1=c=\frac{1}{3}(10-2\sqrt{19})$, $x=\frac{1}{3}$ and $t=1$ in (2.16). For such values of c_1,x and t, Lemma 2.2 elicit $c_2=\frac{2}{27}(97-20\sqrt{19})$ and $c_3=\frac{1}{243}(2050-362\sqrt{19})$. A function $P\in\mathcal{P}$ having the first three coefficients c_1,c_2 and c_3 as above is given by

(2.21)

$$\begin{aligned}P(z)&=(1-2\lambda)\frac{1+z}{1-z}+\lambda\frac{1+uz}{1-uz}+\lambda\frac{1+\overline{u}z}{1-\overline{u}z}\\&=1+\frac{1}{3}(10-2\sqrt{19})z+\frac{2}{27}(97-20\sqrt{19})z^2+\frac{1}{243}(2050-362\sqrt{19})z^3+\cdots,\end{aligned}$$

where $\lambda=\frac{1}{18}(-13+4\sqrt{19})$ and $u=\alpha+i\sqrt{1-\alpha^2}$ with $\alpha=-\frac{1}{9}(1+\sqrt{19})$. Hence the inequality (2.15) is sharp for a function f defined by $(1-z)^2f'(z)=P(z)$, where $P(z)$ is given by (2.21). This completes the proof. ☐

Remark 2.1. In [12], Thomas proved that $|\gamma_3|\le\frac{7}{12}=0.5833$ for functions in the class $\mathcal{K}_0$ with an additional condition that the second coefficient b_2 of the corresponding starlike function g is real. However, in Theorem 2.1 we obtained a much improved bound $|\gamma_3|\le\frac{1}{18}(3+4\sqrt{2})=0.4809$ for functions in the whole class $\mathcal{K}_0$ without assuming any additional condition on functions in the class $\mathcal{K}_0$. While for functions in the class $\mathcal{CR}^+$ (with $1\le a_2\le 2$) we obtained the sharp bound

$|\gamma_3| \leq \frac{1}{243}(28 + 19\sqrt{19}) = 0.4560$. We conjecture that for the whole class $\mathcal{K}_0$ the sharp upper bound for $|\gamma_3|$ is $|\gamma_3| \leq \frac{1}{243}(28 + 19\sqrt{19}) = 0.4560$.

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Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur-721 302, West Bengal, India

E-mail address: `ali.firoz89@gmail.com`

Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur-721 302, West Bengal, India

E-mail address: `alluvasu@maths.iitkgp.ernet.in`