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REFLECTION GROUPS, REFLECTION ARRANGEMENTS, AND INVARIANT REAL VARIETIES

TOBIAS FRIEDL, CORDIAN RIENER, AND RAMAN SANYAL

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ABSTRACT. Let X be a nonempty real variety that is invariant under the action of a reflection group G. We conjecture that if X is defined in terms of the first k basic invariants of G (ordered by degree), then X meets a k-dimensional flat of the associated reflection arrangement. We prove this conjecture for the infinite types, reflection groups of rank at most 3, and F_4 and we give computational evidence for H_4. This is a generalization of Timofte's degree principle to reflection groups. For general reflection groups, we compute nontrivial upper bounds on the minimal dimension of flats of the reflection arrangement meeting X from the combinatorics of parabolic subgroups. We also give generalizations to real varieties invariant under Lie groups.

1. Introduction

A real variety $X \subseteq \mathbb{R}^n$ is the set of real points simultaneously satisfying a system of polynomial equations with real coefficients, that is,

$$X \ = \ \mathrm{V}_{\mathbb{R}}(f_1, \dots, f_m) \ := \ \{\mathbf{p} \in \mathbb{R}^n : f_1(\mathbf{p}) = f_2(\mathbf{p}) = \cdots = f_m(\mathbf{p}) = 0\},$$

for some $f_1, \dots, f_m \in \mathbb{R}[\mathbf{x}] := \mathbb{R}[x_1, \dots, x_n]$. In contrast to working over an algebraically closed field, the question if $X \neq \varnothing$ is considerably more difficult to answer, both theoretically and in practice; see [3]. Timofte [18] studied real varieties invariant under the action of the symmetric group $\mathfrak{S}_n$ and proved an interesting structural result. A $\mathfrak{S}_n$-invariant variety can be defined in terms of **symmetric** polynomials, that is, polynomials $f \in \mathbb{R}[\mathbf{x}]$ such that $f(x_{\tau(1)}, \dots, x_{\tau(n)}) = f(x_1, \dots, x_n)$ for all permutations $\tau \in \mathfrak{S}_n$. Recall that the fundamental theorem of symmetric polynomials states that a polynomial f is symmetric if and only if f is a polynomial in the elementary symmetric polynomials $e_1, \dots, e_n$. Let us call a $\mathfrak{S}_n$-invariant variety X ***k*-sparse** if $X = \mathrm{V}_{\mathbb{R}}(f_1, \dots, f_m)$ for some symmetric polynomials $f_1, \dots, f_m \in \mathbb{R}[e_1, \dots, e_k]$.

Theorem 1 ([18]). *Let $X \subseteq \mathbb{R}^n$ be a nonempty $\mathfrak{S}_n$-invariant real variety. If X is k-sparse, then there is a point $\mathbf{p} \in X$ with at most k distinct coordinates.*

Viewing the symmetric group $\mathfrak{S}_n$ as a reflection group in $\mathbb{R}^n$ yields a sound geometric perspective on this result: As a group of linear transformations, $\mathfrak{S}_n$ is

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generated by reflections in the hyperplanes $H_{ij} = \{\mathbf{p} : p_i = p_j\}$ for $1 \le i < j \le n$. The ambient space $\mathbb{R}^n$ is stratified by the arrangement of reflection hyperplanes $\mathcal{H} = \{H_{ij} : i < j\}$. The closed strata $\mathcal{H}_k \subseteq \mathbb{R}^n$ are the intersections of $n-k$ linearly independent reflection hyperplanes. Timofte's result then states that a k-sparse variety X is nonempty if and only if $X \cap \mathcal{H}_k \neq \varnothing$. Such a point of view can be taken for general real reflection groups, and the aim of this paper is a generalization of Theorem 1.

A (real) **reflection group** G acting on $V \cong \mathbb{R}^n$ is a finite group of orthogonal transformations generated by reflections. The reflection group G is **irreducible** if G is not the product of two nontrivial reflection groups. Associated to G is its **reflection arrangement**

$$\mathcal{H} \;=\; \mathcal{H}(G) \;:=\; \{H = \ker g : g \in G \text{ reflection}\}.$$

The **flats** of $\mathcal{H}$ are the linear subspaces arising from intersections of hyperplanes in $\mathcal{H}$. The arrangement of linear hyperplanes stratifies V with strata given by

$$\mathcal{H}_i \;=\; \mathcal{H}_i(G) \;:=\; \{\mathbf{p} \in V : \mathbf{p} \text{ is contained in a flat of dimension } i\}.$$

In particular, $\mathcal{H}_n = V$. We call G **essential** if G does not fix a nontrivial linear subspace or, equivalently, if $\mathcal{H}_0 = \{0\}$. If G is essential, then the **rank** of G is $\operatorname{rank}(G) := \dim V$. Reflection groups naturally occur in connection with Lie groups/algebras and are well studied from the perspective of geometry, algebra, and combinatorics [4, 5, 11]. A complete classification of reflection groups can be given in terms of Dynkin diagrams (see [11]). There are four infinite families of irreducible reflection groups, $\mathfrak{S}_n \cong A_{n-1}, B_n, D_n, I_2(m)$, and six exceptional reflection groups, H_3, H_4, F_4, E_6, E_7, and E_8.

The linear action of G on V induces an action on the symmetric algebra $\mathbb{R}[V] \cong \mathbb{R}[x_1, \dots, x_n]$ by $g \cdot f(\mathbf{x}) := f(g^{-1} \cdot \mathbf{x})$. Chevalley's Theorem [11, Ch. 3.5] states that the ring $\mathbb{R}[V]^G$ of polynomials invariant under G is generated by algebraically independent homogeneous polynomials $\pi_1, \pi_2, \dots, \pi_n \in \mathbb{R}[V]$. The collection $\pi_1, \dots, \pi_n$ is called a set of **basic invariants** for G. The basic invariants are not unique, but their degrees $d_i(G) := \deg \pi_i$ are. Throughout, we will assume that the basic invariants are labelled such that $d_1 \le d_2 \le \cdots \le d_n$. In accord with $\mathfrak{S}_n$-invariant varieties, we call a G-invariant variety $X = \mathrm{V}_{\mathbb{R}}(f_1, \dots, f_m)$ ***k*-sparse** if $f_1, \dots, f_m$ can be chosen in $\mathbb{R}[\pi_1, \dots, \pi_k]$ for some choice of basic invariants $\pi_1, \dots, \pi_n$ ordered by nondecreasing degrees. The following is the main result of the paper.

Theorem 2. *Let G be a reflection group of type $I_2(m), A_{n-1}, B_n, D_n, H_3$, or F_4 and X a nonempty G-invariant real variety. If X is k-sparse, then $X \cap \mathcal{H}_k(G) \neq \varnothing$.*

Since the first basic invariant of an essential reflection group is a scalar multiple of $p_2(\mathbf{x}) = \|\mathbf{x}\|^2$, Theorem 2 is trivially true for reflection groups of rank ≤ 2. The infinite families A_{n-1}, B_n, and D_n are treated in Section 2. Timofte's original proof and its simplification given by the second author in [15] use properties of the symmetric group that are not shared by all reflection groups (such as D_n), and we highlight this difference in Example 1 and Remark 1. In Section 3, we prove the following general result that implies Theorem 2 in the case $k = n-1$.

Theorem 3. *Let G be an essential reflection group of rank n and let $X = \mathrm{V}_{\mathbb{R}}(f_1, \dots, f_m)$ be nonempty. If there is $j \in \{1, \dots, n\}$ such $f_1, \dots, f_m \in \mathbb{R}[\pi_i : i \neq j]$, then $X \cap \mathcal{H}_{n-1}(G) \neq \varnothing$.*

In particular, this result yields Theorem 2 for all reflection groups of rank ≤ 3. The group F_4 is treated in Section 3, and we provide computational evidence that Theorem 2 also holds for H_4. That supports the following conjecture.

Conjecture 1. *Let G be an irreducible and essential reflection group. Then any nonempty and k-sparse G-invariant real variety X intersects $\mathcal{H}_k(G)$.*

Proposition 8 provides a different geometric perspective on Conjecture 1 in terms of real orbit spaces and implies (Proposition 9) that X will in general not meet $\mathcal{H}_l(G)$ for $l < k$. In Section 4, we prove a weaker form of Conjecture 1 under an extra assumption on the defining polynomials of X. In Section 5, we obtain upper bounds on the dimension of the stratum that meets X in terms of the combinatorics of parabolic subgroups of G. Our results generalize to varieties invariant under the adjoint action of Lie groups, and we explore this connection in Section 6.

Acevedo and Velasco [1] independently considered the related problem of certifying nonnegativity of G-invariant homogeneous polynomials. They show that low-degree forms (where the exact degree depends on the group) are nonnegative if and only if they are nonnegative on $\mathcal{H}_{n-1}(G)$. Questions of nonnegativity of polynomials $f \in \mathbb{R}[V]^G$ are subsumed by our results. Let us call a G-invariant semialgebraic set $S \subseteq V$ k-sparse if S is defined in terms of equations and inequalities with polynomials in $\mathbb{R}[\pi_1, \ldots, \pi_k]$.

Proposition 4. *Let G be a reflection group for which Conjecture 1 holds. Let $S \subseteq V$ be a k-sparse semialgebraic set and let $f \in \mathbb{R}[\pi_1, \ldots, \pi_k]$. Then f is nonnegative/positive on S if and only if f is nonnegative/positive on $\mathcal{H}_k(G) \cap S$.*

Proof. If S is k-sparse, then the G-invariant variety

$$X_k(\mathbf{q}) \;:=\; \{\mathbf{p} \in V : \pi_i(\mathbf{p}) = \pi_i(\mathbf{q}) \text{ for } i = 1, \ldots, k\} \tag{1}$$

is contained in S for any $\mathbf{q} \in S$. Assume that there is a point $\mathbf{q} \in S$ with $f(\mathbf{q}) < 0$. By assumption $f = F(\pi_1, \ldots, \pi_k)$ for some $F \in \mathbb{R}[y_1, \ldots, y_k]$. Hence f is negative (and constant) on $X_k(\mathbf{q}) \subseteq S$. By construction $X_k(\mathbf{q})$ is k-sparse and, since G satisfies Conjecture 1, $X_k(\mathbf{q}) \cap \mathcal{H}_k(G) \neq \varnothing$. □

The proof of Proposition 4 makes use of a key observation: It suffices to consider invariant varieties of the form (1) as any k-sparse variety X contains $X_k(\mathbf{q})$ for all $\mathbf{q} \in X$. We call $X_k(\mathbf{q})$ a **principal** k-sparse variety. Lastly, let us emphasize again that we will work with *real* varieties exclusively. In particular, set-theoretically, every real variety $X = \mathrm{V}_{\mathbb{R}}(f_1, \ldots, f_m)$ is the set of solutions to the equation $f(\mathbf{x}) = 0$ for $f = f_1^2 + f_2^2 + \cdots + f_m^2$.

2. The infinite families A_{n-1}, B_n, and D_n

In this section, we prove Theorem 2 for the reflection groups of types A_{n-1}, B_n, and D_n. The symmetric group $\mathfrak{S}_n$ acts on $\mathbb{R}^n$ but is not essential as it fixes $\mathbb{R}\mathbf{1}$. The restriction to $\{\mathbf{x} \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$ is the essential reflection group of type $\mathbf{A_{n-1}}$. The reflection arrangement $\mathcal{H}(\mathfrak{S}_n)$ was described in the introduction. The k-stratum $\mathcal{H}_k(\mathfrak{S}_n)$ is given by the points $\mathbf{p} \in \mathbb{R}^n$ that have at most k distinct coordinates. A set of basic invariants is given by the **elementary symmetric polynomials**

$$e_k(\mathbf{x}) \;:=\; \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}$$

or, alternatively, by the **power sums**

$$s_k(\mathbf{x}) \ := \ x_1^k + x_2^k + \cdots + x_n^k,$$

for $k = 1, \ldots, n$. The group $\mathbf{B_n} = \mathfrak{S}_n \rtimes \mathbb{Z}_2^n$ acts on $V = \mathbb{R}^n$ by *signed* permutations with reflection hyperplanes $\{x_i = \pm x_j\}$ and $\{x_i = 0\}$ for $1 \leq i < j \leq n$. A point $\mathbf{p}$ lies in $\mathcal{H}_i(B_n)$ if and only if $(|p_1|, \ldots, |p_n|)$ has at most i distinct nonzero coordinates. A set of basic invariants is given by $\pi_i(\mathbf{x}) = s_{2i}(\mathbf{x}) = s_i(x_1^2, \ldots, x_n^2)$. The index-2 subgroup $\mathbf{D_n}$ of B_n given by the semidirect product of $\mathfrak{S}_n$ with 'even sign changes' yields a reflection group with reflection hyperplanes $\{x_i = \pm x_j\}$ for $1 \leq i < j \leq n$. The k-stratum of D_n is a bit more involved to describe: denote by M the set of all $\mathbf{p} \in \mathbb{R}^n$ with exactly one zero coordinate. Then

$$\mathcal{H}_k(D_n) \ = \ (\mathcal{H}_k(B_n) \setminus M) \cup (\mathcal{H}_{k-1}(B_n) \cap M). \tag{2}$$

The invariant that distinguishes D_n from B_n is given by $e_n(\mathbf{x}) = x_1 x_2 \cdots x_n$. A set of basic invariants for D_n are $\pi_1(\mathbf{x}), \ldots, \pi_n(\mathbf{x})$ with

$$\pi_k(\mathbf{x}) \ := \ \begin{cases} s_{2k}(\mathbf{x}) & \text{for } 1 \leq k \leq \lfloor \frac{n}{2} \rfloor, \\ e_n(\mathbf{x}) & \text{for } k = \lfloor \frac{n}{2} \rfloor + 1, \text{ and} \\ s_{2k-2}(\mathbf{x}) & \text{for } \lfloor \frac{n}{2} \rfloor + 1 < k \leq n. \end{cases} \tag{3}$$

We start with the verification of Theorem 2 for $\mathfrak{S}_n$, which is exactly Theorem 1. The proofs for B_n and D_n will rely on the arguments for A_{n-1}.

Proof of Theorem 2 for $A_{n-1} \cong \mathfrak{S}_n$. Let us first assume that $\pi_i(\mathbf{x}) = s_i(\mathbf{x}) = x_1^i + \cdots + x_n^i$ are the power sums for $i = 1, \ldots, n$. It suffices to show the claim for a principal k-sparse variety $X_k(\mathbf{p}_0)$ as defined in (1), i.e. that $X_k(\mathbf{p}_0) \cap \mathcal{H}_k(\mathfrak{S}_n) \neq \varnothing$ for $\mathbf{p}_0 \in X$. Since $\pi_2(\mathbf{p}) = \|\mathbf{p}\|^2$, we conclude that $X_k(\mathbf{p}_0)$ is compact and π_{k+1} attains its maximum over $X_k(\mathbf{p}_0)$ in a point $\mathbf{q}$. At this point, the Jacobian $\operatorname{Jac}_{\mathbf{q}}(\pi_1, \ldots, \pi_{k+1}) = \big(\nabla s_1(\mathbf{q}), \ldots, \nabla s_{k+1}(\mathbf{q})\big)$ has rank $< k+1$. We claim that $\mathbf{q} \in \mathcal{H}_k$ if and only if the Jacobian $J = \operatorname{Jac}_{\mathbf{q}}(\pi_1, \ldots, \pi_{k+1})$ has rank $< k+1$. Indeed, up to scaling columns, J is given by

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ q_1 & q_2 & \cdots & q_n \\ \vdots & & & \vdots \\ q_1^k & q_2^k & \cdots & q_n^k \end{pmatrix}.$$

The $k+1$ minors are thus Vandermonde determinants all of which vanish if and only if $\mathbf{q} \in \mathcal{H}_k$, by the description of k-strata for $\mathfrak{S}_n$. For an arbitrary choice of basic invariants, the result follows from Lemma 5 below. $\square$

Lemma 5. *Fix a reflection group G acting on V. Let $\pi_1, \ldots, \pi_n$ and $\pi'_1, \ldots, \pi'_n$ be two sets of basic invariants and let $1 \leq k \leq n$ be such that $d_{k+1} > d_k$. Then $\mathbb{R}[\pi_1, \ldots, \pi_k] = \mathbb{R}[\pi'_1, \ldots, \pi'_k]$. Moreover,*

$$\operatorname{rank} \operatorname{Jac}_{\mathbf{p}}(\pi_1, \ldots, \pi_k) = \operatorname{rank} \operatorname{Jac}_{\mathbf{p}}(\pi'_1, \ldots, \pi'_k)$$

for all $\mathbf{p} \in V$.

Proof. For every $1 \leq i \leq k$, $\pi_i = F_i(\pi'_1, \ldots, \pi'_n)$ for some polynomial $F_i(y_1, \ldots, y_n)$. Homogeneity and algebraic independence imply that $F_i \in \mathbb{R}[y_1, \ldots, y_k]$. This shows the inclusion $\mathbb{R}[\pi_1, \ldots, \pi_k] \subseteq \mathbb{R}[\pi'_1, \ldots, \pi'_k]$. Note that

$$\operatorname{Jac}_{\mathbf{p}}(\pi_1, \ldots, \pi_k) \ = \ \operatorname{Jac}_{\pi'(\mathbf{p})}(F_1, \ldots, F_k) \cdot \operatorname{Jac}_{\mathbf{p}}(\pi'_1, \ldots, \pi'_k)$$

for every $\mathbf{p} \in V$. The same argument applied to π_i' now proves the first claim and shows that $\mathrm{Jac}_{\pi'(\mathbf{p})}(F_1, \ldots, F_k)$ has full rank, and this proves the second claim. □

We proceed to the reflection groups of type B_n.

Proof of Theorem 2 *for* B_n. By Lemma 5 and the fact that the degrees $d_i(B_n)$ are all distinct, we may assume that $\pi_i(\mathbf{x}) = s_{2i}(\mathbf{x})$ for $i = 1, \ldots, n$. Moreover, we can assume that X is a principal k-sparse variety, that is,

$$X \ = \ X_k(\mathbf{p}) \ = \ \{\mathbf{x} \in \mathbb{R}^n : s_{2i}(\mathbf{x}) = s_{2i}(\mathbf{p}) \text{ for } i = 1, \ldots, k\}.$$

Since $X_k(\mathbf{p}) = X_k(\mathbf{q})$ for all $\mathbf{q} \in X_k(\mathbf{p})$, we can assume that $\mathbf{p} = (p_1, \ldots, p_r, 0, \ldots, 0) \in X$ with the property that $p_1 \cdots p_r \neq 0$ and r is minimal.

If $r = n$, then X does not meet any of the coordinate hyperplanes $\{x_i = 0\}$. Let $\mathbf{q} \in X$ be an extreme point of π_{k+1} over X. At this point, the Jacobian $J = \mathrm{Jac}_{\mathbf{q}}(\pi_1, \ldots, \pi_{k+1})$ does not have full rank and hence every maximal minor of

$$J \ = \ \begin{pmatrix} q_1 & q_2 & \cdots & q_n \\ \vdots & & & \vdots \\ q_1^{2k-1} & q_2^{2k-1} & \cdots & q_n^{2k-1} \end{pmatrix}$$

vanishes. Since $q_i \neq 0$ for all $i = 1, \ldots, n$, the Vandermonde formula implies that $(q_1^2, q_2^2, \ldots, q_n^2)$ has at most k distinct coordinates, which yields the claim.

If $r < n$, we can restrict X to the linear subspace $U = \{\mathbf{x} \in \mathbb{R}^n : x_{r+1} = \cdots = x_n = 0\} \cong \mathbb{R}^r$. The set $X' := X \cap U \subseteq \mathbb{R}^r$ is nonempty and, in particular, a k-sparse B_r-invariant variety that stays away from the coordinate hyperplanes in $\mathbb{R}^r$. By the previous case, there is a point $\mathbf{q}' \in X'$ such that $(|q_1'|, \ldots, |q_r'|)$ has at most k distinct coordinates. By construction, $\mathbf{q} = (\mathbf{q}', \mathbf{0}) \in X \cap \mathcal{H}_k(B_n)$, which proves the claim. □

The key to the proof of Theorem 2 for A_{n-1} and B_n is the strong connection between the strata $\mathcal{H}_k$ and the ranks of the Jacobians $\mathrm{Jac}(\pi_1, \ldots, \pi_{k+1})$.

Corollary 6. *Let* $G \in \{\mathfrak{S}_n, B_n\}$ *and* $\pi_1, \ldots, \pi_n$ *be a set of basic invariants for* G. *Then a point* $\mathbf{q} \in V$ *lies in* $\mathcal{H}_k(G)$ *for* $0 \leq k \leq n-1$ *if and only if* $\mathrm{Jac}_{\mathbf{p}}(\pi_1, \ldots, \pi_{k+1})$ *has rank at most* k.

It is tempting to believe that such a statement holds true for all reflection groups, and, indeed, necessity follows from a well-known result of Steinberg [17]. However, the following example shows that Corollary 6 does not hold in general.

Example 1. Consider the group $G = D_5$ acting on $\mathbb{R}^5$ and the point $\mathbf{p} = (1, 1, 1, 1, 0)$. The point lies in $\mathcal{H}_2(D_5) \setminus \mathcal{H}_1(D_5)$; that is, $\mathbf{p}$ lies on exactly 3 linearly independent reflection hyperplanes. On the other hand, for any choice of basic invariants $\pi_1, \ldots, \pi_5$ the gradients $\nabla_{\mathbf{p}}\pi_1, \nabla_{\mathbf{p}}\pi_2$ are linearly dependent. Indeed, for $\pi_1 = \|x\|^2 = x_1^2 + \cdots + x_5^2$ and $\pi_2 = x_1^4 + \cdots + x_5^4$, this is easy to check, and this extends to all choices of basic invariants using Lemma 5.

Remark 1. Example 1 also serves as a counterexample to generalizations of Corollary 6 to all finite reflection groups; see also [6, Lemma 1'] (without a proof) and [2, Stmt. 3.3]. Moreover, in the language of Acevedo and Velasco [1, Def. 7], it is the first example of a reflection group not satisfying the *minor factorization condition.*

The following proof of Theorem 2 for type D_n does not rely on an extension of Corollary 6.

Proof of Theorem 2 for D_n. Let $\pi_1, \ldots, \pi_n$ be a choice of basic invariants for D_n and let $X = X_k(\mathbf{q}) \subseteq \mathbb{R}^n$ for some $\mathbf{q} \in \mathbb{R}^n$ and $1 \le k < n$. If n is odd or if $k \neq \lfloor \frac{n}{2} \rfloor$, then, by Lemma 5, we can assume that the basic invariants are given by (3). If n is even and $k = \lfloor \frac{n}{2} \rfloor$, then $\pi_{\lfloor \frac{n}{2} \rfloor}(\mathbf{x}) \;=\; \alpha s_n(\mathbf{x}) + \beta e_n(\mathbf{x})$, for some $\beta \neq 0$. We can also assume that $\mathbf{q} = (q_1, \ldots, q_l, 0, \ldots, 0)$ with $q_1 \cdots q_l \neq 0$ and l maximal among all points in $X_k(\mathbf{q})$. We distinguish two cases.

Case $l < n$: In this case, $e_n(\mathbf{x})$ is identically zero on $X_k(\mathbf{q})$ and $X' := X_k(\mathbf{q}) \cap \{\mathbf{x} : x_n = 0\}$ is nonempty. If $k \ge \lfloor \frac{n}{2} \rfloor + 1$, then we can identify

$$X' \;=\; \{\mathbf{x}' \in \mathbb{R}^{n-1} : s_{2i}(\mathbf{x}', \mathbf{0}) = s_{2i}(\mathbf{q}) \text{ for } i = 1, \ldots, k-1\}.$$

Hence X' is a real variety in $\mathbb{R}^{n-1}$ invariant under the action of B_{n-1} and X' is $(k-1)$-sparse. By Theorem 2 for B_{n-1}, $X' \cap \mathcal{H}_{k-1}(B_{n-1}) \neq \varnothing$. The claim now follows the description of $\mathcal{H}_k(D_n)$ given in (2).

If $k < \frac{n}{2}$, consider the Jacobian of $\pi_1 = s_2, \ldots, \pi_k = s_{2k}$ and the $(l+1)$-th elementary symmetric polynomial $e_{l+1}(\mathbf{x})$ at $\mathbf{q}$:

(4)

$$J = \mathrm{Jac}_{\mathbf{q}}(s_2, \ldots, s_{2k}, e_{l+1}) = \begin{pmatrix} q_1 & q_2 & \cdots & q_l & 0 & 0 & \cdots & 0 \\ q_1^3 & q_2^3 & \cdots & q_l^3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ q_1^{2k-1} & q_2^{2k-1} & \cdots & q_l^{2k-1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & q_1 \cdots q_l & 0 & \cdots & 0 \end{pmatrix}.$$

We observe that the $(l+1)$-th elementary symmetric function $e_{l+1}(\mathbf{x})$ is identically zero on $X_k(\mathbf{q})$, and hence the gradients of $\pi_1, \ldots, \pi_k$ and e_{l+1} are linearly dependent on $X_k(\mathbf{q})$. In particular, the Jacobian J has rank $\le k$. Since $q_1 \cdots q_l \neq 0$, the Vandermonde minors imply that

$$\prod_{i,j \in I, i<j} q_i^2 - q_j^2 \;=\; 0$$

for any $I \subseteq \{1, \ldots, l\}$ with $|I| = k$. This shows that $\mathbf{q} \in \mathcal{H}_{k-1}(B_n) \subseteq \mathcal{H}_k(D_n)$.

If $k = \frac{n}{2}$, then n is even and $X_k(\mathbf{q})$ is cut out by $s_2, \ldots, s_{n-2}$ and possibly s_n, since $e_n(\mathbf{x})$ is identically zero on $X_k(\mathbf{q})$. Thus the argument above remains valid.

Case $l = n$: If $k < \frac{n}{2}$, set $f := e_n$. If $k \ge \lfloor \frac{n}{2} \rfloor + 1$, set $f := s_{2k}$. For the special case that n is even and $k = \frac{n}{2}$, we set $f = e_n$ if $\alpha \neq 0$ and $f = s_n$ otherwise. Let $\mathbf{r} \in X_k(\mathbf{q})$ be a maximizer of $|f(\mathbf{x})|$. In particular, $r_1 \cdots r_n \neq 0$. Up to row and column operations, the Jacobian $J = \mathrm{Jac}_{\mathbf{q}}(s_2, s_4, \ldots, s_{2k}, f)$ is of the form

$$J \;=\; \begin{pmatrix} r_1 & r_2 & \cdots & r_n \\ r_1^3 & r_2^3 & \cdots & r_n^3 \\ \vdots & \vdots & & \vdots \\ r_1^{2k-1} & r_2^{2k-1} & \cdots & r_n^{2k-1} \\ \widehat{r_1} r_2 \cdots r_n & r_1 \widehat{r_2} \cdots r_n & \cdots & r_1 r_2 \cdots \widehat{r_n} \end{pmatrix}, \tag{5}$$

where $\widehat{r_i}$ is to be omitted from the product. Multiplying the i-th column by r_i and dividing the last row by $r_1 \cdots r_n$, we get a Vandermonde matrix of rank $\le k$. Hence $(|r_1|, \ldots, |r_k|)$ has at most k distinct entries. Since all entries are nonzero, it follows that $\mathbf{r} \in \mathcal{H}_k(D_n)$. ∎

The proof actually gives stronger implications for the B_n-case.

Corollary 7. *Let $1 \leq k \leq n-2$. Then every nonempty B_n-invariant, k-sparse variety X meets $\mathcal{H}_k(D_n)$.*

Proof. For $X = X_k(\mathbf{q})$, we can assume that $\mathbf{q} = (q_1, \ldots, q_l, 0, \ldots, 0)$ with $q_i \neq 0$ for $1 \leq i \leq l$ and l maximal. If $l \leq k$, then $\mathbf{q} \in \mathcal{H}_k(D_n)$ and we are done. So assume that $k < l \leq n$. We distinguish two cases: If $l \leq n-1$, let $f = e_{l+1}$ and $\mathbf{r} \in X_k(\mathbf{q})$ be arbitrary. If $l = n$, let $f = e_n$ and $\mathbf{r} \in X_k(\mathbf{q})$ be a maximizer of $|f|$. The corresponding Jacobians (4) and (5) for $s_2, \ldots, s_{2k}, f$ at $\mathbf{r}$ yield the claim. ☐

3. Real orbit spaces and reflection arrangements

The reflection arrangement $\mathcal{H}$ decomposes V into relatively open polyhedral cones. The closure σ of a full-dimensional cone in this decomposition serves as a **fundamental domain**: For every $\mathbf{p} \in V$ the orbit $G\mathbf{p}$ meets σ in a unique point; see [11, Thm. 1.12]. On the other hand, the basic invariants define an **orbit map** $\pi : V \to \mathbb{R}^n$ given by $\pi(\mathbf{x}) = (\pi_1(\mathbf{x}), \ldots, \pi_n(\mathbf{x}))$. The basic invariants separate orbits, that is, $\pi(\mathbf{p}) = \pi(\mathbf{q})$ if and only if $\mathbf{q} \in G\mathbf{p}$ for all $\mathbf{p}, \mathbf{q} \in V$. The image $\mathcal{S} := \pi(V)$ is homeomorphic to V/G, and, by abuse of terminology, we call $\mathcal{S}$ the **real orbit space**. Since π is an algebraic map, $\mathcal{S}$ is semialgebraic (with an explicit description given in [14]). Restricted to σ the map $\pi|_\sigma : \sigma \to \mathcal{S}$ is a homeomorphism by [14, Prop. 0.4]. Moreover,

$$\pi^{-1}(\partial\mathcal{S}) \;=\; \bigcup_{\mathbf{p} \in \partial\sigma} G\mathbf{p} \;=\; \mathcal{H}_{n-1}(G), \tag{6}$$

where $\partial\mathcal{S}$ denotes the boundary of $\mathcal{S}$ and where the second equality follows from [11, Thm. 1.12]. Observe that neither the orbit space $\mathcal{S}$ nor the fundamental domain σ is uniquely determined by G.

In terms of the orbit map, Conjecture 1 can be put in a more general context. For $J \subseteq [n] := \{1, \ldots, n\}$, let us write $\pi_J(\mathbf{x}) = (\pi_i(\mathbf{x}) : i \in J)$. For given J, we can ask for the smallest $0 \leq t \leq n$ such that $\pi_J(V) \;=\; \pi_J(\mathcal{H}_t)$.

Proposition 8. *Let G be an irreducible and essential reflection group. Then Conjecture 1 is true for G if and only if for $J = \{1, \ldots, k\}$,*

$$\pi_J(V) \;=\; \pi_J(\mathcal{H}_k).$$

Proof. For $\mathbf{q} \in V$, we have $X_k(\mathbf{q}) = \pi_J^{-1}(\pi_J(\mathbf{q}))$. Hence, $X_k(\mathbf{q}) \cap \mathcal{H}_k \neq \varnothing$ for $\mathbf{q} \in V$ if and only if there is some $\mathbf{p} \in \mathcal{H}_k$ such that $\pi_J(\mathbf{q}) = \pi_J(\mathbf{p})$. ☐

A generalization of Theorem 1 to *J-sparse* symmetric polynomials $f \in \mathbb{R}[\pi_i : i \in J]$ was considered in [16]. The correspondence given in Proposition 8 also shows that the dimensions of strata in Conjecture 1 are best possible.

Proposition 9. *Let $J \subseteq [n]$ and $0 \leq t \leq n$ such that $\pi_J(V) \;=\; \pi_J(\mathcal{H}_t)$. Then $t \geq |J|$.*

Proof. The set $\pi_J(V)$ is the projection of the real orbit space $\mathcal{S}$ onto the coordinates indexed by J and hence is of full dimension $|J|$. By invariance of dimension, this implies that $t = \dim \mathcal{H}_t \geq |J|$. ☐

For the next result recall that, by definition, $G \subset O(V)$ and hence $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$ is an invariant of G.

Lemma 10. *Let G be a finite reflection group and $\pi_1, \ldots, \pi_n$ a choice of basic invariants such that $\pi_i(\mathbf{x}) = \|\mathbf{x}\|^2$ for some i. Then the orbit space $\mathcal{S} = \pi(V)$ is line-free; that is, if $L \subseteq V$ is an affine subspace such that $L \subseteq \mathcal{S}$, then L is a point.*

Proof. Since $\pi_i(\mathbf{x}) = \|\mathbf{x}\|^2 \geq 0$ for all $\mathbf{x} \in V$, the linear function $\ell(\mathbf{y}) = y_i$ is nonnegative on $\mathcal{S} \subset \mathbb{R}^n$. Hence, if $L \subseteq \mathcal{S}$ is an affine subspace, then ℓ is constant on L. Let $\sigma \subseteq V$ be a fundamental domain for G. Then $L = \mathcal{S} \cap L$ is homeomorphic to $\hat{L} := \{\mathbf{p} \in \sigma : \|\mathbf{p}\|^2 = c\}$ for some $c \geq 0$. This implies that L is compact, which proves the claim. $\square$

The following result is a slightly stronger but more technical extension of Theorem 3.

Theorem 11. *Let G be an essential reflection group with a choice of basic invariants $\pi_1, \ldots, \pi_n$. Let $f \in \mathbb{R}[V]^G$ be an invariant polynomial such that f is at most linear in π_k for some k. Then $\mathrm{V}_{\mathbb{R}}(f) \neq \varnothing$ if and only if $\mathrm{V}_{\mathbb{R}}(f) \cap \mathcal{H}_{n-1} \neq \varnothing$.*

If $f_1, \ldots, f_m$ are invariant polynomials that do not depend on π_j for some fixed j, then we can apply Theorem 11 to $f = f_1^2 + \cdots + f_m^2$, which then directly implies Theorem 3.

Proof. Without loss of generality, we can assume that $f(0) < 0$. Since $\mathcal{H}_{n-1}$ is path connected, it suffices to show that there is a point $\mathbf{p}_+ \in \mathcal{H}_{n-1}(G)$ with $f(\mathbf{p}_+) \geq 0$.

We can assume that $\pi_1 = \|\mathbf{x}\|^2$. Indeed, since G is essential, all basic invariants have degree at least 2 and $\|\mathbf{x}\|^2$ is a linear combination of the degree 2 basic invariants. Let $\mathbf{p} \in \mathrm{V}_{\mathbb{R}}(f)$ and define $K = \{\mathbf{q} : \pi_1(\mathbf{q}) = \pi_1(\mathbf{p})\}$, the sphere centered at the origin that contains $\mathbf{p}$. The function f attains its maximum over K in a closed set $M \subseteq K$. We claim that $M \cap \mathcal{H}_{n-1} \neq \varnothing$. Let $\mathbf{p}_0$ be a point in M.

We may pass to the real orbit space $\mathcal{S} = \pi(V)$ associated to G and $\pi_1, \ldots, \pi_n$ and consider the compact set $\overline{K} := \pi(K) = \{\mathbf{y} \in \mathcal{S} : y_1 = \pi_1(\mathbf{p})\}$. We can write $f = F(\pi_1, \ldots, \pi_n)$ for some $F \in \mathbb{R}[y_1, \ldots, y_n]$. In this setting, our assumption states that F is at most linear in y_k. If $\mathbf{p}_0 \in V \setminus \mathcal{H}_{n-1}$, then, by (6), $\overline{\mathbf{p}}_0 := \pi(\mathbf{p}_0)$ is in the interior of $\mathcal{S}$ and hence in the relative interior of $\overline{K}$. Let $L = \{\overline{\mathbf{p}}_0 + te_k : t \in \mathbb{R}\}$ be the affine line through $\overline{\mathbf{p}}_0$ in direction e_k. Restricted to L, the polynomial F has degree at most 1. By Lemma 10 and our choice of $\overline{K}$, the line L meets $\partial\overline{K}$ in two points $\overline{\mathbf{p}}_-, \overline{\mathbf{p}}_+$ and $F(\overline{\mathbf{p}}_-) \leq F(\overline{\mathbf{p}}_0) \leq F(\overline{\mathbf{p}}_+)$. This implies that $\pi^{-1}(\overline{\mathbf{p}}_+) \subseteq M$ and, since $\partial\overline{K} \subseteq \partial\mathcal{S}$, equation (6) shows that $\pi^{-1}(\overline{\mathbf{p}}_+) \subseteq \mathcal{H}_{n-1}$. $\square$

The assumption in Theorem 11 that G is essential is essential. For example, let $G = B_n$ act on $V = \mathbb{R}^n \times \mathbb{R}$ by fixing the last coordinate. A set of basic invariants is given by $\pi_1(\mathbf{x}, x_{n+1}) = x_{n+1}$ and $\pi_i(\mathbf{x}, x_{n+1}) = s_{2i-2}(\mathbf{x})$ for $i = 2, \ldots, n+1$. Pick $\mathbf{p} \in \mathbb{R}^n$ with all coordinates positive and distinct. The variety

$$X = \{(\mathbf{x}, x_{n+1}) \in V : s_{2i}(\mathbf{x}) = s_{2i}(\mathbf{p}) \text{ for } i = 1, \ldots, n\} \tag{7}$$

is defined over $\mathbb{R}[\pi_2, \ldots, \pi_{n+1}]$, but is a collection of affine lines that does not meet the reflection arrangement.

We give two further applications of Theorem 11.

Corollary 12. *Let G be an essential reflection group and let $J \subset [n]$ with $|J| = n-1$. For polynomials $f, f_1, \ldots, f_m \in \mathbb{R}[\pi_i : i \in J]$, the following hold:*

(i) $\mathrm{S} = \{\mathbf{p} : f_1(\mathbf{p}) \geq 0, \ldots, f_m(\mathbf{p}) \geq 0\}$ *is nonempty if and only if* $\mathrm{S} \cap \mathcal{H}_{n-1}(G) \neq \varnothing$.

(ii) $f(\mathbf{q}) \geq 0$ *for all* $\mathbf{q} \in \mathrm{S}$ *if and only if* $f(\mathbf{q}) \geq 0$ *for all* $\mathbf{q} \in \mathrm{S} \cap \mathcal{H}_{n-1}(G)$.

Proof. For $\mathbf{q} \in \mathrm{S}$, it suffices to prove the claim for

$$X \ := \ \{\mathbf{p} \in V : \pi_j(\mathbf{p}) = \pi_j(\mathbf{q}) \text{ for } j \in J\} \ \subseteq \ S.$$

Claim (i) now follows from Theorem 11. As for (ii), assume that $\mathbf{q} \in \mathrm{S} \setminus \mathcal{H}_{n-1}$ and $f(\mathbf{q}) < 0$. Then the same argument applied to $X \cap \{\mathbf{p} : f(\mathbf{p}) = f(\mathbf{q})\}$ finishes the proof. □

If $f \in \mathbb{R}[V]^G$ has degree $\deg(f) < 2\deg(\pi_n)$, then the algebraic independence of the basic invariants implies that Theorem 11 can be applied to prove the following corollary. Under the assumption that f is homogeneous, the second part of the corollary recovers the main result of Acevedo and Velasco [1].

Corollary 13. *Let* $f \in \mathbb{R}[V]^G$ *with* $\deg(f) < 2d_n(G) = 2\deg(\pi_n)$*. Then* $\mathrm{V}_{\mathbb{R}}(f) \neq \varnothing$ *if and only if* $\mathrm{V}_{\mathbb{R}}(f) \cap \mathcal{H}_{n-1} \neq \varnothing$*. In particular,* $f \geq 0$ *on* V *if and only if* $f \geq 0$ *on* $\mathcal{H}_{n-1}$*.*

The bound on the degree is tight: For a point $\mathbf{p} \in V \setminus \mathcal{H}_{n-1}(G)$, the set of solutions to

$$f(\mathbf{x}) \ := \ \sum_{i=1}^{n} (\pi_i(\mathbf{x}) - \pi_i(\mathbf{p}))^2 \ = \ 0$$

is exactly $G\mathbf{p}$, which does not meet $\mathcal{H}_{n-1}(G)$. The defining polynomial $f(\mathbf{x})$ is of degree exactly $2\deg(\pi_n)$. Theorem 11 also allows us to prove Theorem 2 for groups of low rank.

Proof of Theorem 2 *for* $\operatorname{rank}(G) \leq 3$. For $k = \operatorname{rank}(G)$, there is nothing to prove. For $k = 1$, we observe that $X_1(\mathbf{p})$ is the sphere through $\mathbf{p}$, which meets the arrangement $\mathcal{H}_1(G)$ of lines through the origin. Thus, the only nontrivial case is $\operatorname{rank}(G) = 3$ and $k = \operatorname{rank}(G) - 1 = 2$. This is covered by Theorem 3. □

Let G be an essential reflection group of rank ≥ 4. Since G acts on V by orthogonal transformations, we have that $\pi_1(\mathbf{x}) = \|\mathbf{x}\|^2$ and $X_k(\mathbf{p})$ is a subvariety of a sphere centered at the origin. Since the basic invariants are homogeneous, we may assume that $\pi_1(\mathbf{p}) = 1$ and hence $X_k(\mathbf{p}) \subseteq S^{n-1} = \{\mathbf{x} \in V : \|\mathbf{x}\| = 1\}$. To prove Theorem 2 for $k = 2$ we can proceed as follows. Let $\delta_{\min}$ and $\delta_{\max}$ be the minimum and maximum of π_2 over S^{n-1}. Then it suffices to find points $\mathbf{p}_{\min}, \mathbf{p}_{\max} \in \mathcal{H}_2(G) \cap S^{n-1}$ with $\pi_2(\mathbf{p}_{\min}) = \delta_{\min}$ and $\pi_2(\mathbf{p}_{\max}) = \delta_{\max}$. Indeed, since $\mathcal{H}_2(G)$ is connected (for $\operatorname{rank}(G) \geq 3$), this shows that $\pi_J(V) = \pi_J(\mathcal{H}_2(G))$ for $J = \{1, 2\}$, which, by Proposition 8, then proves the claim. For the group F_4, we can implement this strategy.

Proof of Theorem 2 *for* F_4. Since F_4 is of rank 4, we need to consider only the case $k = 2$ and can use the strategy outlined above. Let $\delta_{\min}$ and $\delta_{\max}$ be the minimum and maximum of π_2 over S^3. An explicit description of π_2 for F_4 is

$$\pi_2(\mathbf{x}) \ = \sum_{1 \leq i < j \leq 4} (x_i + x_j)^6 + (x_i - x_j)^6;$$

see, for example, Mehta [13] or [12, Table 5]. The points $\mathbf{p} = (1, 0, 0, 0)$ and $\mathbf{p}' = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)$ are contained in $\mathcal{H}_1(F_4) \subseteq \mathcal{H}_2(F_4)$ and take values $\pi_2(\mathbf{p}) = 1$ and $\pi_2(\mathbf{p}') = \frac{3}{2}$. We claim that these values are exactly $\delta_{\min}$ and $\delta_{\max}$, respectively.

Note that $\pi_2(\mathbf{x}) = g(x_1^2, x_2^2, x_3^2, x_4^2)$ for

$$g(\mathbf{y}) \ = \ 5s_1(\mathbf{y}) \cdot s_2(\mathbf{y}) - 4s_3(\mathbf{y}).$$

Let $\Delta_3 = \{\mathbf{x} \in \mathbb{R}^4 : x_1, \ldots, x_4 \geq 0, x_1 + \cdots + x_4 = 1\}$ be the standard 3-simplex. We have that $\rho(S^3) = \Delta_3$ where $\rho(x_1, \ldots, x_4) := (x_1^2, \ldots, x_4^2)$. Hence,

$$\delta_{\max} \; = \; \max\{g(\mathbf{p}) : \mathbf{p} \in \Delta_3\} \quad \text{and} \quad \delta_{\min} \; = \; \min\{g(\mathbf{p}) : \mathbf{p} \in \Delta_3\}.$$

Now, D_4 is a subgroup of F_4 and $\pi_2 \in \mathbb{R}[s_2, s_4, s_6]$ and does not depend on $e_4(\mathbf{x})$. By Theorem 2 for D_4, the varieties $S^3 \cap \{\pi_2(\mathbf{x}) = \delta_{\min}\}$ and $S^3 \cap \{\pi_2(\mathbf{x}) = \delta_{\max}\}$ both meet $\mathcal{H}_3(D_4)$. Hence, it suffices to minimize or maximize $g(\mathbf{x})$ over

$$\Delta_3 \cap \{\mathbf{x} \in \mathbb{R}^4 : x_1 = x_2\}.$$

This leaves us with the (standard) task to maximize and minimize a bivariate polynomial $g'(s,t)$ of degree 3 over a triangle. In the plane, the polynomial has 3 critical points with values $1, \frac{11}{9}, \frac{11}{9}$. On the boundary, the extreme values are attained at the points given above. $\square$

For the rank-4 reflection group H_4, the invariant $\pi_2(\mathbf{x})$ is a polynomial of degree 12 in four variables; see, for example, [12, Table 6]. Since $(B_1)^4$ is a reflection subgroup of F_4, π_2 is a polynomial in the squares $x_1^2, \ldots, x_4^2$, and, following the argument in the proof above, we are left with minimizing and maximizing a degree-6 polynomial $g(\mathbf{x})$ over the simplex Δ_3. However, finding the critical points is not easy and an extra computational challenge is the fact that $g(\mathbf{x})$ is a polynomial with coefficients in $\mathbb{Q}(\sqrt{5})$. GloptiPoly [10] *numerically* computes $\delta_{\min} = -\frac{5}{16}$ and $\delta_{\max} = 1$. These values are attained at $\mathbf{p}_{\min} = \frac{1}{\sqrt{2}}(1,1,0,0)$ and $\mathbf{p}_{\max} = (1,0,0,0)$, respectively, and both points lie in $\mathcal{H}_2((B_1)^4) \subseteq \mathcal{H}_2(H_4)$. This is strong evidence for the validity of Conjecture 1 for H_4 but, of course, not a rigorous proof.

4. Strata of higher codimension

We have seen in the previous section that every nonempty $(n-1)$-sparse variety meets the hyperplane arrangement $\mathcal{H}_{n-1}$. In this section want to extend this result to k-sparse varieties for $k < n-1$. This case is considerably more difficult, but we can make good use of the techniques and ideas developed in Section 3.

Let G be an essential finite reflection group acting on $V \cong \mathbb{R}^n$. Consider a G-invariant k-sparse variety X with $k < n$. If X is nonempty, then Theorem 11 yields that for some reflection hyperplane $H \in \mathcal{H}$ the variety $X' := X \cap H$ is nonempty. An inductive argument could now replace G by some other reflection subgroup $G' \subseteq G$ that fixes H. If X' remains sparse with respect to G' we can again apply Theorem 11 to obtain a point $\mathbf{p} \in \mathcal{H}_{n-2}(G') \subseteq \mathcal{H}_{n-2}(G)$. However, the results obtained using this strategy are far from optimal. We will briefly illustrate this for $G = \mathfrak{S}_n$: Let X be a nonempty k-sparse $\mathfrak{S}$-invariant variety for $k < n$. The largest subgroup of $\mathfrak{S}_n$ that fixes a given reflection hyperplane $H \in \mathcal{H}$ is $G' \cong \mathfrak{S}_{n-2} \times \mathfrak{S}_2$. Hence Theorem 11 only applies for $X' = X \cap H$ and G' if $k = d_k(G) < d_n(G') = n-2$, in other words, if the original variety X is $(k-3)$-sparse. Inductively, this yields that every nonempty k-sparse $\mathfrak{S}_n$-invariant variety meets $\mathcal{H}_l$ where $l = \lfloor \frac{n+k}{2} \rfloor$. However, applying the above method to the exceptional types gives nontrivial bounds.

Proposition 14. *Let $6 \leq n \leq 8$. Then every nonempty 2-sparse E_n-invariant variety intersects $\mathcal{H}_{n-2}(E_n)$.*

Proof. We exemplify the argument for the case $n = 8$. Let X be a nonempty 2-sparse E_8-invariant variety. By Theorem 11 we find a point $\mathbf{p} \in X \cap \mathcal{H}_7(E_8)$.

The orbit of $\mathbf{p}$ meets every hyperplane in $\mathcal{H}(E_8)$ (see [11, Sect. 2.10]), and hence we may assume that $\mathbf{p}$ lies on the hyperplane $H = \{\mathbf{x} \in \mathbb{R}^8 : x_1 = x_2\}$. Consider the subgroup $G' \cong D_6 \subset E_8$ acting essentially on the coordinates $x_3, \ldots, x_8$. Since $d_6(D_6) = 10 > 8 = d_2(E_8)$, we can apply Theorem 11 to finish the proof. □

By restricting the class of invariant polynomials, we obtain better bounds than those in Proposition 14. In the following, a point $\mathbf{p}$ is called ***G*-general** if it does not lie on any reflection hyperplane of G, and hence $|G\mathbf{p}| = |G|$.

Definition 15. For a positive integer d, let $\delta_G(d)$ be the largest number ℓ such that for every $p \in \mathcal{H}_{\ell+1}$ there is a reflection subgroup $G' \subseteq G$ such that p is G'-general and $2d_n(G') > d$. Moreover, we define $\sigma_G(k) := \delta_G(2d_k(G))$. That is, $\sigma_G(k)$ is the largest $0 \le \ell \le d$ such that for every $p \in \mathcal{H}_{\ell+1}$, there is a reflection subgroup $G' \subseteq G$ such that p is G'-general and $d_n(G') > d_k(G)$.

We call an invariant polynomial $f \in \mathbb{R}[V]^G$ ***G*-finite** if either $V_\mathbb{R}(f) = \varnothing$ or if there is a point $\mathbf{p} \in V_\mathbb{R}(f)$ such that f has finitely many extreme points restricted to the sphere $K = \{\mathbf{q} \in V : \|\mathbf{q}\| = \|\mathbf{p}\|\}$.

Theorem 16. *Let $f \in \mathbb{R}[V]^G$ be a G-finite polynomial and $X = \mathrm{V}_\mathbb{R}(f)$. If $f \in \mathbb{R}[\pi_1, \ldots, \pi_k]$, then*

$$X \neq \varnothing \quad \text{if and only if} \quad X \cap \mathcal{H}_{\sigma_G(k)} \neq \varnothing.$$

If $d = \deg(f)$, then

$$X \neq \varnothing \quad \text{if and only if} \quad X \cap \mathcal{H}_{\delta_G(d)} \neq \varnothing.$$

Proof. We give a proof only for the second result. The proof of the first is analogous. Suppose $\mathrm{V}_\mathbb{R}(f) \neq \varnothing$. We may assume that $f(0) \le 0$, and, since $\mathcal{H}_{\delta_G(d)}$ is connected, it suffices to show that there is some point $\mathbf{p}_+ \in \mathcal{H}_{\delta_G(d)}$ with $f(\mathbf{p}_+) \ge 0$.

By assumption, there is a zero $\mathbf{p}_0 \in \mathrm{V}_\mathbb{R}(f)$ such that f has only finitely many extreme points restricted to $K = \{\mathbf{q} : \|\mathbf{q}\| = \|\mathbf{p}_0\|\}$. Let $\mathbf{p}_+ \in K$ be a point maximizing f over K and hence $f(\mathbf{p}_+) \ge f(\mathbf{p}_0) \ge 0$. We claim that $\mathbf{p}_+ \in \mathcal{H}_{\delta_G(d)}$. Otherwise, there is a reflection subgroup $G' \subset G$ such that $\mathbf{p}_+ \notin \mathcal{H}_{n-1}(G')$ and $2d_n(G') > d$. Let $\pi'_1, \ldots, \pi'_n$ be a choice of basic invariants of G' and, without loss of generality, $\pi'_1(\mathbf{x}) = \|\mathbf{x}\|^2$. Thus, $\overline{\mathbf{p}}_+ = \pi'(\mathbf{p}_+)$ is in the interior of $\mathcal{S} = \pi'(V)$. We can write $f = F(\pi'_1, \ldots, \pi_n)$ for some $F \in \mathbb{R}[y_1, \ldots, y_n]$. On the level of orbit spaces, our assumption states that restricted to $\overline{K} = \pi(L) = \{\mathbf{y} \in \mathcal{S} : y_1 = \pi_1(\mathbf{p}_+)\}$, the polynomial F has only finitely many extreme points. However, F is linear in y_n, and thus $\overline{\mathbf{p}}_+$ is a maximum only if $\overline{\mathbf{p}}_+ \in \partial\overline{K} \subseteq \partial\mathcal{S}$. This is a contradiction. □

5. Bounds from parabolic subgroups

The numbers $\delta_G(d)$ and $\sigma_G(k)$ defined in Section 4 are difficult to compute in general. In this section, we compute upper bounds on these numbers coming from parabolic subgroups. Let G be a finite irreducible reflection group. A fundamental domain $\sigma \subset V$, as defined in Section 3, is a simplicial cone of dimension $n = \dim V$. Let $H_1, \ldots, H_n \in \mathcal{H}(G)$ be the reflection hyperplanes that are facet-defining for σ and let $\Delta = \{s_1, \ldots, s_n\} \subset G$ be the corresponding reflections. For $i \neq$, we denote by $m(i, j)$ the order of the cyclic group generated by $s_i s_j$. The **Dynkin diagram** D of G is the labelled graph with vertex set $\{1, \ldots, n\}$ and edges ij whenever $m(i, j) \ge 3$ and edge labelling $m(i, j)$; see [11, Sect. 2.1] for details. A subgroup of G is **parabolic** if it is conjugate to a subgroup generated by a subset of the reflections in Δ; cf. [11, Sect. 1.10].

Lemma 17. *Fix a finite irreducible reflection group G with Dynkin diagram D. Let $D' \subset D$ be a subdiagram obtained by removing a node from D and let $H \in \mathcal{H}(G)$ be a reflection hyperplane. Then there is a parabolic subgroup $W \subset G$ with Dynkin diagram D', and H is not a reflection hyperplane of W.*

Proof. Let W be a parabolic subgroup with Dynkin diagram D'. Since every parabolic subgroup with Dynkin diagram D' is conjugate to W, it suffices to show that for every $H \in \mathcal{H}(G)$ there is a $g \in G$ such that $gH \notin \mathcal{H}(W)$.

Unless G is of type F_4, B_n or $I_2(2m)$ for $m > 1$, the hyperplanes in $\mathcal{H}(G)$ form a single G-orbit (see [11, Sect. 2.9, Sect. 2.10]). Hence, the claim follows, since $\mathcal{H}(W) \subsetneq \mathcal{H}(G)$. For F_4, the possible proper parabolic subgroups are B_3 and $A_1 \times A_2$ and the result follows by inspection. For $I_2(2m)$, there are two orbits each with $2m \geq 4$ roots, whereas the only nontrivial proper parabolic subgroup is A_1. For B_n, this follows from counting the number of elements in each of the two orbits. □

The lemma yields the following result about finite reflection groups that might be interesting in its own right.

Proposition 18. *Let G be a finite irreducible reflection group with Dynkin diagram D acting on a real vector space V. For $k \geq 1$, let $\mathbf{p} \in \mathcal{H}_k \setminus \mathcal{H}_{k-1}$ and $D' \subset D$ be a connected subdiagram on k nodes. Then there is a parabolic subgroup $W \subset G$ with Dynkin diagram D' such that $\mathbf{p}$ is W-general.*

Proof. We argue by induction on $s = \dim V - k$. For $s = 0$, $\mathbf{p} \in V \setminus \mathcal{H}_{\dim V-1}$ and $\mathbf{p}$ is by definition G-general. Otherwise, let $D_1 \subset D$ be a subdiagram obtained by removing a leaf such that $D' \subseteq D_1$ and let H_1 be a reflection hyperplane of G containing $\mathbf{p}$. We may use Lemma 17 to obtain a parabolic subgroup W_1 with Dynkin diagram D_1 and not containing H_1 as a reflection hyperplane. In particular, $\mathbf{p}$ is contained in precisely $s-1$ linearly independent reflection hyperplanes of W_1. By induction, there is a parabolic subgroup $W \subseteq W_1$ with Dynkin diagram D' for which $\mathbf{p}$ is W-general. In particular, W is a parabolic subgroup of G, which concludes the proof. □

For $I \subseteq \Delta$, let W_I be the parabolic subgroup generated by the reflections I. We define

$$\tilde{\delta}_G(d) \ := \ \min\{|I| - 1 : I \subseteq \Delta, 2d_n(W_I) > d\},$$

and, analogously, we define $\tilde{\sigma}_G(k) := \tilde{\delta}_G(2d_k(G))$. Since G acts transitively on the set of fundamental domains, these definitions do not depend on the choice of σ. Proposition 18 implies the following bound on δ_G.

Corollary 19. $\delta_G(d) \leq \tilde{\delta}_G(d)$, *for all $d \geq 0$.*

The clear advantage is a simple way to compute upper bounds on $\delta_G(d)$ from the knowledge of parabolic subgroups of reflection groups; cf. [11]. The explicit values are given in Table 1. However, not every reflection subgroup is parabolic (e.g. $D_n \subseteq B_n$, $I_2(m) \subseteq I_2(2m)$). Nevertheless, we conjecture that $\delta_G(d)$ is attained at a parabolic subgroup.

Conjecture 2. *For any finite reflection group G, $\delta_G(d) \ = \ \tilde{\delta}_G(d)$ for all d.*

G	d	$\tilde{\delta}_G(d)$	W	$d_n(W)$	k	$\tilde{\sigma}_G(k)$
$A_{n-1}/\mathfrak{S}_n$	$0 — 2n-1$	$\lfloor d/2 \rfloor$	$A_{\lfloor d/2 \rfloor}$	$\lfloor d/2 \rfloor + 1$	$0 — n-1$	k
B_n	$0 — 4n-1$	$\lfloor d/4 \rfloor$	$B_{\lfloor d/4 \rfloor + 1}$	$2(\lfloor d/4 \rfloor + 1)$	$0 — n-1$	k
D_n	$0 — 4n-5$	$\lfloor d/4 \rfloor + 1$	$D_{\lfloor d/4 \rfloor + 2}$	$2(\lfloor d/4 \rfloor + 1)$	$0 — \lfloor \frac{n}{2} \rfloor$	$k+1$
					$\lfloor \frac{n}{2} \rfloor + 1 — n$	k
$I_2(m)$	$1 — 2m-1$	1	$I_2(m)$	m	1	1
E_6	1 — 5	1	A_2	3	1	1
	6 — 7	2	A_3	4	2	3
	8 — 11	3	D_4	6	3	4
	12 — 15	4	D_5	8	4	5
	16 — 23	5	E_6	12	5	5
E_7	1 — 5	1	A_2	3	1	1
	6 — 7	2	A_3	4	2	4
	8 — 11	3	D_4	6	3	5
	12 — 15	4	D_5	8	4	5
	16 — 23	5	E_6	12	5	6
	24 — 35	6	E_7	18	6	6
E_8	1 — 5	1	A_2	3	1	1
	6 — 7	2	A_3	4	2	5
	8 — 11	3	D_4	6	3	6
	12 — 15	4	D_5	8	4	6
	16 — 23	5	E_6	12	5	7
	24 — 35	6	E_7	18	6	7
	36 — 59	7	E_8	30	7	7
F_4	1 — 7	1	B_2	4	1	1
	8 — 11	2	B_3	6	2	3
	12 — 23	3	F_4	12	3	3
H_3	1 — 9	1	$I_2(5)$	5	1	1
	10 — 19	2	H_3	10	2	2
H_4	1 — 9	1	$I_2(m)$	5	1	1
	10 — 19	2	H_3	10	2	3
	20 — 59	3	H_4	30	3	3

TABLE 1. Computation for $\tilde{\delta}_G(d)$ and $\tilde{\sigma}_G(k)$. $a — b$ refers to the range $a, a+1, \dots, b$. The column W gives the parabolic subgroup that attains $\tilde{\delta}_G$.

6. Adjoint representations of Lie groups

In this last section, we extend some of our results to polynomials invariant under the action of a Lie group. More precisely, we consider the case of a real simple Lie group G with the adjoint action on its Lie algebra $\mathfrak{g}$. We illustrate our results for the case $G = \mathrm{SL}_n$. Its Lie algebra $\mathfrak{sl}_n$ is the vector space of real n-by-n matrices of trace 0. The adjoint action of SL_n on $\mathfrak{sl}_n$ is by conjugation: $g \in \mathrm{SL}_n$ acts on $A \in \mathfrak{sl}_n$ by $g \cdot A := gAg^{-1}$. The following description of its ring of invariants is well-known. We briefly recall the standard proof, which immediately suggests a connection to our treatment of reflection groups; cf. [7, Ch. 12.5.3].

Theorem 20. *For $n \geq 1$, $\mathbb{R}[\mathfrak{sl}_n]^G = \mathbb{R}[s_2, \dots, s_n]$, where $s_k(A) = \mathrm{tr}(A^k)$ for $k = 2, \dots, n$. Moreover, $s_2, \dots, s_n$ are algebraically independent.*

Proof. We write $D \subset \mathfrak{sl}_n$ for the set of diagonalizable matrices and we denote by $\lambda(A) = (\lambda_1, \dots, \lambda_n)$ the eigenvalues of $A \in D$. Then for any $A \in D$,

$$s_k(A) \;=\; s_k(\lambda(A)) \;=\; \lambda_1(A)^k + \lambda_2(A)^k + \cdots + \lambda_n(A)^k,$$

and $s_2, \dots, s_n$ are simply the power sums restricted to the linear subspace $\Delta \subset D$ of diagonal matrices. This shows that $s_2, \dots, s_n$ are algebraically independent. Now for a polynomial $f(\mathbf{X}) \in \mathbb{R}[\mathfrak{sl}_n]$ invariant under the action of SL_n, the restriction to $\Delta \cong \mathbb{R}^{n-1}$ is a polynomial $f(\mathbf{x})$ that is invariant under A_{n-1}. Hence

$f(\mathbf{x}) = F(s_2(\mathbf{x}), \ldots, s_n(\mathbf{x}))$ for some $F \in \mathbb{R}[y_2, \ldots, y_n]$. The polynomial $\tilde{f}(\mathbf{X}) = F(s_2(\mathbf{X}), \ldots, s_n(\mathbf{X}))$ is invariant under SL_n and agrees with f on D. Since D contains a nonempty open set, $f = \tilde{f}$ as required. □

In general, let $T \subseteq G$ be a maximal torus with Lie algebra $\mathfrak{t} \subseteq \mathfrak{g}$. If $N \subseteq G$ is the normalizer of T in G, then $W = N/T$ is a reflection group, the **Weyl group**, that acts on $\mathfrak{t}$. By Chevalley's Restriction Theorem (see [8, Lem. 7]), the restriction $\mathbb{R}[\mathfrak{g}] \to \mathbb{R}[\mathfrak{t}]$ extends to an isomorphism of invariant rings. This yields that $\mathbb{R}[\mathfrak{g}]^G$ is generated by homogeneous and algebraically independent polynomials $\pi_1, \ldots, \pi_m$ whose restriction to $\mathfrak{t}$ gives a set of basic invariants for W. We call a G-invariant variety $X \subseteq \mathfrak{g}$ k-sparse if $X = \mathrm{V}_{\mathbb{R}}(f_1, \ldots, f_m)$ for some $f_1, \ldots, f_m \in \mathbb{R}[\pi_1, \ldots, \pi_k]$. The **discriminant locus** $\mathcal{D} \subset \mathfrak{g}$ of G is the Zariski closure of the orbit of the reflection arrangement $\mathcal{H}(W) \subset \mathfrak{t}$ under G. By the result of Steinberg [17] and the restriction theorem, this is a real G-invariant hypersurface given by the vanishing of a single polynomial, called the **discriminant** of G. Since $G \cdot \mathfrak{t}$ is dense in $\mathfrak{g}$, $\mathcal{D}$ is the closure of the set of points with nontrivial stabilizer; see [9, Ch. 6]. This yields a stratification of $\mathfrak{g}$ by defining $\mathcal{D}_i$ to be the closure of the orbit of $\mathcal{H}_i$, which corresponds to the points for which the discriminant vanishes up to order $n - i$. Hence, the results from the previous sections generalize to Lie groups.

For the special orthogonal group SO_n, its Lie algebra $\mathfrak{so}_n \subset \mathfrak{sl}_n$ is the vector space of skew-symmetric n-by-n matrices on which SO_n acts by conjugation. If $n = 2k+1$, then the corresponding Weyl group is B_k and D_k if $n = 2k$. Hence $\mathbb{R}[\mathfrak{so}_{2k+1}]^{\mathrm{SO}_{2k+1}}$ is generated by $s_2(\mathbf{X}), s_4(\mathbf{X}), \ldots, s_{2k}(\mathbf{X})$. For $n = 2k$, a minimal generating set is given by $s_2(\mathbf{X}), s_4(\mathbf{X}), \ldots, s_{2n-2}(\mathbf{X})$ and the Pfaffian $\mathrm{pf}(\mathbf{X}) = \sqrt{\det \mathbf{X}}$. Theorem 2 yields the following.

Theorem 21. *Let $G \in \{\mathrm{SL}_k, \mathrm{SO}_k : k \in \mathbb{Z}_{\geq 1}\}$ and let $X \subseteq \mathfrak{g}$ be G-invariant and k-sparse. If X is nonempty it intersects $\mathcal{D}_k$.*

For suitable Lie groups G, Theorem 21 gives a first relation between real varieties invariant under the action of G and the discriminant locus, and Conjecture 1 is reasonable for this setting. It would be very interesting to explore this connection further.

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Fachbereich Mathematik und Informatik, Freie Universität Berlin, 14195 Berlin, Germany

E-mail address: tfriedl@zedat.fu-berlin.de

Aalto Science Institute, P.O. Box 11000, FI-00076 Aalto, Finland

E-mail address: cordian.riener@aalto.fi

Institut für Mathematik, Goethe-Universität Frankfurt, 60325 Frankfurt, Germany

E-mail address: sanyal@math.uni-frankfurt.de