A NOTE ON SURGERY OBSTRUCTIONS
AND HYPERBOLIC INTEGER HOMOLOGY SPHERES

JENNIFER HOM AND TYE LIDMAN

(Communicated by David Futer)

ABSTRACT. Auckly gave two examples of irreducible integer homology spheres (one toroidal and one hyperbolic) which are not surgery on a knot in the three-sphere. Using Heegaard Floer homology, the authors and Karakurt provided infinitely many small Seifert fibered examples. In this note, we extend those results to give infinitely many hyperbolic examples, as well as infinitely many examples with arbitrary JSJ decomposition.

Lickorish [Lic62] and Wallace [Wal60] proved that any closed, oriented three-manifold can be obtained by surgery on a link in the three-sphere. Thus, a natural question to ask is which manifolds can be described via the simplest possible surgery description, i.e., as surgery on a knot. Irreducible integer homology spheres are a particularly interesting family to consider, since the simplest obstructions (e.g., [BL90]) to being surgery on a knot all vanish. Note that Gordon and Luecke [GL89] showed that a reducible integer homology sphere can never be surgery on a knot. Auckly [Auc97] provided the first two examples (one toroidal and one hyperbolic) of irreducible integer homology spheres which are not surgery on a knot, answering [Kir95, Problem 3.6(C)] in the affirmative. For over 15 years, these two manifolds were the only known examples in the literature, although Auckly certainly knew how to construct infinitely many examples [Auc17]. The authors and Karakurt [HKL16] provided an infinite family of small Seifert fibered integer homology spheres which are not surgery on a knot. In this note, we refine that result to give new examples of irreducible homology spheres which are not surgery on a knot.

Theorem 1. *There exist infinitely many hyperbolic integer homology spheres which are not surgery on a knot in S^3. Similarly, one can construct infinitely many examples with arbitrarily complicated JSJ decomposition. Finally, one can arrange that none of these examples are rationally homology cobordant.*

Theorem 1 is a consequence of the following results about the behavior of Heegaard Floer homology under Dehn surgery. The following results were originally proved for knots in S^3, but hold more generally for knots in arbitrary integer homology sphere L-spaces.

Proposition 2 ([Gai17, Theorem 3]). *Let Y be an integer homology sphere L-space and $K \subset Y$ a genus one knot. Then $U^2 \cdot HF_{red}(Y_{1/n}(K)) = 0$ for any n.*

Proposition 3. *Let Y be an integer homology sphere L-space and $K \subset Y$ a genus one knot. Then $|d(Y) - d(Y_{1/n}(K))| \leq 2$ for any n.*

Proof. By [NW15, Proposition 1.6], we have that

$$(1) \qquad\qquad d(Y_{1/n}(K)) - d(Y) = -2V_0 \text{ for } n > 0,$$

where V_0 is the integer-valued knot invariant defined in [Ras03, Definition 7.1] (see also [NW15, Section 2.2]).

Since being an integer homology sphere L-space is preserved under orientation reversal, by (1), it suffices to prove that $V_0 \leq 1$ for any genus one knot in an integer homology sphere L-space. This is well-known for knots in S^3 (see, for instance, [Ras04, Theorem 2.3]), and the same arguments apply in general. $\qquad\square$

Proof of Theorem 1. Let $Y_j = \#_j \Sigma(2, 3, 5)$ for $j \geq 6$, where $\Sigma(2, 3, 5)$ is oriented as the boundary of the positive-definite E_8 plumbing. Recall that Y_j is an L-space and that $d(Y_j) = -2j$. Consider a genus one knot K_j in Y_j and let $Y_{j,n}$ be the manifold obtained by $1/n$-surgery on K_j. By Proposition 3, $d(Y_{j,n}) \leq -10$, and by Proposition 2, $U^2 \cdot HF_{red}(Y_{j,n}) = 0$. The same arguments used to prove Theorem 1.2 of [HKL16] imply that if $Y_{j,n}$ is surgery on a knot in S^3 with $d(Y_{j,n}) \leq -10$, then $U^2 \cdot HF_{red}(Y_{j,n}) \neq 0$. Therefore, we obtain a contradiction.

It is now straightforward to construct $Y_{j,n}$ of the desired forms. Indeed, by [Tsu03, Proposition 5.4], there exists a hyperbolic knot K_j in Y_j which is genus one. By Thurston's hyperbolic Dehn surgery theorem, $Y_{j,n}$ will be hyperbolic for all but finitely many n. For a more complicated JSJ decomposition, simply take the knot K_j and perform any number of cables and/or Whitehead doubles of this knot, making sure that the last step is a Whitehead double. The resulting knot will be genus one, and the number of hyperbolic pieces (respectively Seifert pieces) of the knot exterior will be determined by the number of Whitehead doubling operations (respectively cabling operations). Again, $Y_{j,n}$ will have the same JSJ decomposition as the satellite knot exterior for all but finitely many n.

Finally, by taking $j \to \infty$, we can guarantee that the different $Y_{j,n}$ have different d-invariants and thus live in different rational homology cobordism classes. $\qquad\square$

Remark 4. A similar argument can be used to show that for any integer homology sphere Y, there are infinitely many hyperbolic and irreducible toroidal homology spheres that cannot be obtained by surgery on a knot in Y. This comes from combining Propositions 2 and 3 with the arguments used in the proof of [HKL16, Theorem 1.7].

References

[Auc97] David Auckly, *Surgery numbers of 3-manifolds: a hyperbolic example*, Geometric topology (Athens, GA, 1993), AMS/IP Stud. Adv. Math., vol. 2, Amer. Math. Soc., Providence, RI, 1997, pp. 21–34. MR1470719

[Auc17] ———, private communication, 2017.

[BL90]   Steven Boyer and Daniel Lines, *Surgery formulae for Casson's invariant and extensions to homology lens spaces*, J. Reine Angew. Math. **405** (1990), 181–220. MR1041002

[Gai17]   Fyodor Gainullin, *The mapping cone formula in Heegaard Floer homology and Dehn surgery on knots in S3*, Algebr. Geom. Topol. **17** (2017), no. 4, 1917–1951, DOI 10.2140/agt.2017.17.1917. MR3685598

[GL89]   C. McA. Gordon and J. Luecke, *Knots are determined by their complements*, J. Amer. Math. Soc. **2** (1989), no. 2, 371–415, DOI 10.2307/1990979. MR965210

[HKL16]   Jennifer Hom, Çağrı Karakurt, and Tye Lidman, *Surgery obstructions and Heegaard Floer homology*, Geom. Topol. **20** (2016), no. 4, 2219–2251, DOI 10.2140/gt.2016.20.2219. MR3548466

[Kir95]   Robion Kirby, *Problems in low-dimensional topology*, 1995, `https://math.berkeley.edu/~kirby/problems.ps.gz`.

[Lic62]   W. B. R. Lickorish, *A representation of orientable combinatorial 3-manifolds*, Ann. of Math. (2) **76** (1962), 531–540, DOI 10.2307/1970373. MR0151948

[NW15]   Yi Ni and Zhongtao Wu, *Cosmetic surgeries on knots in S^3*, J. Reine Angew. Math. **706** (2015), 1–17, DOI 10.1515/crelle-2013-0067. MR3393360

[Ras03]   Jacob A. Rasmussen, *Floer homology and knot complements*, ProQuest LLC, Ann Arbor, MI, Thesis (Ph.D.)–Harvard University, 2003. MR2704683

[Ras04]   Jacob Rasmussen, *Lens space surgeries and a conjecture of Goda and Teragaito*, Geom. Topol. **8** (2004), 1013–1031, DOI 10.2140/gt.2004.8.1013. MR2087076

[Tsu03]   Yukihiro Tsutsumi, *Universal bounds for genus one Seifert surfaces for hyperbolic knots and surgeries with non-trivial JSJT-decompositions*, Proceedings of the Winter Workshop of Topology/Workshop of Topology and Computer (Sendai, 2002/Nara, 2001), Interdiscip. Inform. Sci. **9** (2003), no. 1, 53–60, DOI 10.4036/iis.2003.53. MR2023107

[Wal60]   Andrew H. Wallace, *Modifications and cobounding manifolds*, Canad. J. Math. **12** (1960), 503–528, DOI 10.4153/CJM-1960-045-7. MR0125588

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332
Email address: `hom@math.gatech.edu`

Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27607
Email address: `tlid@math.ncsu.edu`