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A NOTE ON BAND-LIMITED MINORANTS OF AN EUCLIDEAN BALL

FELIPE GONÇALVES

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Abstract. We study the Beurling-Selberg problem of finding band-limited L^1-functions that lie below the indicator function of an Euclidean ball. We compute the critical radius of the support of the Fourier transform for which such construction can have a positive integral.

1. Introduction

For a given $r > 0$ we denote by $B^d(r)$ the closed Euclidean ball in $\mathbb{R}^d$ centered at the origin with radius $r > 0$. We simply write B^d when $r = 1$. Define the following quantity:

$$\beta(d, r) = \sup_F \int_{\mathbb{R}^d} F(\boldsymbol{x})dx, \tag{1.1}$$

where the supremum is taken among functions $F \in L^1(\mathbb{R}^d)$ such that:

(1) The Fourier transform of $F(x)$,

$$\widehat{F}(\xi) = \int_{\mathbb{R}^d} F(x)e^{2\pi i x\cdot\xi}\mathrm{d}x,$$

is supported in $B^d(r)$;

(2) $F(x) \leq \mathbf{1}_{B^d}(x)$ for all $x \in \mathbb{R}^d$.

We call such a function $\beta(d, r)$-admissible. A trivial observation is that $F \equiv 0$ is $\beta(d, r)$-admissible, hence $\beta(d, r) \geq 0$. Heuristically, such function $F(x)$ should exist and its mass should be close to $\mathrm{vol}(B^d)$ when r is large. On the other hand, if r is small, the mass of $F(x)$ should be close to zero and a critical $r_d > 0$ should exist such that no function can beat the identically zero function for $r \leq r_d$. For this reason we define

$$r_d = \inf\{r > 0 : \beta(d, r) > 0\}$$

and it is this critical radius that we want to study in this paper.

The problem stated in (1.1) has its origins with Beurling and Selberg which studied one-sided band-limited approximations for many different functions other than indicator functions with the purpose of using them to derive sharp estimates in analytic number theory (see the introduction of [9] for a nice first view). Although Selberg was one of the first to study the higher dimensional problem, it was first systematically analyzed by Holt and Vaaler in the remarkable paper [7]. They were able to construct non-zero $\beta(d, r)$-admissible functions for any $r > 0$ and, most

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importantly, they established a fascinating connection of the d-dimensional problem with the theory of Hilbert spaces of entire functions contructed by de Branges (see [1]). They reduced the higher dimensional problem, after a radialization argument, to a weighted one-dimensional problem where the weight was given by a special function of Hermite-Biehler class, which in turn allowed them to use the machinery of homogeneous de Branges spaces to attack the problem. This new connection established by Holt and Vaaler started a new way of thinking about problems of this kind and ultimately inspired Littmann to completely solve the one-dimensional problem in [8] by using a cleaver argument based on a special structure of certain de Branges spaces. Finally, using the ideas introduced by Littmann in [8], the problem of minorizing the indicator function of a symmetric interval was completely solved in [2] in the de Branges space setting.

This paper was mainly motivated by the related problem where balls are substituted by boxes $Q(r) = [-r, r]^d$ and where practically nothing is known (see [3]). The box minorant problem is harder since it is a truly higher dimensional problem, whereas for the ball we can make radial reductions that transform it in a one-dimensional problem. Another interesting similar question, connected with upper bounds for sphere packings in $\mathbb{R}^d$, is studied in [6] (see also [4]), where the author constructs a minorant $F(x) \leq \mathbf{1}_{B^d}(x)$ with Fourier transform non-negative and supported in $B^d(j_{d/2,1})$ and such that it maximizes $\widehat{F}(0)$ among all functions with these properties.[1]

1.1. **Main result.** For any given parameter $\nu > -1$ let J_ν denote the classical Bessel function of the first kind. We also denote by $\{j_{\nu,n}\}_{n\geq 1}$ its positive zeros listed in increasing order. The Bessel function of the first kind J_ν can be defined in a number of ways. We follow the treatise [10] and define it for $\nu > -1$ and $\Re(z) > 0$ by

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n}}{n!\,\Gamma(\nu+n+1)}. \tag{1.2}$$

For these values of ν, one can check that on the half space $\{\Re(z) > 0\}$ the Bessel functions defined by (1.2) satisfy the differential equation

$$z^2 J_\nu''(z) + zJ_\nu'(z) + (z^2 - \nu^2)J_\nu(z) = 0,$$

and that the following recursion relations hold:

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2J_\nu'(z),$$
$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z).$$

In particular we have $J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos(z)$ and $J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin(z)$, which implies that $j_{-1/2,1} = \pi/2$ and $j_{1/2,1} = \pi$. The following is the main result of this paper.

Theorem 1. *We have*

$$r_d = \frac{j_{d/2-1,1}}{\pi}.$$

[1]It is not the intention of this paper to give a survey of related articles on the subject, which is very rich and full of subtleties; the purpose here is to draw a straight line between what we have so far and what we want to show.

Moreover, if $j_{d/2-1,1} < \pi r < j_{d/2,1}$ *then*

$$\beta(d,r) = \frac{(2/r)^d}{|\mathbb{S}^{d-1}|}\frac{\gamma_{\pi r}}{1+\gamma_{\pi r}/d},$$

where $\gamma_{\pi r} = -\frac{\pi r J_{d/2-1}(\pi r)}{J_{d/2}(\pi r)} > 0$*. In particular we have*

$$\beta(d,r) = \frac{\pi^2 2^d}{r^{d-1}|\mathbb{S}^{d-1}|}\left(r - \frac{j_{d/2-1,1}}{\pi}\right) + O_d\left(r - \frac{j_{d/2-1,1}}{\pi}\right)^2$$

for r *close to* $\frac{j_{d/2-1,1}}{\pi}$.

Remarks.

(1) It is known that $j_{\nu,1} = \nu + 1.855757\nu^{1/3} + O(\nu^{-1/3})$ as $\nu \to \infty$ (see [5, Section 1.3]). This implies that $r_d = \frac{d}{2\pi} + \frac{1.855757}{2^{1/3}\pi}d^{1/3} + O(d^{-1/3})$ as $d \to \infty$. Heuristically, this means that if one wishes to non-trivially minorate (that is, beat the zero function) the indicator function of a ball of radius of order $\sqrt{d}$, then one needs frequencies of order at least $\sqrt{d}$.

(2) The first 5 values of r_d rounded up to 4 significant digits are the following: $r_1 = 1/2$, $r_2 = 0.7655$, $r_3 = 1$, $r_4 = 1.220$ and $r_5 = 1.431$.

(3) Explicit expressions for $\beta(d,r)$ can also be tracked from [2, Theorem 5], but they involve sums of Bessel functions evaluated at Bessel zeros that can be quite complicated to grasp. Moreover, this is the case only when πr is a zero of $J_{d/2-1}(z)$ or $J_{d/2}(z)$. If that is not the case, then writing a formula for $\beta(d,r)$ becomes pointless, since it will involve zeros of more complicated functions related to Bessel functions and this is not the purpose here.

2. Proof of Theorem 1

Step 1. The first step is to reduce the higher dimensional by considering only radial functions. We can apply [7, Lemmas 18 and 19] to reduce the d-dimensional problem to the following weighted one-dimensional problem:

$$\beta(d,r) = \frac{|\mathbb{S}^{d-1}|}{2}\sup_F \int_{\mathbb{R}} F(x)|x|^{d-1}dx, \tag{2.1}$$

where $|\mathbb{S}^{d-1}|$ denotes the surface area of the unit sphere in $\mathbb{R}^d$ and the supremum is taken among functions $F \in L^1(\mathbb{R}, |x|^{d-1}\mathrm{d}x)$ such that:

(1) $F(x)$ is the restriction to the real axis of an even entire function $F(z)$ of exponential type at most $2\pi r$, that is,

$$|F(z)| \le Ce^{2\pi r|\mathrm{Im}\, z|}, \quad z \in \mathbb{C},$$

for some constant $C > 0$;

(2) $F(x) \le \mathbf{1}_{[-1,1]}(x)$ for all $x \in \mathbb{R}$.

In this framework the problem becomes treatable with the theory of de Branges spaces of entire functions. The latter generalize the well-known Paley-Wiener spaces by using weighted norms given by Hermite-Biehler functions. In what follows we briefly review the construction of a special family of de Branges spaces called *homogeneous spaces* which were introduced by de Branges (see [1, Section 50] and [7, Section 5]). We refer to [7, Section 3] for a brief description of the general theory and also to [1, Chapter 2] for the full theory.

Step 2. Let $\nu > -1$ be a parameter and consider the real entire functions $A_\nu(z)$ and $B_\nu(z)$ given by

$$A_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}z\right)^{2n}}{n!(\nu+1)(\nu+2)\dots(\nu+n)} = \Gamma(\nu+1)\left(\tfrac{1}{2}z\right)^{-\nu} J_\nu(z)$$

and

$$B_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}z\right)^{2n+1}}{n!(\nu+1)(\nu+2)\dots(\nu+n+1)} = \Gamma(\nu+1)\left(\tfrac{1}{2}z\right)^{-\nu} J_{\nu+1}(z).$$

If we write

$$E_\nu(z) = A_\nu(z) - iB_\nu(z),$$

then $E_\nu(z)$ is a Hermite–Biehler function, that is, it satisfies the following fundamental inequality:

$$|E_\nu(\overline{z})| < |E_\nu(z)|$$

for all $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$. It is also known that this function does not have real zeros, that $E(iy) \in \mathbb{R}$ for all real y (that is, $B_\nu(z)$ is odd and $A_\nu(z)$ is even), that $E_\nu(z)$ is of bounded type in the upper-half plane (that is, $\log|E_\nu(z)|$ has a positive harmonic majorant in the upper-half plane) and $E_\nu(z)$ is of exponential type 1. We also have that

$$c|x|^{2\nu+1} \le |E_\nu(x)|^{-2} \le C|x|^{2\nu+1}$$

for all $|x| \ge 1$ and for some $c, C > 0$. The homogeneous space $\mathcal{H}(E_\nu)$ is then defined as the space of entire functions $F(z)$ of exponential type at most 1 and such that[2]

$$\int_{\mathbb{R}} |F(x)|^2 \, |E_\nu(x)|^{-2} \, dx < \infty.$$

Using standard asymptotic expansions for Bessel functions one can show that $A_\nu, B_\nu \notin \mathcal{H}(E_\nu)$. As a particular case, observing that $E_{-1/2} = e^{-iz}$ we can deduce that $\mathcal{H}(E_{-1/2})$ coincides with the Paley-Wiener space of square integrable entire functions of exponential type at most 1.

These spaces are relevant to our problem since we have the following magical identity:

$$a_\nu \int_{\mathbb{R}} |F(x)|^2 \, |E_\nu(x)|^{-2} \, \mathrm{d}x = \int_{\mathbb{R}} |F(x)|^2 \, |x|^{2\nu+1} \, \mathrm{d}x \tag{2.2}$$

for each $F \in \mathcal{H}(E_\nu)$, where $a_\nu = 2^{2\nu+1}\,\Gamma(\nu+1)^2/\pi$. For our purposes we will need an identity analogous of (2.2), but which allows us to compute integrals instead of L^2-norms. It can be derived as follows. Let $F(z)$ be an entire function of exponential type at most 2 such that $F(x) \le \mathbf{1}_{[-t,t]}(x)$ for some $t > 0$ and $F \in L^1(\mathbb{R}, |x|^{2\nu+1}\mathrm{d}x)$. Since $G_n(x) = (\frac{\sin(x/n)}{x/n})^n$ belongs to $\mathcal{H}(E_\nu)$ for large n and it converges to 1 uniformly in compact sets as $n \to \infty$, we have that $4G_n(x)^2 - F(x) \ge 0$ for all real x (if n is large and even) and this function has exponential type at most 2. This implies that $4G_n(x)^2 - F(x) = H_n(z)\overline{H_n(\overline{z})}$ for all $z \in \mathbb{C}$, for some entire

[2] As a historical note, de Branges originally defined this space in another way but, in [7, Lemma 16], the authors showed that this is an equivalent definition.

function $H_n(z)$ of exponential type at most 1 (see [1, Theorem 13]). We conclude that $H_n \in \mathcal{H}(E_\nu)$ and we obtain

$$
\begin{aligned}
a_\nu \int_{\mathbb{R}} F(x)\,|E_\nu(x)|^{-2}\mathrm{d}x &= a_\nu \int_{\mathbb{R}} \{4G_n(x)^2 - |H_n(x)|^2\}\,|E_\nu(x)|^{-2}\mathrm{d}x \\
&= \int_{\mathbb{R}} \{4G_n(x)^2 - |H_n(x)|^2\}|x|^{2\nu+1}\mathrm{d}x \qquad (2.3)\\
&= \int_{\mathbb{R}} F(x)|x|^{2\nu+1}\mathrm{d}x.
\end{aligned}
$$

Step 3. Taking $\nu = d/2-1$, we can apply the change of variables $x \mapsto x/(\pi r)$ in (2.1) and use identity (2.3) to reduce the problem of minorizing the indicator function of an Euclidean ball to the following final form:

$$\beta(d,r) = \frac{(2/r)^d}{\pi|\mathbb{S}^{d-1}|}\Lambda^-_{E_{d/2-1}}(\pi r),$$

where

$$\Lambda^-_{E_{d/2-1}}(\pi r) = \sup_F \int_{\mathbb{R}} F(x)|E_{d/2-1}(x)|^{-2}\mathrm{d}x$$

and the supremum is taken among functions $F \in L^1(\mathbb{R}, |E_{d/2-1}(x)|^{-2}\mathrm{d}x)$ such that:

(1) $F(x)$ is the restriction to the real axis of an even entire function $F(z)$ of exponential type at most 2;
(2) $F(x) \le \mathbf{1}_{[-\pi r,\pi r]}(x)$ for all $x \in \mathbb{R}$.

The above problem was completely solved in [2]. By all the previous discussion in Step 2, we can apply [2, Theorem 5 (i) and (iv)] to the function $E_{d/2-1}(z)$ (it actually can be applied to any $E_\nu(z)$) to derive that $\pi r_d = j_{d/2-1,1}$. Moreover, if $j_{d/2-1,1} < \pi r < j_{d/2,1}$, then [2, Theorem 5 (iv)] also give us that

$$\Lambda^-_{E_{d/2-1}}(\pi r) = \frac{\pi\gamma_{\pi r}}{1+\gamma_{\pi r}/d},$$

where $\gamma_{\pi r} = -\frac{\pi r J_{d/2-1}(\pi r)}{J_{d/2}(\pi r)} > 0$. A simple Taylor expansion leads to

$$\Lambda^-_{E_{d/2-1}}(\pi r) = \pi^2 r(\pi r - j_{d/2-1,1}) + O(\pi r - j_{d/2-1,1})^2$$

and we finally obtain that

$$\beta(d,r) = \frac{\pi^2 2^d}{r^{d-1}|\mathbb{S}^{d-1}|}\left(r - \frac{j_{d/2-1,1}}{\pi}\right) + O_d\left(r - \frac{j_{d/2-1,1}}{\pi}\right)^2.$$

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Mathematical and Statistical Sciences, University of Alberta, CAB 632, Edmonton, Alberta, T6G 2G1 Canada

Email address: `felipe.goncalves@ualberta.ca`

URL: `sites.ualberta.ca/~goncalve`