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ON COMPLETE MONOTONICITY OF CERTAIN SPECIAL FUNCTIONS

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Abstract. Given an entire function $f(z)$ that has only negative zeros, we shall prove that all the functions of type $f^{(m)}(x)/f^{(n)}(x),\ m>n$ are completely monotonic. Examples of this type are given for Laguerre polynomials, ultraspherical polynomials, Jacobi polynomials, Stieltjes-Wigert polynomials, q-Laguerre polynomials, Askey-Wilson polynomials, Ramanujan function, q-exponential functions, q-Bessel functions, Euler's gamma function, Airy function, modified Bessel functions of the first and the second kind, and the confluent basic hypergeometric series.

1. Introduction

An infinitely differentiable real function $f(x)$ on $(0,\infty)$ is called completely monotonic if $(-1)^n f^{(n)}(x)\ge 0$ for $n=0,1,\dots$ and all $x\in(0,\infty)$. It is well known that a function $f(x)$ is completely monotonic if and only if there exists a unique finite positive measure μ on $(0,\infty)$ such that for each $x>0$ we have $f(x)=\int_0^\infty e^{-xy}\mu(dy),\quad x\in(0,\infty)$, [8, 26, 31]. Completely monotonic functions have many important applications in mathematics; for example, see the classics [8,31], a more recent monograph [26], and articles such as [1,2,6,11,13,18,19,28,29]. In this work we shall prove if a genus 0 entire function has only negative zeros, then all the quotients of its higher order derivatives over lower ones are completely monotonic. Examples of this type are provided for Laguerre polynomials, ultraspherical polynomials, Jacobi polynomials, Stieltjes-Wigert polynomials, q-Laguerre polynomials, Askey-Wilson polynomials, Ramanujan function, q-exponential functions, q-Bessel functions, Euler's gamma function, the modified Bessel functions of the first and second kind, the confluent basic hypergeometric series, and Airy function. Throughout this work we shall follow the standard notation used in [17] and [5].

Order 0 entire functions are extremely important in the theory of orthogonal polynomials and special functions. For example, let $\mathcal{I}=\left\{f\,\middle|\,f(z)=\sum_{n=0}^\infty c_n q^{\alpha n^2}z^n\right\}$ where $0<q<1$, $\alpha>0$ and let $\{c_n\}$ be a bounded sequence of complex numbers. Then $\mathcal{I}$ is a subclass of order 0 entire functions, and every non-polynomial member of $\mathcal{I}$ has infinitely many zeros, [17]. In [12] Hayman proved an asymptotic formula for the zeros of such functions. The motivation for this work is to show certain

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functions related to class $\mathcal{I}$ are completely monotonic. To see why class $\mathcal{I}$ is very interesting, we only need to look at several of its members. Obviously, it contains all the complex polynomials as well as the Ramanujan type functions, [34]

$$A_q^{(\alpha)}(a;z)=\sum_{n=0}^{\infty}\frac{(a;q)_n\,q^{\alpha n^2}z^n}{(q;q)_n}. \tag{1.1}$$

This entire function $A_q^{(\alpha)}(a;z)$ is interesting already; for $a=0$ we obtain the q-exponential function [17]

$$E_q^{(\alpha)}(z)=A_q^{(\alpha)}(0;z)=\sum_{n=0}^{\infty}\frac{q^{\alpha n^2}z^n}{(q;q)_n}. \tag{1.2}$$

It is clear that $E_q^{(1/2)}(z)=E_q\left(-q^{1/2}z\right)$ and $E_q^{(1)}(z)=A_q(-z)$, where [5, 10, 17]

$$E_q(z)=(z;q)_\infty=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(-z)^n}{(q;q)_n},\ A_q(z)=\sum_{n=0}^{\infty}\frac{q^{n^2}(-z)^n}{(q;q)_n}. \tag{1.3}$$

They are the Euler entire q-exponential function and the Ramanujan entire function respectively. The Ramanujan function $A_q(z)$ plays a fundamental role in combinatorics and in asymptotics of q-orthogonal polynomials; for example, see [3, 4, 14–16, 32, 33]. Both q-Bessel functions, $J_\nu^{(2)}(z;q)$ and $J_\nu^{(3)}(z;q)$, [5, 10, 17, 21, 22]

$$\frac{J_\nu^{(2)}(z;q)}{(z/2)^\nu}=\frac{\left(q^{\nu+1};q\right)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{\left(-z^2q^{\nu+n}/4\right)^n}{(q,q^{\nu+1};q)_n}, \tag{1.4}$$

$$\frac{J_\nu^{(3)}(z;q)}{(z/2)^\nu}=\frac{\left(q^{\nu+1};q\right)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{\left(-z^2q^{\frac{n+1}{2}}/4\right)^n}{(q,q^{\nu+1};q)_n} \tag{1.5}$$

are also in class $\mathcal{I}$. They are entire q-analogues of the Bessel function, [5, 17]

$$\frac{J_\nu(z)}{(z/2)^\nu}=\sum_{n=0}^{\infty}\frac{(-z^2/4)^n}{\Gamma(n+\nu+1)n!}. \tag{1.6}$$

The q-Bessel functions $J_\nu^{(k)}(z;q)$, $k=2,3$ are important in the study of q-orthogonal polynomials and representation theory of quantum groups, [9, 20]. More generally, for $r,\ s\in\mathbb{N}_0$, $0<q<1$, $\alpha>0$, and $a_1,\dots,a_r,\ b_1,\dots,b_s\in\mathbb{C}$, the infinite series [34]

$${}_rA_s^{(\alpha)}(a_1,\dots a_r;\ b_1,\dots,\ b_s\ ;\ q;\ z)=\sum_{n=0}^{\infty}\frac{(a_1,\dots,\ a_r;\ q)_n}{(b_1,\dots,b_s;\ q)_n}q^{\alpha n^2}z^n \tag{1.7}$$

is also in class $\mathcal{I}$. Evidently, it includes all the rescaled confluent basic hypergeometric series, [5, 10, 17]

$${}_r\phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix}\ \middle|\ q,(-q)^{\frac{s+1-r}{2}}z\right)=\sum_{n=0}^{\infty}\frac{(a_1,\dots,a_r;q)_n\,z^n}{(q,b_1,\dots,b_s;q)_n}q^{\frac{n^2(s+1-r)}{2}},\quad s\ge r, \tag{1.8}$$

as special cases.

2. Main results

Let $f(z)$ be an entire function of finite genus. We define the degree $d(f)$ of $f(z)$ to be $d(f) = n$ if $f(z)$ is a polynomial of degree n, and $d(f) = \infty$ otherwise.

Lemma 1. *Let G be the set of all genus 0 entire functions that have only negative roots. If $f(z), g(z) \in G$, then $f(z)g(z) \in G$. If $f(z) \in G$, then $f(az+b) \in G$ for all $a > 0$ and $b \geq 0$. Let $f(z)$ be an even entire function that $f(0) \neq 0$ and its genus is at most 1. If it has only real zeros, then the entire function obtained from $f(i\sqrt{z})$, $|\arg(z)| < \pi$ by the analytic continuation is in G. Let $f(z)$ be an entire function of genus is at most 1. If it has only real zeros, then for all $a > 0$, the entire function obtained from $f(a - i\sqrt{z})f(a + i\sqrt{z})$ $|\arg(z)| < \pi$ by analytic continuation is also in G.*

Lemma 2. *Let G be defined as in Lemma 1; then G is closed under differentiation. Furthermore, for each $f(z) \in G$, if $m, n+1 \in \mathbb{N}$ and $d(f) \geq m > n$, then the function $\frac{f^{(m)}(x)}{f^{(n)}(x)}$ is completely monotonic in x.*

We now apply Lemma 2 to polynomials:

Corollary 3. *We have the following results on polynomials:*

(1) *Let $\{p_n(x)\}_{n\geq 0}$ be a set of polynomials orthogonal with respect to a probability measure $d\mu(x)$ on the real line. For any $a \in \mathbb{R}$ if the support of the measure is inside (a, ∞) and*

$$A_k \leq a,\ \alpha_k > 0,\ 1 \leq k \leq K,\ K, n_k+1, i_1, i_2+1 \in \mathbb{N},\ \sum_{k=1}^{K} n_k \geq i_1 > i_2, \tag{2.1}$$

then the rational function $\frac{\partial_x^{i_1}\left(\prod_{k=1}^{K} p_{n_k}(A_k - \alpha_k x)\right)}{\partial_x^{i_2}\left(\prod_{k=1}^{K} p_{n_k}(A_k - \alpha_k x)\right)}$ is completely monotonic in x. For any $b \in \mathbb{R}$ if the support is contained inside $(-\infty, b)$ and

$$B_k \geq b,\ \alpha_k > 0,\ 1 \leq k \leq K,\ K, n_k+1, i_1, i_2+1 \in \mathbb{N},\ \sum_{k=1}^{K} n_k \geq i_1 > i_2, \tag{2.2}$$

then the rational function $\frac{\partial_x^{i_1}\left(\prod_{k=1}^{K} p_{n_k}(\alpha_k x + B_k)\right)}{\partial_x^{i_2}\left(\prod_{k=1}^{K} p_{n_k}(\alpha_k x + B_k)\right)}$ is completely monotonic in x. If the support of the measure is inside (a, b) and

$$B_k \geq b > a \geq A_k,\ \alpha_k > 0,\ 1 \leq k \leq K,\ K, n_k+1, i_1, i_2+1 \in \mathbb{N},\ \sum_{k=1}^{K} n_k \geq i_1 > i_2, \tag{2.3}$$

then the function $\frac{\partial_x^{i_1}\left(\prod_{k=1}^{K} w_{n_k}(\alpha_k x)\right)}{\partial_x^{i_2}\left(\prod_{k=1}^{K} w_{n_k}(\alpha_k x)\right)}$ is completely monotonic in x where $w_{n_k}(x)$ may be either $p_{n_k}(A_k - x)$ or $p_{n_k}(x + B_k)$. If the measure is symmetric and

$$K, n_k+1, i_1, i_2+1 \in \mathbb{N},\ \alpha_k > 0,\ 1 \leq k \leq K,\ \sum_{k=1}^{K} n_k \geq i_1 > i_2, \tag{2.4}$$

then the rational function $\frac{\partial_x^{i_1}\left(\prod_{k=1}^{K} u_{n_k}(\alpha_k x)\right)}{\partial_x^{i_2}\left(\prod_{k=1}^{K} u_{n_k}(\alpha_k x)\right)}$ is completely monotonic in x where $u_{n_k}(x)$ may be $p_{2n_k}(i\sqrt{x})$ or $p_{2n_k+1}(i\sqrt{x}) / (i\sqrt{x})$.

(2) *For* $m,\ell\in\mathbb{N}_0,\ n_j\in\mathbb{N},\ 0<q_j<1,\ 1\le j\le m,$ *and* $\beta_r>0,\ 0<q_r<1,\ 1\le r\le\ell,$ *let*

$$P_1(x)=\sum_{k=0}^{\min\{n_j|1\le j\le m\}}\prod_{j=1}^{m}\frac{\left(q_j^{-n_j};q_j\right)_k}{(q_j;q_j)_k}\frac{q^{\alpha k^2}\left((-1)^m x\right)^k}{\prod_{r=1}^{\ell}\left(q_r,q_r^{\beta_r};q_r\right)_k}. \tag{2.5}$$

Then for $i_1,\ i_2+1\in\mathbb{N}$ *and* $\min\{n_j|1\le j\le m\}\ge i_1>i_2$, *the rational function* $\frac{\partial_x^{i_1}P_1(x)}{\partial_x^{i_2}P_1(x)}$ *is completely monotonic in* x.

(3) *For* $m,\ell\in\mathbb{N}_0,\ n_j\in\mathbb{N},\ 1\le j\le m,$ *and* $\beta_r>0,\ 1\le r\le\ell,$ *let*

$$P_2(x)=\sum_{k=0}^{\min\{n_j|1\le j\le m\}}\frac{\prod_{j=1}^{m}(-n_j)_k}{\prod_{r=1}^{\ell}(\beta_r)_k}\frac{\left((-1)^m x\right)^k}{(k!)^{m+\ell}}. \tag{2.6}$$

Then for $i_1,\ i_2+1\in\mathbb{N}$ *and* $\min\{n_j|1\le j\le m\}\ge i_1>i_2$, *the rational function* $\frac{\partial_x^{i_1}P_2(x)}{\partial_x^{i_2}P_2(x)}$ *is completely monotonic in* x.

We now apply Lemma 2 to entire functions:

Corollary 4. *Let* $0<q<1$ *and* $\alpha>0$. *Then,*

(1) *for* $m,\ell\in\mathbb{N},\ a_j\ge0,\ 0<q_j<1,\ 1\le j\le m,$ *and* $\beta_r>0,\ 0<q_r<1,\ 1\le r\le\ell,$ *let*

$$f_1(z)=\sum_{k=0}^{\infty}\prod_{j=1}^{m}\frac{(-a_j;q_j)_k}{(q_j;q_j)_k}\frac{q^{\alpha k^2}z^k}{\prod_{r=1}^{\ell}\left(q_r,q_r^{\beta_r};q_r\right)_k}. \tag{2.7}$$

Then for $i_1,\ i_2+1\in\mathbb{N}$ *and* $i_1>i_2$ *the function* $\frac{f_1^{(i_1)}(x)}{f_1^{(i_2)}(x)}$ *is completely monotonic in* x.

(2) *For* $m\in\mathbb{N}_0,\ \ell\in\mathbb{N}$ *and* $\beta_r\ge1,\ 1\le r\le\ell,$ *let*

$$f_2(z)=\sum_{k=0}^{\infty}\frac{z^k}{(k!)^{m+\ell}\prod_{r=1}^{\ell}(\beta_r)_k}. \tag{2.8}$$

Then for $i_1,\ i_2+1\in\mathbb{N}$ *and* $i_1>i_2$ *the function* $\frac{f_2^{(i_1)}(x)}{f_2^{(i_2)}(x)}$ *is completely monotonic in* x.

(3) *Let* ${}_rA_s^{(\alpha)}(a_1,\dots,a_r;\ b_1,\dots,\ b_s\ ;\ q;\ z)$ *be as defined in* (1.7). *If it has only negative zeros and for* $m,\ n+1\in\mathbb{N}$ *such that* $m>n$, *then the function*

$$\frac{\partial_x^m\,{}_rA_s^{(\alpha)}(a_1,\dots,a_r;\ b_1,\dots,b_s\ ;\ q;\ x)}{\partial_x^n\,{}_rA_s^{(\alpha)}(a_1,\dots,a_r;\ b_1,\dots,b_s\ ;\ q;\ x)} \tag{2.9}$$

is completely monotonic in x. *In particular, if* $r,\ s\in\mathbb{N}_0,\ 0<q<1,\ \alpha>0,$ $1>a_j,\ b_k>0,\ 1\le j\le r,\ 1\le k\le s,$ *and*

$$\prod_{j=1}^{r}(1-a_j)\prod_{k=1}^{s}(1-b_kq)\ge4q^{2\alpha}, \tag{2.10}$$

and for $m, n+1 \in \mathbb{N}$ such that $m > n$, then the function in (2.9) *is completely monotonic in x, which is equivalent to that the function*

$$\frac{\sum_{k=0}^{\infty}(a_1q^m, \ldots, a_rq^m; q)_k \left(q^{\alpha(2m+k)}x\right)^k (m)_k / (b_1q^m, \ldots, b_sq^m; q)_k}{\sum_{k=0}^{\infty}(a_1q^n, \ldots, a_rq^n; q)_k \left(q^{\alpha(2n+k)}x\right)^k (n)_k / (b_1q^n, \ldots, b_sq^n; q)_k} \tag{2.11}$$

is completely monotonic in x.

2.1. **Examples.** Let $\mathcal{CM}$ be the set of completely monotonic functions. Then [8, 26, 31]

- if $f(x),\ g(x) \in \mathcal{CM}$ and $\alpha > 0,\ \beta \geq 0$, then $\alpha f(x) + \beta g(x) \in \mathcal{CM}$,
- if $f(x) \in \mathcal{CM}$ and $\alpha > 0,\ \beta \geq 0$, then $f(\alpha x + \beta) \in \mathcal{CM}$,
- if $f(x),\ g(x) \in \mathcal{CM}$, then $f(x)g(x) \in \mathcal{CM}$,
- let $\{f_n(x)\}_{n=1}^{\infty} \subset \mathcal{CM}$. If the sequence converges to $f(x)$ on $(0, \infty)$, then $f(x) \in \mathcal{CM}$.

The properties above and Lemmas 1 and 2 allow us to generate even more completely monotonic functions from our corollaries and examples listed below.

2.1.1. *Laguerre polynomials.* For $\alpha > -1$ the Laguerre polynomials [5, 17]

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k x^k}{(\alpha+1)_k k!} \tag{2.12}$$

are orthogonal over $(0, \infty)$. By Corollary 3 and the identity $\partial_x L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x)$, we see that for $i_1,\ i_2+1,\ n \in \mathbb{N}$ and $n \geq i_1 > i_2$, the function $\frac{L_{n-i_1}^{(\alpha+i_1)}(-x)}{L_{n-i_2}^{(\alpha+i_2)}(-x)}$ is completely monotonic in x. Let

$$\alpha_j + 1, a_j > 0,\ b_j \geq 0,\ N, n_j + 1 \in \mathbb{N},\ 1 \leq j \leq N. \tag{2.13}$$

Then for $i_1,\ i_2+1 \in \mathbb{N}$ and $\sum_{j=1}^{N} n_j \geq i_1 > i_2$, the function $\frac{\partial_x^{i_1}\left(\prod_{j=1}^{N} L_{n_j}^{(\alpha_j)}(-a_jx-b_j)\right)}{\partial_x^{i_2}\left(\prod_{j=1}^{N} L_{n_j}^{(\alpha_j)}(-a_jx-b_j)\right)}$ is completely monotonic in x.

2.1.2. *Ultraspherical polynomials.* For $\nu > -\frac{1}{2}$ the ultraspherical polynomials [5, 17]

$$C_n^{\nu}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(2\nu)_n x^{n-2k}(x^2-1)^k}{2^{2k} k!\,(\nu+1/2)_k\,(n-2k)!} \tag{2.14}$$

are orthogonal with respect to a symmetric measure over the interval $(-1, 1)$. Then for $\nu > -\frac{1}{2}$, $i_1,\ i_2+1,\ n \in \mathbb{N}$ and $n \geq i_1 > i_2$, the functions

$$\frac{\partial_x^{i_1} C_{2n}^{\nu}\left(i\sqrt{x}\right)}{\partial_x^{i_2} C_{2n}^{\nu}\left(i\sqrt{x}\right)}, \quad \frac{\partial_x^{i_1}\left(C_{2n+1}^{\nu}\left(i\sqrt{x}\right)/(i\sqrt{x})\right)}{\partial_x^{i_2}\left(C_{2n+1}^{\nu}\left(i\sqrt{x}\right)/(i\sqrt{x})\right)} \tag{2.15}$$

are completely monotonic in x. Let

$$e_n^{\nu}(x) = C_{2n}^{\nu}\left(i\sqrt{x}\right), \quad o_n^{\nu}(x) = C_{2n+1}^{\nu}\left(i\sqrt{x}\right)/(i\sqrt{x}), \tag{2.16}$$

and

$$\alpha_j + \frac{1}{2}, a_j > 0,\ N, n_j + 1 \in \mathbb{N},\ 1 \leq j \leq N; \tag{2.17}$$

then for $i_1,\ i_2+1\in\mathbb{N}$ and $\sum_{j=1}^N n_j\ge i_1>i_2$, the function $\frac{\partial_x^{i_1}\left(\prod_{j=1}^N c_{n_j}^{(\alpha_j)}(a_jx)\right)}{\partial_x^{i_2}\left(\prod_{j=1}^N c_{n_j}^{(\alpha_j)}(a_jx)\right)}$ is completely monotonic in x where $c_{n_j}^{(\alpha_j)}(x)$ may be $e_{n_j}^{\alpha_j}(x)$ or $o_{n_j}^{\alpha_j}(x)$.

2.1.3. *Jacobi polynomials.* For $\alpha,\beta>-1$ the Jacobi polynomials [5,17]

$$P_n^{(\alpha,\beta)}(x)=\frac{(\alpha+1)_n}{n!}\sum_{k=0}^n\frac{(-n,\alpha+\beta+1)_k}{(\alpha+1)_k k!}\left(\frac{1-x}{2}\right)^k \tag{2.18}$$

are orthogonal over $(-1,1)$. Then by Corollary 3 and the formula

$$\partial_x P_n^{(\alpha,\beta)}(x)=\frac{n+\alpha+\beta+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{2.19}$$

we get that for $i_1,\ i_2+1,\ n\in\mathbb{N}$ and $n\ge i_1>i_2$, the functions

$$\frac{P_{n-i_1}^{(\alpha+i_1,\beta+i_1)}(x+1)}{P_{n-i_2}^{(\alpha+i_2,\beta+i_2)}(x+1)},\quad (-1)^{i_1-i_2}\frac{P_{n-i_1}^{(\alpha+i_1,\beta+i_1)}(1-x)}{P_{n-i_2}^{(\alpha+i_2,\beta+i_2)}(1-x)}, \tag{2.20}$$

are completely monotonic in x. Let

$$pp_n^{(\alpha,\beta)}(x)=P_n^{(\alpha,\beta)}(x+1),\ pm_n^{(\alpha,\beta)}(x)=P_n^{(\alpha,\beta)}(1-x) \tag{2.21}$$

and

$$\alpha_j+1,\beta_j+1,a_j>0,\ b_j\ge 0,\ N,n_j+1\in\mathbb{N},\ 1\le j\le N; \tag{2.22}$$

then for $i_1,\ i_2+1\in\mathbb{N}$ and $\sum_{j=1}^N n_j\ge i_1>i_2$ the function $\frac{\partial_x^{i_1}\left(\prod_{j=1}^N c_{n_j}^{(\alpha_j,\beta_j)}(a_jx+b_j)\right)}{\partial_x^{i_2}\left(\prod_{j=1}^N c_{n_j}^{(\alpha_j,\beta_j)}(a_jx+b_j)\right)}$ is completely monotonic in x, where $c_{n_j}^{(\alpha_j,\beta_j)}(x)$ may be $pp_{n_j}^{(\alpha_j,\beta_j)}(x)$ or $pm_{n_j}^{(\alpha_j,\beta_j)}(x)$.

2.1.4. *Stieltjes-Wigert polynomials.* For $0<q<1$ the Stieltjes-Wigert polynomials [5,17]

$$S_n(x;q)=\frac{1}{(q;q)_n}\sum_{k=0}^n\begin{bmatrix}n\\k\end{bmatrix}_q q^{k^2}(-x)^k \tag{2.23}$$

are orthogonal over $(0,\infty)$. Then for $i_1,\ i_2+1,\ n\in\mathbb{N}$ and $n\ge i_1>i_2$ the function $\frac{\partial_x^{i_1}S_n(-x;q)}{\partial_x^{i_2}S_n(-x;q)}$ is completely monotonic in x, which is equivalent to that the function

$$\frac{\sum_{k=0}^{n-i_1}\begin{bmatrix}n\\k+i_1\end{bmatrix}_q(i_1+1)_k\left(q^{2i_1+k}x\right)^k}{\sum_{k=0}^{n-i_2}\begin{bmatrix}n\\k+i_2\end{bmatrix}_q(i_2+1)_k\left(q^{2i_2+k}x\right)^k} \tag{2.24}$$

is completely monotonic in x. Let

$$q_j\in(0,1),\ a_j>0,\ b_j\ge 0,\ N,n_j+1\in\mathbb{N},\ 1\le j\le N; \tag{2.25}$$

then for $i_1,\ i_2+1\in\mathbb{N}$ and $\sum_{j=1}^N n_j\ge i_1>i_2$ the function $\frac{\partial_x^{i_1}\left(\prod_{j=1}^N S_{n_j}(-a_jx-b_j;q_j)\right)}{\partial_x^{i_2}\left(\prod_{j=1}^N S_{n_j}(-a_jx-b_j;q_j)\right)}$ is completely monotonic in x.

2.1.5. *q-Laguerre polynomials.* For $\alpha > -1$ and $0 < q < 1$ the q-Laguerre polynomials [5, 10, 17]

$$L_n^{(\alpha)}(x;q) = \frac{\left(q^{\alpha+1};q\right)_n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{\alpha k+k^2}(-x)^k}{\left(q^{\alpha+1};q\right)_k} \tag{2.26}$$

are orthogonal over $(0,\infty)$. Then for $i_1,\ i_2+1,\ n \in \mathbb{N}$ and $n \geq i_1 > i_2$, the function $\frac{\partial_x^{i_1} L_n^{(\alpha)}(-x;q)}{\partial_x^{i_2} L_n^{(\alpha)}(-x;q)}$, which is equivalent to the function

$$\frac{\displaystyle\sum_{k=0}^{n-i_1}\begin{bmatrix} n \\ k+i_1 \end{bmatrix}_q (i_1+1)_k \left(q^{\alpha+2i_1+k}x\right)^k / \left(q^{\alpha+i_1+1};q\right)_k}{\displaystyle\sum_{k=0}^{n-i_2}\begin{bmatrix} n \\ k+i_2 \end{bmatrix}_q (i_2+1)_k \left(q^{\alpha+2i_2+k}x\right)^k / \left(q^{\alpha+i_2+1};q\right)_k}, \tag{2.27}$$

is completely monotonic in x. Let

$$q_j \in (0,1),\ \alpha_j+1, a_j > 0,\ b_j \geq 0,\ N, n_j+1 \in \mathbb{N},\ 1 \leq j \leq N; \tag{2.28}$$

then for $i_1,\ i_2+1 \in \mathbb{N}$ and $\sum_{j=1}^{N} n_j \geq i_1 > i_2$, the function $\frac{\partial_x^{i_1}\left(\prod_{j=1}^{N} L_{n_j}^{(\alpha_j)}(-a_jx-b_j;q_j)\right)}{\partial_x^{i_2}\left(\prod_{j=1}^{N} L_{n_j}^{(\alpha_j)}(-a_jx-b_j;q_j)\right)}$ is completely monotonic in x.

2.1.6. *Askey-Wilson polynomials.* For simplicity we assume the parameters satisfying $t_1,\ t_2,\ t_3\ t_4 \in (-1,1)$ and $t_1 \neq 0$. The Askey-Wilson polynomials [5, 10, 17]

$$\frac{p_n(x;\ t_1,t_2,t_3,t_4|q)t_1^n}{(t_1t_2,t_1t_3,t_1t_4;q)_n} = {}_4\phi_3\left(\begin{matrix} q^{-n},\ t_1t_2t_3t_4q^{n-1},\ t_1e^{i\theta},\ t_1e^{-i\theta} \\ t_1t_2,\ t_1t_3,\ t_1t_4 \end{matrix}\middle| q,q\right) \tag{2.29}$$

are polynomials of degree n in $x = \cos\theta,\ \theta \in (0,\pi)$, and they are orthogonal over $x \in (-1,1)$. Then for $i_1,\ i_2+1,\ n \in \mathbb{N}$ and $n \geq i_1 > i_2$, the functions

$$\frac{\partial_x^{i_1} p_n(x+1;\ t_1,t_2,t_3,t_4|q)}{\partial_x^{i_2} p_n(x+1;\ t_1,t_2,t_3,t_4|q)},\quad \frac{\partial_x^{i_1} p_n(1-x;\ t_1,t_2,t_3,t_4|q)}{\partial_x^{i_2} p_n(1-x;\ t_1,t_2,t_3,t_4|q)}, \tag{2.30}$$

are completely monotonic in x. Let

$$ap_n(x;\ t_1,t_2,t_3,t_4|q) = p_n(x+1;\ t_1,t_2,t_3,t_4|q), \tag{2.31}$$

$$am_n(x;\ t_1,t_2,t_3,t_4|q) = p_n(1-x;\ t_1,t_2,t_3,t_4|q), \tag{2.32}$$

and for $N \in \mathbb{N}$, $1 \leq j \leq N$ and

$$t_1^{(j)}, t_2^{(j)}, t_3^{(j)}, t_4^{(j)} \in (-1,1),\ q_j \in (0,1),\ \alpha_j > 0,\ \beta_j \geq 0,\ n_j+1 \in \mathbb{N}. \tag{2.33}$$

Then for $i_1,\ i_2+1 \in \mathbb{N}$ and $\sum_{j=1}^{N} n_j \geq i_1 > i_2$, the function

$$\frac{\partial_x^{i_1}\left(\prod_{j=1}^{N} c_{n_j}(\alpha_jx+\beta_j;\ t_1^{(j)},t_2^{(j)},t_3^{(j)},t_4^{(j)}|q_j)\right)}{\partial_x^{i_2}\left(\prod_{j=1}^{N} c_{n_j}(\alpha_jx+\beta_j;\ t_1^{(j)},t_2^{(j)},t_3^{(j)},t_4^{(j)}|q_j)\right)} \tag{2.34}$$

is completely monotonic in x where $c_{n_j}(x;\ t_1^{(j)},t_2^{(j)},t_3^{(j)},t_4^{(j)}|q_j)$ may be $ap_{n_j}(x;\ t_1^{(j)},t_2^{(j)},t_3^{(j)},t_4^{(j)}|q_j)$ or $am_{n_j}(x;\ t_1^{(j)},t_2^{(j)},t_3^{(j)},t_4^{(j)}|q_j)$.

2.1.7. *Ramanujan type function.* For $0 < q < 1$ and $\alpha,\ a > 0$ the Ramanujan type function $A_q^{(\alpha)}(-a;z)$,

$$A_q^{(\alpha)}(-a;z) = \sum_{n=0}^{\infty} \frac{(-a;q)_n q^{\alpha n^2}}{(q;q)_n} z^n, \tag{2.35}$$

is an order 0 entire function that has only negative zeros, [34]. Then for $a, \beta \geq 0$, $\alpha > 0$, $i_1,\ i_2+1 \in \mathbb{N}$ and $i_1 > i_2$, the function, $\frac{\partial_x^{i_1} A_q^{(\alpha)}(-a;\alpha x+\beta)}{\partial_x^{i_2} A_q^{(\alpha)}(-a;\alpha x+\beta)}$, which is essentially the same as the function

$$\frac{\sum_{k=0}^{\infty} (-aq^{i_1};q)_k (i_1)_k \left(q^{\alpha(2i_1+k)}(\alpha x+\beta)\right)^k / \left(q^{i_1+1};q\right)_k}{\sum_{k=0}^{\infty} (-aq^{i_2};q)_k (i_2)_k \left(q^{\alpha(2i_2+k)}(\alpha x+\beta)\right)^k / \left(q^{i_2+1};q\right)_k}, \tag{2.36}$$

is completely monotonic in x. Let

$$q_j \in (0,1),\ \alpha_j, a_j, c_j > 0,\ d_j \geq 0,\ N \in \mathbb{N},\ 1 \leq j \leq N; \tag{2.37}$$

then for $i_1,\ i_2+1 \in \mathbb{N}$ and $i_1 > i_2$ the function $\frac{\partial_x^{i_1}\left(\prod_{j=1}^N A_{q_j}^{(\alpha_j)}(-a_j;c_j x+d_j)\right)}{\partial_x^{i_2}\left(\prod_{j=1}^N A_{q_j}^{(\alpha_j)}(-a_j;c_j x+d_j)\right)}$ is completely monotonic in x. Moreover, for all $m,\ n+1 \in \mathbb{N}$, $m > n$ and

$$a_j, b_j, c_j, \alpha_j > 0,\ d_j \geq 0,\ q_j \in (0,1),\ 1 \leq j \leq N,\ N \in \mathbb{N}, \tag{2.38}$$

the function

$$\frac{\partial_x^m \left(\prod_{j=1}^N \left|A_{q_j}^{(\alpha_j)}(-a_j, b_j + i\sqrt{c_j x + d_j})\right|^2\right)}{\partial_x^n \left(\prod_{j=1}^N \left|A_{q_j}^{(\alpha_j)}(-a_j, b_j + i\sqrt{c_j x + d_j})\right|^2\right)} \tag{2.39}$$

is completely monotonic in x.

2.1.8. *q-Bessel functions.* Since

$$j_\nu^{(2)}(z;q) = z^{-\nu} J_\nu^{(2)}(z;q), \quad j_\nu^{(3)}(z;q) = z^{-\nu} J_\nu^{(3)}(z;q), \tag{2.40}$$

are order 0 even entire functions that have only real zeros, then [17, 21, 34]

$$\frac{I_\nu^{(2)}(\sqrt{z};q)}{(\sqrt{z}/2)^\nu} = \frac{\left(q^{\nu+1};q\right)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{\left(zq^{\nu+n}/4\right)^n}{(q, q^{\nu+1};q)_n} \tag{2.41}$$

and

$$\frac{I_\nu^{(3)}(\sqrt{z};q)}{(\sqrt{z}/2)^\nu} = \frac{\left(q^{\nu+1};q\right)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{\left(zq^{\frac{n+1}{2}}/4\right)^n}{\left(q, q^{\nu+1};q\right)_n} \tag{2.42}$$

are order 0 entire functions that have only negative zeros. Let

$$b_2(x;\ \nu|q) = \frac{I_\nu^{(2)}(\sqrt{z};q)}{(\sqrt{z}/2)^\nu}, \quad b_3(x;\ \nu|q) = \frac{I_\nu^{(3)}(\sqrt{z};q)}{(\sqrt{z}/2)^\nu}, \tag{2.43}$$

and

$$\nu_j + 1, \alpha_j > 0,\ \beta_j \geq 0,\ q_j \in (0,1),\ N \in \mathbb{N},\ 1 \leq j \leq N; \tag{2.44}$$

then for $i_1,\ i_2+1\in\mathbb{N}$ and $i_1>i_2$, the function $\frac{\partial_x^{i_1}\left(\prod_{j=1}^{N} c(\alpha_j x+\beta_j;\ \nu_j|q_j)\right)}{\partial_x^{i_2}\left(\prod_{j=1}^{N} c(\alpha_j x+\beta_j;\ \nu_j|q_j)\right)}$ is completely monotonic in x, where $c(x;\ \nu_j|q_j)$ may be either $b_2(x;\ \nu_j|q_j)$ or $b_3(x;\ \nu_j|q_j)$. Let

$$q_j\in(0,1),\ a_j,\nu_j+1,\alpha_j>0,\ \beta_j\ge 0,\ N\in\mathbb{N},\ 1\le j\le N; \tag{2.45}$$

then for $i_1,\ i_2+1\in\mathbb{N}$ and $i_1>i_2$, the function

$$\frac{\partial_x^{i_1}\left(\prod_{j=1}^{N}\left|j_{\nu_j}(a_j+i\sqrt{\alpha_j x+\beta_j};q_j)\right|^2\right)}{\partial_x^{i_2}\left(\prod_{j=1}^{N}\left|j_{\nu_j}(a_j+i\sqrt{\alpha_j x+\beta_j};q_j)\right|^2\right)} \tag{2.46}$$

is completely monotonic in x, where $j_{\nu_j}(z;q)$ may be any of $j_{\nu_j}^{(k)}(z;q),\ k=2,3$.

2.1.9. *Euler's gamma function.* Since the order 1 entire function $1/\Gamma(z)$ has zeros $0,-1,\dots$. Then for all $m,\ n+1\in\mathbb{N}$, and $m>n$ and

$$a_j,\alpha_j>0,\ \beta_j\ge 0,\ 1\le j\le N,\ N\in\mathbb{N}, \tag{2.47}$$

the function

$$\frac{\partial_x^{m}\left(\prod_{j=1}^{N}\left|\Gamma(a_j+i\sqrt{\alpha_j x+\beta_j})\right|^{-2}\right)}{\partial_x^{n}\left(\prod_{j=1}^{N}\left|\Gamma(a_j+i\sqrt{\alpha_j x+\beta_j})\right|^{-2}\right)} \tag{2.48}$$

is completely monotonic in x.

2.1.10. *Airy function.* The Airy function [24, 27]

$$\mathrm{Ai}(z)=\frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3}+zt\right)dt, \tag{2.49}$$

is an order $\frac{3}{2}$ entire function that has only negative zeros. Thus for all

$$m,\ n+1,\ m>n,\ N\in\mathbb{N},\ b_j,\alpha_j>0,\ \beta_j\ge 0,\ 1\le j\le N, \tag{2.50}$$

the function

$$\frac{\partial_x^{m}\left(\left|\prod_{j=1}^{N}\mathrm{Ai}(b_j+i\sqrt{\alpha_j x+\beta_j})\right|^2\right)}{\partial_x^{n}\left(\left|\prod_{j=1}^{N}\mathrm{Ai}(b_j+i\sqrt{\alpha_j x+\beta_j})\right|^2\right)} \tag{2.51}$$

is completely monotonic in x.

2.1.11. *Modified Bessel functions of the first kind.* For $\nu>-1$ the function [5, 17–19, 23, 24, 30]

$$\Gamma(\nu+1)\left(z^{1/2}/2\right)^{-\nu} I_\nu\left(z^{1/2}\right)=\sum_{n=0}^{\infty}\frac{(z/4)^n}{(\nu+1)_n n!}=\prod_{n=1}^{\infty}\left(1+\frac{z}{j_{\nu,n}^2}\right) \tag{2.52}$$

is an order 1/2 entire function. It clearly has only negative zeros where $j_{\nu,n},\ n\in\mathbb{N}$ are positive zeros of $J_\nu(z)$. Then for $\nu>-1,\ m,\ n+1\in\mathbb{N}$ and $m>n$, the function $\frac{\partial_x^m\left(x^{-\nu/2}I_\nu\left(x^{1/2}\right)\right)}{\partial_x^n\left(x^{-\nu/2}I_\nu\left(x^{1/2}\right)\right)}$ is completely monotonic in x. By applying the differential-recurrence formulas we see that it is equivalent to $\frac{x^{n/2}I_{\nu+m}\left(\sqrt{x}\right)}{x^{m/2}I_{\nu+n}\left(\sqrt{x}\right)}$ and is completely monotonic in x. Let

$$\nu_j+1,\alpha_j,m_j-n_j>0,\ \beta_j\ge 0,\ m_j,n_j+1\in\mathbb{N},\ 1\le j\le N,\ N\in\mathbb{N}; \tag{2.53}$$

then the function

$$\prod_{j=1}^{N} \frac{(\alpha_j x+\beta_j)^{n_j/2} I_{\nu_j+m_j}\left(\sqrt{\alpha_j x+\beta_j}\right)}{(\alpha_j x+\beta_j)^{m_j/2} I_{\nu_j+n_j}\left(\sqrt{\alpha_j x+\beta_j}\right)} \tag{2.54}$$

is also completely monotonic in x. On the other hand, let

$$\nu_j+1, \alpha_j>0,\ \beta_j\ge 0,\ 1\le j\le N,\ N\in\mathbb{N}; \tag{2.55}$$

then for $m,\ n+1\in\mathbb{N}$ and $m>n$, the function

$$\frac{\partial_x^m\left((\alpha_j x+\beta_j)^{-(\sum_{j=1}^N \nu_j)/2}\prod_{j=1}^N I_{\nu_j}\left(\sqrt{\alpha_j x+\beta_j}\right)\right)}{\partial_x^n\left((\alpha_j x+\beta_j)^{-(\sum_{j=1}^N \nu_j)/2}\prod_{j=1}^N I_{\nu_j}\left(\sqrt{\alpha_j x+\beta_j}\right)\right)} \tag{2.56}$$

is completely monotonic in x. Since Bessel function $j_\nu(z)=z^{-\nu}J_\nu(z)$ is an order 1 entire function having only real zeros, then for all $m, n+1, m>n,\ N\in\mathbb{N}$, and

$$\beta_j\ge 0,\ a_j,\alpha_j,\nu_j+1>0,\ 1\le j\le N, \tag{2.57}$$

the function

$$\frac{\partial_x^m\left(\prod_{j=1}^N \left|j_{\nu_j}(a_j+i\sqrt{\alpha_j x+\beta_j})\right|^2\right)}{\partial_x^n\left(\prod_{j=1}^N \left|j_{\nu_j}(a_j+i\sqrt{\alpha_j x+\beta_j})\right|^2\right)} \tag{2.58}$$

is completely monotonic in x.

2.1.12. *Modified Bessel functions of the second kind.* Given a positive number $a>0$, then even entire function [5, 23, 24, 30]

$$K_{iz}(a)=\int_0^\infty \exp(-a\cosh t)\cos(zt)\,dt \tag{2.59}$$

is of order 1 that has only real zeros. Then for $m,\ n+1\in\mathbb{N}$, $a,b>0,\ c\ge 0$, and $m>n$ the function $\frac{\partial_x^m K_{\sqrt{bx+c}}(a)}{\partial_x^n K_{\sqrt{bx+c}}(a)}$ is completely monotonic in x. More generally, let

$$a_j,\alpha_j>0,\ \beta_j\ge 0,\ 1\le j\le N,\ N\in\mathbb{N}; \tag{2.60}$$

then for $m,\ n+1\in\mathbb{N}$, and $m>n$, the function $\frac{\partial_x^m\left(\prod_{j=1}^N K_{\sqrt{\alpha_j x+\beta_j}}(a_j)\right)}{\partial_x^n\left(\prod_{j=1}^N K_{\sqrt{\alpha_j x+\beta_j}}(a_j)\right)}$ is completely monotonic in x. Furthermore, for $m, n+1, m>n,\ N\in\mathbb{N}$, and

$$\alpha_j, a_j, b_j>0,\ \beta_j\ge 0,\ 1\le j\le N, \tag{2.61}$$

the function

$$\frac{\partial_x^m\left(\prod_{j=1}^N \left|K_{\sqrt{\alpha_j x+\beta_j}+ib_j}(a_j)\right|^2\right)}{\partial_x^n\left(\prod_{j=1}^N \left|K_{\sqrt{\alpha_j x+\beta_j}+ib_j}(a_j)\right|^2\right)} \tag{2.62}$$

is completely monotonic in x.

3. Proofs

Recall that the order $\rho(f)$ of an entire function $f(z)$ is defined by [7, 17]

$$\rho(f) = \limsup_{r\uparrow\infty} \frac{\log\log M(r)}{\log r}, \quad M(r) = \sup_{|z|\le r} |f(z)|, \tag{3.1}$$

and it can be computed by

$$\rho(f) = \left(1 - \limsup_{n\uparrow\infty} \frac{\log\left(|f^{(n)}(z_0)|\right)}{n\log n}\right)^{-1}, \tag{3.2}$$

where $z_0 \in \mathbb{C}$ is an arbitrary point and $f^{(n)}(z)$ is the n-th derivative of $f(z)$. From formulas (3.1) and (3.2), it is clear that f and f' share the same order. Every entire function of finite order has Hadamard's canonical representation, [7]

$$f(z) = z^m e^{p(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n} + \ldots + \frac{1}{k}\left(\frac{z}{z_n}\right)^k\right), \tag{3.3}$$

where $p(z)$ is a polynomial of degree j, $j,\ k,\ m$ are non-negative integers, $\{z_n\}$ are the non-zero roots of $f(z)$, and k is the smallest non-negative integer such that the series $\sum_{n=1}^{\infty} \frac{1}{|z_n|^{k+1}}$ converges. The genus $g(f)$ of $f(z)$ is defined by $g(f) = \max\{j, k\}$. It is known that $g(f)$ is the integer part of $\rho(f)$, i.e., $g(f) = \lfloor \rho(f) \rfloor$. Thus if an entire function of order $\rho(f) \in [0, 1)$, then $f(z)$ and all its derivatives must be of genus $g(f) = 0$. The last statement also implies the derivatives of a genus 0 entire function is also of genus 0.

3.1. **Proof of Lemma 1.** The first two claims are straightforward. If $f(z)$ is an even entire function that $f(0) \neq 0$ and its genus is at most 1, then it has order strictly less than 2. Let $f(z) = \sum_{n=0}^{\infty} c_n z^{2n}$; then $\sum_{n=0}^{\infty} (-1)^n c_n z^n$ defines an entire function of order strictly less than 1. Furthermore, we have $\sum_{n=0}^{\infty} (-1)^n c_n z^n = f(i\sqrt{z}), \quad |\arg z| < 2\pi$. It is clear that it has only negative zeros and it's of genus 0. Let $f(z)$ be an entire function of genus at most 1. If it has only real zeros, then for all $a > 0$, the entire function obtained from $f(a-z)f(a+z)$ is an even entire function of genus at most 1 such that it has only real zeros. Then the last claim follows from the last one.

3.2. **Proof of Lemma 2.** Since $f(z)$ is an entire function of genus 0 that has only negative zeros, then

$$f(z) = f(0) \prod_{k=1}^{d(f)} \left(1 + \frac{z}{z_k}\right), \quad \sum_{n=1}^{d(f)} \frac{1}{z_k} < \infty,$$

where $0 < z_1 \le z_2 \le \ldots$ and $\{-z_k\}_{k\ge 1}$ are negative roots of $f(z)$ where its degree $d(f)$ may be finite or infinite. Thus for $x \ge 0,\ n+1 \in \mathbb{N}$ we have

$$(-1)^n \frac{d^n}{dx^n}\left(\frac{f'(x)}{f(x)}\right) = \sum_{k=1}^{d(f)} \frac{n!}{(x+z_k)^{n+1}} > 0,$$

which shows that $\frac{f'(x)}{f(x)}$ is completely monotonic in x. By Laguerre's theorem on separation of zeros in [7], we see that all the zeros of $f'(z)$ also are negative. Furthermore, $f'(z)$ is also of genus 0 because $f(z)$ has genus 0, thus $f'(z) \in G$. Then the same argument $\frac{f'(x)}{f(x)}$ is completely monotonic also shows that $\frac{f^{(2)}(x)}{f^{(1)}(x)}$ is also completely monotonic in x. By induction we see that for all $m,\ n+1\in\mathbb{N}$, and $d(f)\geq m>n$, all the functions $\frac{f^{(n+1)}(x)}{f^{(n)}(x)},\ \cdots,\ \frac{f^{(m)}(x)}{f^{(m-1)}(x)}$ are completely monotonic in x. Therefore,

$$\frac{f^{(m)}(x)}{f^{(n)}(x)}=\frac{f^{(n+1)}(x)}{f^{(n)}(x)}\cdot\dots\cdot\frac{f^{(m)}(x)}{f^{(m-1)}(x)}$$

is also completely monotonic in x.

3.3. **Proof of Corollary 3.** Since $p_n(x)$ is an orthogonal polynomial on the real line, then all its coefficients are real, and all its zeros are inside the convex hull of the support of its orthogonal measures. If the orthogonal measure is symmetric, then $p_{2n}(x)$ and $p_{2n+1}(x)$ are even and odd polynomials respectively. The first assertion follows from Lemma 2 and the observation that all the polynomials have negative zeros under their corresponding conditions. The second and the third assertion follow from the polynomial case in Lemma 2 and assertions 2 and 3 of Theorem 5 in [34].

3.4. **Proof of Corollary 4.** The first assertion is a straightforward application of Lemma 2 and the first assertion in Theorem 6 of [34]. To prove the second assertion we only need to show that $f_2(z)$ is of order at most 1/2; then it would follow from Lemma 2 and the third assertion of Theorem 6 of [34]. For all $z\in\mathbb{C}$ we have

$$|f_2(z)|\leq\sum_{k=0}^{\infty}\frac{|z|^k}{(k!)^{m+\ell}\prod_{r=1}^{\ell}(\beta_r)_k}\leq\sum_{k=0}^{\infty}\frac{|z|^k}{(k!)^2}=I_0(2\sqrt{|z|}).$$

Since $I_0(\sqrt{z})$ is of order 1/2, then $f_2(z)$ is of order at most 1/2. Since

$${}_rA_s^{(\alpha)}(a_1,\dots a_r;\ b_1,\dots,\ b_s\ ;\ q;\ z)$$

is of order 0 if all of its zeros are negative, then the first assertion follows from Lemma 2. From Theorem 7 of [34] we know that under the condition $r,\ s\in\mathbb{N}_0$, $0<q<1,\ \alpha>0,\ 1>a_j,\,b_k>0,\ 1\leq j\leq r,\ 1\leq k\leq s$, and (2.10), the entire function ${}_rA_s^{(\alpha)}(a_1,\dots,a_r;\ b_1,\dots,b_s\ ;\ q;\ z)$ has only negative zeros. Then, by Lemma 2, the function

$$\frac{\partial_x^m\,{}_rA_s^{(\alpha)}(a_1,\dots,a_r;\ b_1,\dots,b_s\ ;\ q;\ x)}{\partial_x^n\,{}_rA_s^{(\alpha)}(a_1,\dots,a_r;\ b_1,\dots,b_s\ ;\ q;\ x)}$$

is completely monotonic for $m,n+1\in\mathbb{N}$ and $m>n$.

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