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OSCILLATION AND NONOSCILLATION CRITERIA FOR SECOND-ORDER NONLINEAR DIFFERENCE EQUATIONS OF EULER TYPE

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Abstract. This paper deals with the oscillatory behavior of solutions of difference equations corresponding to second-order nonlinear differential equations of Euler type. The obtained results are represented as a pair of oscillation and nonoscillation criteria, and are best possible in a certain sense. Linear difference equations corresponding to the Riemann–Weber version of the Euler differential equation and its extended equations play an important role in proving our results. The proofs of our results are based on the Riccati technique and the phase plane analysis of a system.

1. Introduction

We are concerned with oscillation and nonoscillation criteria for the second-order nonlinear difference equation

$$\Delta^2 x(n) + c(n) f(x(n)) = 0, \quad n \in \mathbb{N}, \tag{1.1}$$

where Δ is the forward difference operator $\Delta x(n) = x(n+1) - x(n)$, $\Delta^2 x(n) = \Delta(\Delta x(n))$, $c(n) > 0$ for $n \in \mathbb{N}$, and $f(x)$ is continuous on $\mathbb{R}$ and satisfies the signum condition

$$x f(x) > 0, \quad x \neq 0. \tag{1.2}$$

For simplicity, we use the notation $I_{\mathbb{N}} = I \cap \mathbb{N}$ for the interval $I \subset \mathbb{R}$. Then a nontrivial solution $x(n)$ is said to be *oscillatory* if for every $N \in \mathbb{N}$ there exists $n \in [N, \infty)_{\mathbb{N}}$ such that $x(n)x(n+1) \leq 0$. Otherwise, the solution is said to be *nonoscillatory*. Hence, a nonoscillatory solution $x(n)$ is either eventually positive or eventually negative.

Equation (1.1) naturally includes the linear difference equation

$$\Delta^2 x(n) + \frac{\lambda}{n(n+1)} x(n) = 0, \quad n \in \mathbb{N}, \tag{1.3}$$

as a special case, where $\lambda > 0$. It is known that the condition $\lambda > 1/4$ is necessary and sufficient for all nontrivial solutions of equation (1.3) to be oscillatory (for example, see [7, 13]). In other words, $1/4$ is the lower bound for all nontrivial solutions of equation (1.3) to be oscillatory. Such a number is generally called the *oscillation constant*.

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Equation (1.3) is one of the discrete equations of the Cauchy–Euler differential equation

$$y'' + \frac{\lambda}{t^2} y = 0, \quad t > 0. \tag{1.4}$$

In fact, putting $t = hn$ and $y(t) = x(n)$, equation (1.3) is transformed into the equation

$$\frac{y(t+2h) - 2y(t+h) + y(t)}{h^2} + \frac{\lambda}{t(t+h)} y(t) = 0,$$

and therefore, letting $h \to 0$, we have equation (1.4). Hence, difference equation (1.3) and differential equation (1.4) have similar properties, e.g., these two equations have the same oscillation constant. Such results can be found in [3, 6, 7].

Let us add a perturbation to equation (1.4) with the oscillation constant and consider the linear differential equation

$$y'' + \frac{1}{t^2}\left\{\frac{1}{4} + \frac{1}{4}\sum_{k=1}^{m-1} \frac{1}{\mathrm{Log}_k^2(t)} + \frac{\lambda}{\mathrm{Log}_m^2(t)}\right\} y = 0, \tag{1.5}$$

where

$$\mathrm{Log}_k(t) = \prod_{j=1}^{k} \log_j(t), \quad \log_j(t) = \log(\log_{j-1}(t)), \quad \log_0(t) = t.$$

Then, from the Liouville transformation $s = \log t$, $u(s) = t^{-1/2} y(t)$ successively, we can transform equation (1.5) into equation (1.4), and therefore, the oscillation constant for equation (1.5) is also $1/4$ (for example, see [4, 5, 10, 11, 14]). Note that in the case of $m = 1$, equation (1.5) is called the Riemann–Weber version of the Euler differential equation.

As for a perturbation of difference equation (1.3) with $\lambda = 1/4$, recently, Hongyo and the author [6] considered the linear difference equation

$$\Delta^2 x(n) + \frac{1}{n(n+1)}\left\{\frac{1}{4} + \delta_m(n)\right\} x(n) = 0, \tag{1.6}$$

where

$$\delta_m(n) = \frac{1}{4}\sum_{k=1}^{m-1}\left(\prod_{j=1}^{k} \frac{1}{l_j(n) l_j(n+1)}\right) + \lambda \prod_{j=1}^{m} \frac{1}{l_j(n) l_j(n+1)},$$

$l_0(n) = n$, and $l_j(n)$ is positive and satisfies

$$\Delta l_j(n) = \left(\frac{l_{j-1}(n)}{\Delta l_{j-1}(n)} + \frac{1}{2}\right)^{-1} \tag{1.7}$$

for $n \in \mathbb{N}$. We note that equation (1.6) can be regarded as a discrete equation of (1.5). In fact, as shown in Section 2 below, $l_j(n)$ has similar properties of the logarithm function $\log_j(n)$. Moreover, we see that the oscillation constant for equation (1.6) is also $1/4$; see [6].

The oscillation constant for equation (1.6) plays an important role in the oscillation problem for nonlinear difference equations of the form

$$\Delta^2 x(n) + \frac{1}{n(n+1)} f(x(n)) = 0, \quad n \in \mathbb{N}. \tag{1.8}$$

Indeed, using the oscillation constant for equation (1.6) with $m = 1$, the author [13] presented the following pair of oscillation and nonoscillation criteria.

Theorem A. *Assume* (1.2) *and suppose that there exists* λ *with* $\lambda > 1/4$ *such that*

$$\frac{f(x)}{x} \geq \frac{1}{4} + \frac{\lambda}{\log^2(x^2)}$$

for $|x|$ *sufficiently large. Then all nontrivial solutions of equation* (1.8) *are oscillatory.*

Theorem B. *Assume* (1.2) *and suppose that*

$$\frac{f(x)}{x} \leq \frac{1}{4} + \frac{1}{4\log^2(x^2)}$$

for $|x|$ *sufficiently large. Then equation* (1.8) *has a nonoscillatory solution.*

Here let us consider the second-order nonlinear differential equation of Euler type

$$y'' + \frac{1}{t^2} f(y) = 0, \quad t > 0, \tag{1.9}$$

which corresponds to difference equation (1.8). The study of the oscillation behavior of solutions of equation (1.9) was started by Sugie and Hara [8], and they gave a pair of oscillation and nonoscillation criteria for equation (1.9). After that their results were improved by many authors. For example, see [1, 2, 9, 10, 12, 14] and the references cited therein. In particular, oscillation and nonoscillation criteria given by Sugie and the author [10] can be applied even to the critical case $f(x)/x \to 1/4$ or the more delicate case

$$(\log(x^2))^2 \left\{ \frac{f(x)}{x} - \frac{1}{4} \right\} \to \frac{1}{4} \tag{1.10}$$

as $|x| \to \infty$.

On the other hand, as for difference equation (1.8), by means of Theorems A and B, we cannot judge whether solutions of equation (1.8) are oscillatory or not when $f(x)$ satisfies (1.10). The purpose of this paper is to settle this problem and to improve Theorems A and B. Our main results are stated as follows.

Theorem 1.1. *Assume* (1.2) *and suppose that* $c(n)$ *satisfies*

$$n(n+1)c(n) \geq 1 \tag{1.11}$$

for $n \in \mathbb{N}$ *sufficiently large and that there exist* $m \in \mathbb{N}$ *and* λ *with* $\lambda > 1/4$ *such that*

$$\frac{f(x)}{x} \geq \frac{1}{4} + \frac{1}{4} \sum_{k=1}^{m-1} \frac{1}{\mathrm{Log}_k^2(x^2)} + \frac{\lambda}{\mathrm{Log}_m^2(x^2)} \tag{1.12}$$

for $|x|$ *sufficiently large. Then all nontrivial solutions of equation* (1.1) *are oscillatory.*

Theorem 1.2. *Assume* (1.2) *and suppose that* $c(n)$ *satisfies*

$$n(n+1)c(n) \leq 1 \tag{1.13}$$

for $n \in \mathbb{N}$ *sufficiently large and that there exists* $m \in \mathbb{N}$ *such that*

$$\frac{f(x)}{x} \leq \frac{1}{4} + \frac{1}{4} \sum_{k=1}^{m} \frac{1}{\mathrm{Log}_k^2(x^2)} \tag{1.14}$$

for $|x|$ *sufficiently large. Then equation* (1.1) *has a nonoscillatory solution.*

Remark 1.3. When $c(n) = 1/(n(n+1))$ and $m = 1$, Theorems 1.1 and 1.2 become Theorems A and B, respectively.

2. Preliminaries

In this section, we prepare some lemmas which are useful for proving our results. To begin with, we give the relation between $l_j(n)$ and $\log_j(n)$, and the oscillation and nonoscillation criteria for equation (1.6).

Lemma 2.1 ([6, Lemma 2.3])**.** *Let $j \in \mathbb{N}$. Then there exists $C > 1$ such that*

$$|l_j(n) - \log_j(n)| < C$$

for $n \in \mathbb{N}$ sufficiently large, where $l_j(n)$ is defined by (1.7)*.*

Remark 2.2. From the definition of $l_j(n)$ and Lemma 2.1, we see that $l_j(n)$ has the following properties:

(i) $l_j(n)$ is increasing because $\Delta l_j(n)$ is represented as

$$\Delta l_j(n) = \frac{2\Delta l_{j-1}(n)}{2l_{j-1}(n) + \Delta l_{j-1}(n)} = \prod_{i=0}^{j-1} \frac{2}{l_i(n) + l_i(n+1)} > 0.$$

(ii) $l_j(n) \to \infty$ as $n \to \infty$.

(iii) We can choose $l_j(n)$ such that either $l_j(n) \geq \log_j(n)$ or $l_j(n) \leq \log_j(n)$ for $n \in \mathbb{N}$ sufficiently large. In fact, by Lemma 2.1, there exists $C > 1$ such that

$$0 < l_j(n) - C < \log_j(n) < l_j(n) + C$$

for $n \in \mathbb{N}$ sufficiently large. Let $\tilde{l}_j(n) = l_j(n) + C$. Then $\tilde{l}_j(n)$ satisfies $\tilde{l}_j(n) \geq \log_j(n)$ and $\Delta\tilde{l}_j(n) = \Delta l_j(n)$ for $n \in \mathbb{N}$ sufficiently large. Similarly, let $\hat{l}_j(n) = l_j(n) - C$. Then we have $\hat{l}_j(n) \leq \log_j(n)$ and $\Delta\hat{l}_j(n) = \Delta l_j(n)$ for $n \in \mathbb{N}$ sufficiently large.

Lemma 2.3 ([6, Corollary 4.2])**.** *Equation* (1.6) *can be classified into two types as follows:*

(i) *if $\lambda > 1/4$, then all nontrivial solutions of equation* (1.6) *are oscillatory;*

(ii) *if $0 < \lambda \leq 1/4$, then all nontrivial solutions of equation* (1.6) *are nonoscillatory.*

We next consider the Riccati inequalities

$$\Delta w(n) + \frac{w^2(n) + \delta(n)}{w(n) + n + (1/2)} \leq 0 \tag{2.1}$$

and

$$\Delta w(n) + \frac{w^2(n) + \delta(n)}{w(n) + n + (1/2)} \geq 0, \tag{2.2}$$

where $\delta(n) \geq 0$ for $n \in [n_0, \infty)_{\mathbb{N}}$. Then we have the following lemmas.

Lemma 2.4. *Suppose that $w(n)$ satisfies inequality* (2.1) *for $n \in \mathbb{N}$ sufficiently large and that $\{w(n)\}$ is bounded below. Then the solution is nonincreasing and tends to* 0 *as $n \to \infty$.*

As in the proof of Lemma 3.2 in [13], we can easily prove this lemma, and therefore we omit the proof.

Lemma 2.5. *Let $\delta(n) = C/\log^2 n$, where $C > 0$. Suppose that $w(n)$ is positive and satisfies inequality (2.1) for $n \in \mathbb{N}$ sufficiently large. Then we have*

$$w(n) \le \frac{2}{\log n}$$

for $n \in \mathbb{N}$ sufficiently large.

Proof. Since $w(n)$ is eventually positive and satisfies inequality (2.1) with $\delta(n) = C/\log^2 n$, we see that assumptions in Lemma 2.4 hold. Hence, we have $0 < w(n) < 1/2$ for $n \in \mathbb{N}$ sufficiently large. Let $z(n) = (\log^2 n)w(n) - \log n$. Then, using inequality (2.1) with $\delta(n) = C/\log^2 n$, we have

$$\begin{aligned}
\Delta z(n) &= (\Delta(\log^2 n))w(n) + \log^2(n+1)\Delta w(n) - \Delta \log n \\
&\le (\Delta(\log^2 n))w(n) - \frac{\log^2(n+1)}{w(n)+n+(1/2)}\left\{w^2(n) + \frac{C}{\log^2 n}\right\} - \Delta\log n \\
&\le -\frac{\log^2(n+1)}{n+1}w^2(n) + (\Delta(\log^2 n))w(n) - \frac{C\log^2(n+1)}{(n+1)\log^2 n} - \Delta \log n \\
&= -\frac{\log^2(n+1)}{n+1}\left\{w(n) - \frac{(n+1)(\Delta(\log^2 n))}{2\log^2(n+1)}\right\}^2 \\
&\quad + \frac{(n+1)(\Delta(\log^2 n))^2}{4\log^2(n+1)} - \frac{C\log^2(n+1)}{(n+1)\log^2 n} - \Delta\log n \\
&\le \frac{(n+1)\{(\log n + \log(n+1))\Delta\log n\}^2}{4\log^2(n+1)} - \frac{C\log^2(n+1)}{(n+1)\log^2 n} - \Delta \log n,
\end{aligned}$$

and therefore, we obtain

$$\begin{aligned}
(n+1)\Delta z(n) &\le \left(\frac{(\log n + \log(n+1))(n+1)\Delta\log n}{2\log(n+1)}\right)^2 \\
&\quad - C\left(\frac{\log(n+1)}{\log n}\right)^2 - (n+1)\Delta\log n \\
&\to -C
\end{aligned}$$

as $n \to \infty$. Note that $(n+1)\Delta\log n = \log(1+1/n)^{n+1} \to 1$ as $n \to \infty$. Hence, we have $\Delta z(n) < 0$ for $n \in \mathbb{N}$ sufficiently large. Thus, we obtain

$$w(n) = \frac{1}{\log n} + \frac{z(n)}{\log^2 n} \le \frac{2}{\log n}$$

for $n \in \mathbb{N}$ sufficiently large. □

Lemma 2.6. *Suppose that there exists $n_0 \in \mathbb{N}$ such that $w(n)$ is positive and satisfies inequality (2.1) for $n \in [n_0,\infty)_{\mathbb{N}}$. Then the linear difference equation*

$$\Delta^2 y(n) + \frac{1}{n(n+1)}\left\{\frac{1}{4} + \delta(n)\right\}y(n) = 0 \tag{2.3}$$

has a nonoscillatory solution.

Proof. Let $v(n)$ be a sequence satisfying $v(n_0) = w(n_0) > 0$ and

$$\Delta v(n) + \frac{v^2(n)+\delta(n)}{v(n)+n+(1/2)} = 0 \tag{2.4}$$

for $n \in [n_0, \infty)_{\mathbb{N}}$. Let $F : \mathbb{N} \times [0, \infty) \to \mathbb{R}$ be a function defined by

$$F(n,t) = t - \frac{t^2 + \delta(n)}{t + n + (1/2)}.$$

Then $v(n)$ satisfies $v(n+1) = F(n, v(n))$. Using mathematical induction on n, we show that $v(n)$ is well defined and satisfies $v(n) \geq w(n) > 0$ for $n \in [n_0, \infty)_{\mathbb{N}}$. It is clear that the assertion is true for $n = n_0$. Assume that the assertion is true for $n = p$. Then $v(p+1) = F(p, v(p))$ exists because $v(p) > 0$. Since

$$\frac{d}{dt} F(p,t) = \frac{\{p + (1/2)\}^2 + \delta(p)}{(t + p + (1/2))^2} > 0,$$

$F(p,t)$ is increasing with respect to $t \in (0, \infty)$ for each fixed p. Hence, together with inequality (2.1), we have

$$v(p+1) = F(p, v(p)) \geq F(p, w(p)) \geq w(p+1) > 0.$$

Thus, the assertion is also true for $n = p + 1$.

Let

$$y(n) = \prod_{j=n_0}^{n-1} \left(1 + \frac{1}{2j} + \frac{v(j)}{j}\right), \quad n \in [n_0, \infty)_{\mathbb{N}}.$$

Then we have

$$\Delta y(n) = \left(\frac{1}{2n} + \frac{v(n)}{n}\right) y(n).$$

Hence, we obtain

$$v(n) = \frac{n \Delta y(n)}{y(n)} - \frac{1}{2} = \frac{n y(n+1)}{y(n)} - n - \frac{1}{2},$$

and therefore we get

$$\begin{aligned}
\Delta v(n) &= \frac{\Delta(n \Delta y(n)) y(n) - n (\Delta y(n))^2}{y(n) y(n+1)} \\
&= \frac{\{\Delta y(n) + (n+1) \Delta^2 y(n)\} y(n) - n (\Delta y(n))^2}{y(n) y(n+1)} \\
&= \frac{y(n)}{n y(n+1)} \left\{ \frac{n \Delta y(n)}{y(n)} + \frac{n(n+1) \Delta^2 y(n)}{y(n)} - \left(\frac{n \Delta y(n)}{y(n)}\right)^2 \right\} \\
&= \frac{1}{v(n) + n + (1/2)} \left\{ -v^2(n) + \frac{1}{4} + \frac{n(n+1) \Delta^2 y(n)}{y(n)} \right\}.
\end{aligned}$$

Together with (2.4), we see that $y(n)$ is a positive solution of equation (2.3). $\square$

Lemma 2.7. *Let $v(n)$ be a positive solution of equation* (2.4). *Suppose that $w(n)$ satisfies $w(n_0) = v(n_0)$ and inequality* (2.2) *for $n \in [n_0, n_1)_{\mathbb{N}}$. Then $w(n) \geq v(n)$ for $n \in [n_0, n_1]_{\mathbb{N}}$.*

Proof. We use mathematical induction on n. It is clear that the assertion is true for $n = n_0$. Assume that $w(n) \geq v(n)$ for $n = p \in [n_0, n_1)_{\mathbb{N}}$. Let

$$F(n,t) = t - \frac{t^2 + \delta(n)}{t + n + (1/2)}.$$

Then, as in the proof of Lemma 2.6, we can show that $F(n,t)$ is increasing with respect $t \in [0,\infty)$ for each fixed n. Hence, we have

$$w(p+1) \geq F(p,w(p)) \geq F(p,v(p)) = v(p+1).$$

Thus, the assertion is also true for $n = p+1$. This completes the proof. $\square$

3. Proof of the main theorems

In this section, we prove the oscillation and nonoscillation criteria, Theorems 1.1 and 1.2. Using the following lemma, we first give the proof of Theorem 1.1.

Lemma 3.1. *Assume* (1.2) *and suppose that* $c(n)$ *satisfies* (1.11) *for* $n \in \mathbb{N}$ *sufficiently large, and that equation* (1.1) *has a positive solution. Then the solution is increasing for* $n \in \mathbb{N}$ *sufficiently large and it tends to* ∞ *as* $n \to \infty$.

Proof. Let $x(n)$ be a positive solution of equation (1.1). Then there exists $n_0 \in \mathbb{N}$ such that $x(n) > 0$ for $n \in [n_0,\infty)_{\mathbb{N}}$. Hence, by (1.2), we have

$$\Delta^2 x(n) = -c(n)f(x(n)) < 0 \tag{3.1}$$

for $n \in [n_0,\infty)_{\mathbb{N}}$.

We first show that $\Delta x(n) > 0$ for $n \in [n_0,\infty)_{\mathbb{N}}$. By way of contradiction, we suppose that there exists $n_1 \in [n_0,\infty)_{\mathbb{N}}$ such that $\Delta x(n_1) \leq 0$. Then, using (3.1), we have $\Delta x(n) < \Delta x(n_1) \leq 0$ for $n \in (n_1,\infty)_{\mathbb{N}}$. Using (3.1) again, we get $\Delta x(n) \leq \Delta x(n_1+1) < 0$ for $n \in (n_1,\infty)_{\mathbb{N}}$. Hence, we obtain

$$x(n) \leq \Delta x(n_1+1)(n-(n_1+1)) + x(n_1+1) \to -\infty$$

as $n \to \infty$, which is a contradiction to the assumption that $x(n)$ is positive for $n \in [n_0,\infty)_{\mathbb{N}}$. Thus, $x(n)$ is increasing for $n \in [n_0,\infty)_{\mathbb{N}}$.

We next suppose that $\{x(n)\}$ is bounded from above. Then there exists a positive number L such that $\lim_{n\to\infty} x(n) = L$. Since $f(x)$ is continuous on $\mathbb{R}$, we have $\lim_{n\to\infty} f(x(n)) = f(L)$, and therefore there exists $n_2 \in [n_0,\infty)_{\mathbb{N}}$ such that $0 < f(L)/2 < f(x(n))$ for $n \in [n_2,\infty)_{\mathbb{N}}$. Hence, together with (1.11), we have

$$\Delta x(n) = \Delta x(p) + \sum_{j=n}^{p-1} c(j)f(x(j)) > \frac{f(L)}{2}\sum_{j=n}^{p-1}\frac{1}{j(j+1)} = \frac{f(L)}{2}\left(\frac{1}{n}-\frac{1}{p}\right)$$

for $n,p \in [n_2,\infty)_{\mathbb{N}}$ with $p > n$. Taking the limit of this inequality as $p \to \infty$, we get $\Delta x(n) \geq f(L)/(2n)$ for $n \in [n_2,\infty)_{\mathbb{N}}$, and therefore we obtain

$$x(n) \geq \frac{f(L)}{2}\sum_{j=n_2}^{n-1}\frac{1}{j} + x(n_2) \to \infty$$

as $n \to \infty$. This contradicts the assumption that $\{x(n)\}$ is bounded from above. Thus, we have $\lim_{n\to\infty} x(n) = \infty$. $\square$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $R > 0$ be a large number satisfying (1.12) for $|x| \geq R$. Since $\lambda > 1/4$, there exists $\varepsilon_0 > 0$ such that

$$\frac{1}{4} < \frac{1}{4} + \varepsilon_0 < \lambda. \tag{3.2}$$

The proof is by contradiction. Suppose that equation (1.1) has a nonoscillatory solution $x(n)$. Then, without loss of generality, we may assume that $x(n)$ is eventually positive. By Lemma 3.1, we see that $x(n) \geq R$ and $\Delta x(n) > 0$ for $n \in \mathbb{N}$ sufficiently large. Put

$$w(n) = \frac{n\Delta x(n)}{x(n)} - \frac{1}{2}.$$

Then we have $w(n) > -1/2$ for $n \in \mathbb{N}$ sufficiently large. Moreover, using (1.11) and (1.12), we also have

$$\begin{aligned}
\Delta w(n) &= \frac{\Delta(n\Delta x(n))x(n) - n(\Delta x(n))^2}{x(n)x(n+1)} \\
&= \frac{\{\Delta x(n) + (n+1)\Delta^2 x(n)\}x(n)}{x(n)x(n+1)} - \frac{n(\Delta x(n))^2}{x(n)x(n+1)} \\
&\leq \frac{\{\Delta x(n) - f(x(n))/n\}x(n)}{x(n)x(n+1)} - \left(\frac{n\Delta x(n)}{x(n)}\right)^2 \frac{x(n)}{nx(n+1)} \qquad (3.3)\\
&= -\frac{x(n)}{nx(n+1)}\left\{-\frac{n\Delta x(n)}{x(n)} + \frac{f(x(n))}{x(n)} + \left(\frac{n\Delta x(n)}{x(n)}\right)^2\right\} \\
&\leq \frac{-1}{w(n)+n+(1/2)}\left\{w^2(n) + \frac{1}{4}\sum_{k=1}^{m-1}\frac{1}{\mathrm{Log}_k^2(x^2(n))} + \frac{\lambda}{\mathrm{Log}_m^2(x^2(n))}\right\}
\end{aligned}$$

for $n \in \mathbb{N}$ sufficiently large. Hence, by Lemma 2.4, there exists $n_0 \in \mathbb{N}$ such that

$$0 < w(n) \leq \frac{\varepsilon_0}{4}$$

for $n \in [n_0, \infty)_{\mathbb{N}}$. Hence, we have

$$\frac{x(n+1)}{x(n)} \leq 1 + \frac{1}{2n}\left(1 + \frac{\varepsilon_0}{2}\right),$$

and therefore we obtain

$$\frac{x(n)}{x(n_0)} = \prod_{j=n_0}^{n-1}\frac{x(j+1)}{x(j)} \leq \prod_{j=n_0}^{n-1}\left\{1 + \frac{1}{2j}\left(1 + \frac{\varepsilon_0}{2}\right)\right\}$$

for $n \in [n_0, \infty)_{\mathbb{N}}$. Since $\log(1+z) \leq z$ for $z > -1$, we have

$$\begin{aligned}
\log x(n) &\leq \sum_{j=n_0}^{n-1}\log\left\{1 + \frac{1}{2j}\left(1 + \frac{\varepsilon_0}{2}\right)\right\} + \log x(n_0) \\
&\leq \sum_{j=n_0}^{n-1}\frac{1}{2j}\left(1 + \frac{\varepsilon_0}{2}\right) + \log x(n_0) = \frac{1}{2}\left(1 + \frac{\varepsilon_0}{2}\right)\sum_{j=n_0}^{n-1}\frac{1}{j} + \log x(n_0) \\
&\leq \frac{1}{2}\left(1 + \frac{\varepsilon_0}{2}\right)\sum_{j=n_0}^{n-1}\int_{j-1}^{j}\frac{dt}{t} + \log x(n_0) \leq \frac{1}{2}(1+\varepsilon_0)\log n
\end{aligned}$$

for $n \in \mathbb{N}$ sufficiently large. Hence, by (3.3), we have

$$\begin{aligned}\Delta w(n) &\le -\frac{1}{w(n)+n+(1/2)}\left\{w^2(n)+\frac{1}{4}\sum_{k=1}^{m-1}\frac{1}{\mathrm{Log}_k^2(x^2(n))}+\frac{\lambda}{\mathrm{Log}_m^2(x^2(n))}\right\}\\ &\le -\frac{1}{w(n)+n+(1/2)}\left\{w^2(n)+\frac{1}{4(\log x^2(n))^2}\right\}\\ &\le -\frac{1}{w(n)+n+(1/2)}\left\{w^2(n)+\frac{1}{4(1+\varepsilon_0)^2\log^2 n}\right\}\end{aligned}$$

for $n \in \mathbb{N}$ sufficiently large. Hence, from Lemma 2.5, we obtain $w(n) \le 2/\log n$, and therefore there exists $n_1 \in [n_0,\infty)_{\mathbb{N}}$ such that

$$\frac{x(n+1)}{x(n)} \le 1+\frac{1}{2n}+\frac{2}{n\log n}$$

for $n \in [n_1,\infty)_{\mathbb{N}}$. Thus, we have

$$\frac{x(n)}{x(n_1)} = \prod_{j=n_1}^{n-1}\frac{x(j+1)}{x(j)} \le \prod_{j=n_1}^{n-1}\left(1+\frac{1}{2j}+\frac{2}{j\log j}\right)$$

for $n \in [n_1,\infty)_{\mathbb{N}}$. Hence, there exists $M>0$ such that

$$\begin{aligned}\log x(n) &\le \sum_{j=n_1}^{n-1}\log\left(1+\frac{1}{2j}+\frac{2}{j\log j}\right)+\log x(n_1)\\ &\le \sum_{j=n_1}^{n-1}\left(\frac{1}{2j}+\frac{2}{j\log j}\right)+\log x(n_1)\\ &\le \sum_{j=n_1}^{n-1}\left(\frac{1}{2}\int_{j-1}^{j}\frac{dt}{t}+2\int_{j-1}^{j}\frac{dt}{t\log t}\right)+\log x(n_1)\\ &\le \frac{1}{2}\log n+\frac{M}{2}\log_2(n)\end{aligned}$$

for $n \in \mathbb{N}$ sufficiently large. Hence, we have

$$\log_j(x^2(n)) \le \log_j(n)\left(1+\frac{M\log_2(n)}{\log n}\right), \quad j \in [1,m]_{\mathbb{N}}, \tag{3.4}$$

for $n \in \mathbb{N}$ sufficiently large. Indeed, we can easily check this inequality by using mathematical induction on j. It is clear that (3.4) is true for $j=1$. Assume that (3.4) with $j \in \mathbb{N}$ holds. Then we have

$$\begin{aligned}\log_{j+1}(x^2(n)) &= \log(\log_j(x^2(n))) \le \log\left\{\log_j(n)\left(1+\frac{M\log_2(n)}{\log n}\right)\right\}\\ &= \log(\log_j(n))+\log\left(1+\frac{M\log_2(n)}{\log n}\right) \le \log_{j+1}(n)+\frac{M\log_2(n)}{\log n}\\ &\le \log_{j+1}(n)\left(1+\frac{M\log_2(n)}{\log n}\right)\end{aligned}$$

for $n \in \mathbb{N}$ sufficiently large. Since $(1+z)^{-\alpha} \geq 1 - \alpha z$ for $z > 0$ and $\alpha > 0$, we have the inequality

$$\begin{aligned}
\frac{1}{\mathrm{Log}_k^2(x^2(n))} &= \left(\prod_{j=1}^{k} \log_j(x^2(n))\right)^{-2} \geq \left[\prod_{j=1}^{k} \left\{\log_j(n)\left(1 + \frac{M\log_2(n)}{\log n}\right)\right\}\right]^{-2} \\
&= \left(\prod_{j=1}^{k} \log_j(n)\right)^{-2} \left(1 + \frac{M\log_2(n)}{\log n}\right)^{-2k} \\
&\geq \left(\prod_{j=1}^{k} \frac{1}{\log_j^2(n)}\right)\left(1 - \frac{2mM\log_2(n)}{\log n}\right) \\
&\geq \prod_{j=1}^{k} \frac{1}{\log_j^2(n)} - \frac{\log_2^2(n)}{\log^3 n}, \quad k \in [1, m]_{\mathbb{N}},
\end{aligned}$$

for $n \in \mathbb{N}$ sufficiently large. Moreover, from Remark 2.2(iii), we can choose $l_j(n)$ satisfying $\log_j(n) \leq l_j(n) < l_j(n+1)$ for $n \in \mathbb{N}$ sufficiently large. Hence, by (3.3), we have

$$\begin{aligned}
&\left(w(n) + n + \frac{1}{2}\right)\Delta w(n) \\
&\leq -\left\{w^2(n) + \frac{1}{4}\sum_{k=1}^{m-1} \frac{1}{\mathrm{Log}_k^2(x^2(n))} + \frac{\lambda}{\mathrm{Log}_m^2(x^2(n))}\right\} \\
&\leq -\left\{w^2(n) + \frac{1}{4}\sum_{k=1}^{m-1}\left(\prod_{j=1}^{k}\frac{1}{\log_j^2(n)}\right) + \lambda\prod_{j=1}^{m}\frac{1}{\log_j^2(n)} - \left(\frac{m-1}{4} + \lambda\right)\frac{\log_2^2(n)}{\log^3 n}\right\} \\
&\leq -\left\{w^2(n) + \frac{1}{4}\sum_{k=1}^{m-1}\left(\prod_{j=1}^{k}\frac{1}{\log_j^2(n)}\right) + (\lambda - \varepsilon_0)\prod_{j=1}^{m}\frac{1}{\log_j^2(n)}\right\} \\
&\leq -\left\{w^2(n) + \tilde{\delta}(n)\right\}
\end{aligned}$$

for $n \in \mathbb{N}$ sufficiently large, where

$$\tilde{\delta}(n) = \frac{1}{4}\sum_{k=1}^{m-1}\left(\prod_{j=1}^{k}\frac{1}{l_j(n)l_j(n+1)}\right) + (\lambda - \varepsilon_0)\prod_{j=1}^{m}\frac{1}{l_j(n)l_j(n+1)}.$$

Hence, by Lemma 2.6, the linear difference equation

$$\Delta^2 y(n) + \frac{1}{n(n+1)}\left\{\frac{1}{4} + \tilde{\delta}(n)\right\} y(n) = 0 \tag{3.5}$$

has a nonoscillatory solution. However, by (3.2), we have $\lambda - \varepsilon_0 > 1/4$. Hence, from Lemma 2.3, we see that all nontrivial solutions of equation (3.5) are oscillatory. This is a contradiction. The proof is now complete. □

We next prove Theorem 1.2.

Proof of Theorem 1.2. We only give the proof of the case that (1.14) holds for $x \geq R$, where R is a large number. This is because the other case is carried out in the same manner.

Let us consider the linear difference equation

$$\Delta^2 u(n) + \frac{1}{n(n+1)}\left\{\frac{1}{4} + \frac{1}{4}\sum_{k=1}^{m}\left(\prod_{j=1}^{k}\frac{1}{l_j(n)l_j(n+1)}\right)\right\}u(n) = 0, \tag{3.6}$$

which is equivalent to equation (1.6) with $\lambda = 1/4$. Note that, from Remark 2.2(iii), we can choose $l_j(n)$ satisfying

$$2\log R \le \log_1(n) \le l_1(n) \quad \text{and} \quad l_j(n) \le \log_j(n), \quad j \in [2,m]_{\mathbb{N}} \tag{3.7}$$

for $n \in \mathbb{N}$ sufficiently large. From Lemma 2.3(ii), equation (3.6) has a positive solution $u(n)$. As in the proof of Lemma 3.1, we can show that $\Delta u(n) > 0$ for $n \in \mathbb{N}$ sufficiently large and that $u(n) \to \infty$ as $n \to \infty$. Put

$$v(n) = \frac{n\Delta u(n)}{u(n)} - \frac{1}{2}.$$

Then we see that $v(n)$ satisfies

$$\Delta v(n) = -\frac{1}{v(n) + n + (1/2)}\left\{v^2(n) + \frac{1}{4}\sum_{k=1}^{m}\left(\prod_{j=1}^{k}\frac{1}{l_j(n)l_j(n+1)}\right)\right\}$$

for $n \in \mathbb{N}$ sufficiently large. Moreover, we see that $\{v(n)\}$ is bounded below because $u(n)$ and $\Delta u(n)$ are positive for $n \in \mathbb{N}$ sufficiently large. Hence, from Lemma 2.4, we have that $v(n) \searrow 0$ as $n \to \infty$, and therefore we obtain

$$0 < v(n) < \frac{1}{2} \tag{3.8}$$

for $n \in [n_0, \infty)_{\mathbb{N}}$. From the facts above, there exists $n_0 \in \mathbb{N}$ such that (1.13), (3.7), and (3.8) hold for $n \in [n_0, \infty)_{\mathbb{N}}$. Since equation (3.6) is linear, $(e^{l_1(n_0+1)/2}/u(n_0))u(n)$ is also a positive solution of equation (3.6). Hence, without loss of generality, we may assume that $u(n)$ satisfies

$$R \le e^{l_1(n_0+1)/2} = u(n_0) \le u(n) \tag{3.9}$$

for $n \in \mathbb{N}$ sufficiently large.

By way of contradiction, we suppose that all nontrivial solutions of equation (1.1) are oscillatory. Let $x(n)$ be the oscillatory solution of equation (1.1) satisfying the initial condition

$$x(n_0) = u(n_0) \quad \text{and} \quad \Delta x(n_0) = \Delta u(n_0) \tag{3.10}$$

and let $y(n) = n\Delta x(n) - x(n)$. Then $(x(n), y(n))$ satisfies

$$\begin{cases} n\Delta x(n) = y(n) + x(n), \\ n\Delta y(n) = -n(n+1)c(n)f(x(n)). \end{cases} \tag{3.11}$$

From (3.8) and (3.10), we have $y(n_0) = n_0\Delta x(n_0) - x(n_0) > -x(n_0)/2$, and therefore we get

$$(x(n_0), y(n_0)) \in \{(x,y) : x \ge R,\ -x/2 \le y < 0\} \stackrel{\text{def}}{=} D.$$

Since $x(n)$ is oscillatory, $(x(n), y(n))$ cannot stay in D. Moreover, we also see that if $(x(n), y(n)) \in D$, then $(x(n+1), y(n+1))$ is in the 4th quadrant. In fact, by

$n\Delta x(n) > 0$ and $n\Delta y(n) < 0$, we obtain $x(n+1) > x(n)$ and $y(n+1) < y(n)$. Hence, there exists $n_1 \in (n_0, \infty)_{\mathbb{N}}$ such that $(x(n), y(n)) \in D$ for $n \in [n_0, n_1)_{\mathbb{N}}$ and

$$(x(n_1), y(n_1)) \in \{(x,y) : y < -x/2 < 0\}. \tag{3.12}$$

Then, we obtain

$$n\Delta x(n) = y(n) + x(n) \geq -\frac{x(n)}{2} + x(n) = \frac{x(n)}{2},$$

and therefore we have

$$\frac{x(n)}{x(n_0)} = \prod_{j=n_0}^{n-1} \frac{x(j+1)}{x(j)} \geq \prod_{j=n_0}^{n-1} \left(1 + \frac{1}{2j}\right)$$

for $n \in [n_0, n_1)_{\mathbb{N}}$. Hence, by using (3.7), (3.9), (3.10), and the relation $\log(1+z) \geq z - z^2/2$ for $z \geq 0$, we obtain

$$\begin{aligned}
\log x(n) \geq& \sum_{j=n_0}^{n-1} \log\left(1 + \frac{1}{2j}\right) + \log x(n_0) \geq \sum_{j=n_0}^{n-1} \left\{\frac{1}{2j} - \frac{1}{2}\left(\frac{1}{2j}\right)^2\right\} + \frac{l_1(n_0+1)}{2} \\
=&\frac{1}{2}\sum_{j=n_0}^{n-1}\left(\frac{1}{j} - \frac{1}{4j^2}\right) + \frac{l_1(n_0+1)}{2} \geq \frac{1}{2}\sum_{j=n_0}^{n-1} \frac{2}{2j+3} + \frac{l_1(n_0+1)}{2} \\
=&\frac{1}{2}\left\{\sum_{j=n_0}^{n-1} \Delta l_1(j+1) + l_1(n_0+1)\right\} = \frac{l_1(n+1)}{2} \geq \frac{\log_1(n+1)}{2}
\end{aligned}$$

for $n \in [n_0, n_1)_{\mathbb{N}}$. Hence, by (3.7), we have $\log_j(x^2(n)) \geq \log_j(n+1) \geq l_j(n+1)$ for $j \in [2, m]_{\mathbb{N}}$, and therefore we obtain

$$\begin{aligned}
\mathrm{Log}_k^2(x^2(n)) = \left(\prod_{j=1}^{k} \log_j(x^2(n))\right)^2 \geq \left(\prod_{j=1}^{k} l_j(n+1)\right)^2 \\
\geq \prod_{j=1}^{k} l_j(n) l_j(n+1)
\end{aligned} \tag{3.13}$$

for $n \in [n_0, n_1)_{\mathbb{N}}$.

We define

$$w(n) = \frac{y(n)}{x(n)} + \frac{1}{2}.$$

Then, using (1.13), (1.14), (3.11), and (3.13), we have

$$\begin{aligned}
\Delta w(n) =& \frac{(\Delta y(n))x(n) - y(n)\Delta x(n)}{x(n)x(n+1)} \geq \frac{-f(x(n))x(n) - y(n)(y(n) + x(n))}{nx(n)x(n+1)} \\
=& -\frac{x(n)}{nx(n+1)}\left\{\frac{f(x(n))}{x(n)} + \left(\frac{y(n)}{x(n)}\right)^2 + \frac{y(n)}{x(n)}\right\} \\
\geq& -\frac{1}{w(n) + n + (1/2)}\left\{w^2(n) + \frac{1}{4}\sum_{k=1}^{m} \frac{1}{\mathrm{Log}_k^2(x^2(n))}\right\} \\
\geq& -\frac{1}{w(n) + n + (1/2)}\left\{w^2(n) + \frac{1}{4}\sum_{k=1}^{m} \left(\prod_{j=1}^{k} \frac{1}{l_j(n) l_j(n+1)}\right)\right\}
\end{aligned}$$

for $n \in [n_0, n_1)_{\mathbb{N}}$. Note that, from (3.10) and (3.12), $w(n)$ satisfies

$$w(n_0) = \frac{y(n_0)}{x(n_0)} + \frac{1}{2} = \frac{n_0 \Delta u(n_0)}{u(n_0)} - \frac{1}{2} = v(n_0) \quad \text{and} \quad w(n_1) < 0. \tag{3.14}$$

Hence, it follows from Lemma 2.7 that $v(n) \leq w(n)$ for $n \in [n_0, n_1]_{\mathbb{N}}$. Thus, together with (3.8) and (3.14), we have $0 < v(n_1) \leq w(n_1) < 0$, which is a contradiction. This completes the proof. □

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