PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 5, May 2018, Pages 1879–1893
http://dx.doi.org/10.1090/proc/13893
Article electronically published on December 4, 2017

THE DIMENSION OF AUTOMORPHISM GROUPS OF ALGEBRAIC VARIETIES WITH PSEUDO-EFFECTIVE LOG CANONICAL DIVISORS

FEI HU

(Communicated by Lev Borisov)

Abstract. Let (X, D) be a log smooth pair of dimension n, where D is a reduced effective divisor such that the log canonical divisor $K_X + D$ is pseudo-effective. Let G be a connected algebraic subgroup of $\mathrm{Aut}(X, D)$. We show that G is a semi-abelian variety of dimension $\leq \min\{n - \bar{\kappa}(V), n\}$ with $V := X \setminus D$. In the dimension two, Iitaka claimed in his 1979 Osaka J. Math. paper that $\dim G \leq \bar{q}(V)$ for a log smooth surface pair with $\bar{\kappa}(V) = 0$ and $\bar{p}_g(V) = 1$. We (re-)prove and generalize this classical result for all surfaces with $\bar{\kappa} = 0$ without assuming Iitaka's classification of logarithmic Iitaka surfaces or logarithmic $K3$ surfaces.

1. Introduction

Throughout this paper, unless otherwise stated, we work over the field $\mathbb{C}$ of complex numbers. Let V be an algebraic variety. By Nagata, there is a complete algebraic variety $\overline{V}$ containing V as a Zariski-dense open subvariety. Then by Hironaka, there exist a *log smooth* pair (X, D), i.e., X is a smooth projective variety and D is a reduced effective divisor with only simple normal crossing (SNC) singularities, and a projective birational morphism $\pi\colon X \to \overline{V}$ such that $D = \pi^{-1}(\overline{V} \setminus V)$ and $X \setminus D = \pi^{-1}(V)$. Such a pair is called a *log smooth completion* of V, and D is called the *boundary divisor*. We then define:

$$\text{the } \textit{logarithmic irregularity } \bar{q}(V) := h^0(X, \Omega_X^1(\log D)),$$
$$\text{the } \textit{logarithmic geometric genus } \bar{p}_g(V) := h^0(X, K_X + D),$$
$$\text{the } \textit{logarithmic Kodaira dimension } \bar{\kappa}(V) := \kappa(X, K_X + D),$$

where $\Omega_X^1(\log D)$ is the logarithmic differential sheaf, $h^i(-)$ denotes the complex dimension of $H^i(-)$ and κ denotes the Iitaka D-dimension. It is known that these numerical invariants are independent of the choice of the log smooth completion (X, D). See [10, §11] for details.

Let G be a connected algebraic group. By Chevalley's structure theorem on algebraic groups, there exists a unique connected affine normal subgroup G_{aff} of G such that the quotient group $A_G := G/G_{\mathrm{aff}}$ is an abelian variety. Moreover, the quotient morphism is the Albanese morphism alb_G of G. If G_{aff} is an algebraic

Received by the editors February 16, 2017, and, in revised form, June 28, 2017 and June 30, 2017.

2010 *Mathematics Subject Classification.* Primary 14J50, 14L10, 14L30.

Key words and phrases. Automorphism, semi-abelian variety, group action, logarithmic Kodaira dimension.

torus, denoted as T_G, then G is called a *semi-abelian variety*; i.e., there is an exact sequence of connected algebraic groups:

$$1 \longrightarrow T_G \longrightarrow G \xrightarrow{\mathrm{alb}_G} A_G \longrightarrow 1. \tag{1.1}$$

It is known that such G is a commutative algebraic group (see e.g. [1, Proposition 3.1.1]).

Due to Serre [23, Théorèmes 5 and 7], there exist an abelian variety $\mathrm{Alb}(V)$ (resp. a semi-abelian variety $\mathcal{A}_V$) and a morphism $\mathrm{alb}_V : V \to \mathrm{Alb}(V)$ (resp. a morphism $\alpha_V : V \to \mathcal{A}_V$) such that any morphism from V to an abelian variety (resp. a semi-abelian variety) factors, uniquely up to translations, through this $\mathrm{Alb}(V)$ (resp. $\mathcal{A}_V$). Then $\mathrm{Alb}(V)$ (resp. alb_V) is called the *Albanese variety* (resp. the *Albanese morphism*) of V, and $\mathcal{A}_V$ (resp. α_V) is called the *quasi-Albanese variety* (resp. the *quasi-Albanese morphism*) of V. Note, however, that this construction of the Albanese morphism is, in general, not of a birational nature. Alternatively, one can *birationally* define the Albanese variety and the Albanese map (which is only a rational map; cf. [16, Chapter II, §3]). See [23, Théorème 6] for the relation between these two definitions. From the viewpoint of birational geometry, they are the same in characteristic zero for normal projective varieties with only rational singularities (cf. [12, Lemma 8.1]).

Let V be a smooth algebraic variety with some log smooth completion (X, D) obtained by blowing up subvarieties of the boundary such that $V = X \setminus D$. Then the Albanese varieties of V and X are isomorphic to each other and the Albanese morphism alb_V of V is just the restriction of the Albanese morphism alb_X of X. Also, the Albanese morphism alb_V of V factors through the quasi-Albanese morphism α_V of V. That is, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{A}_V \arrow[drr, "\mathrm{alb}_{\mathcal{A}_V}"'] & & V \arrow[ll, "\alpha_V"'] \arrow[r, hook, "j"] \arrow[d, "\mathrm{alb}_V"] & X \arrow[d, "\mathrm{alb}_X"] \\
& & \mathrm{Alb}(V) \arrow[r, "\simeq", "\mathrm{alb}(j)"'] & \mathrm{Alb}(X).
\end{tikzcd}$$

Further, the quasi-Albanese variety $\mathcal{A}_V$ of V can be constructed using the space of logarithmic 1-forms $H^0(X, \Omega^1_X(\log D))$. See [5, 7] for more details about this construction, which depends on Deligne's mixed Hodge theory for smooth complex algebraic varieties (unlike Serre's construction [23], which is valid over an algebraically closed field of arbitrary characteristic). It is known that $\dim \mathcal{A}_V = \bar{q}(X)$ and $\dim \mathrm{Alb}(V) = q(X) := h^1(X, \mathcal{O}_X)$. If we assume further that V is projective, then the quasi-Albanese morphism α_V of V is just the original Albanese morphism alb_V of V.

We shall refer to [15] for the standard definitions, notation, and terminologies in birational geometry. For instance, see [15, Definitions 2.34, 2.37, and 5.8] for the definitions of Kawamata log terminal singularity (*klt*), log canonical singularity (*lc*), divisorial log terminal singularity (*dlt*), and *rational singularity.*

Theorem 1.1. *Let (X, D) be a projective $\mathbb{Q}$-factorial dlt pair of dimension n, where D is a reduced effective divisor such that $K_X + D$ is pseudo-effective. Let* $\mathrm{Aut}(X, D)$ *denote the stabilizer of the boundary D (viewed as a subset of X) in the automorphism group* $\mathrm{Aut}(X)$ *of X. Let G be a connected algebraic subgroup of* $\mathrm{Aut}(X, D)$*. Then the following assertions hold.*

(1) *G is a semi-abelian variety sitting in the exact sequence* (1.1) *of dimension at most*
$$\min\{n - \kappa(X, K_X + D), n\}.$$
(2) *When* $\dim G = n$, *X is a G-equivariant compactification of G such that* $K_X + D \sim 0$.
(3) *Suppose further that $\kappa(X, K_X + D) \geq 0$. Then we have:*
   (a) $\dim G \leq n$ *and the equality holds only if $\kappa(X, K_X + D) = 0$ and the dimension of the abelian variety A_G equals $q(X)$;*
   (b) $\dim T_G \leq n$ *and the equality holds only if $\kappa(X, K_X + D) = 0$ and* $\dim A_G = q(X) = 0$.

A *logarithmic Iitaka surface* is a smooth algebraic surface V such that the logarithmic Kodaira dimension $\bar{\kappa}(V) = 0$ and the logarithmic geometric genus $\bar{p}_g(V) = 1$. In this case by Kawamata [11, Corollary 29], we know that the logarithmic irregularity $\bar{q}(V) \leq \dim V = 2$. If we assume further that $\bar{q}(V) = 0$, we then call V a *logarithmic $K3$ surface*. See [9] for details.

Next, we (re-)prove and generalize [9, Theorem 5], in which Iitaka provided an upper bound of the dimension of automorphism groups of certain logarithmic Iitaka surfaces. However, his (implicit) proof depends heavily on his classification of logarithmic Iitaka surfaces and logarithmic $K3$ surfaces, so that we are not able to follow his proof completely. Here we offer a classification-free proof for all smooth surfaces with vanishing logarithmic Kodaira dimension.

Theorem 1.2. *Let (X, D) be a log smooth pair of dimension 2 with $V := X \setminus D$ such that $\bar{\kappa}(V) = 0$. Let G be a connected algebraic subgroup of* $\mathrm{Aut}(X, D)$. *Then G is a semi-abelian variety of dimension at most $\bar{q}(V)$. If we assume further that $\bar{p}_g(V) = 0$, then* $\dim G \leq q(X)$.

Remark 1.3. It is known that for an abelian variety A acting faithfully on a smooth algebraic variety X, the induced group homomorphism $A \to \mathrm{Alb}(X)$ has a finite kernel by the Nishi–Matsumura theorem (cf. [17]). In particular, we have $\dim A \leq \dim \mathrm{Alb}(X) = q(X)$. However, for a semi-abelian variety G acting faithfully on a smooth algebraic variety V, by Brion's example[1] below one cannot try to prove $G \to \mathcal{A}_V$ has a finite kernel and to deduce $\dim G \leq \bar{q}(V)$.

Let X be the projective plane $\mathbb{P}^2$ and D the union of a smooth conic and a transversal line. In homogeneous coordinates, one can take for D the union of $(xy = z^2)$ and $(z = 0)$. Then the neutral component of the automorphism group of (X, D) is a one-dimensional algebraic torus, acting via $t \cdot [x : y : z] = [tx : t^{-1}y : z]$. Also, $V := X \setminus D$ is the complement of the conic $(xy = 1)$ in the affine plane $\mathbb{A}^2$ with coordinates x, y. So the quasi-Albanese variety $\mathcal{A}_V$ of V is a one-dimensional algebraic torus too, and the quasi-Albanese morphism α_V is just given by $xy - 1$ (which generates the group of all invertible regular functions on V modulo constants). Then α_V is G-invariant, and hence G does not act on $\mathcal{A}_V$ with a finite kernel.

The following two corollaries are direct consequences of our main theorems, Sumihiro's equivariant completion theorem (cf. [24, Theorem 3]), and the equivariant

[1]The author is grateful to Professor Michel Brion for a conversation about his (counter) example.

resolution theorem (see [14, Proposition 3.9.1 and Theorem 3.36] for a modern description). Indeed, let V be a normal algebraic variety, and let G be a *linear* algebraic subgroup of $\mathrm{Aut}(V)$. Sumihiro's theorem asserts that there exists a G-equivariant completion $\overline{V}$ of V. Let (X, D) be a G-equivariant resolution of singularities of $\overline{V}$. Thus we may identify G with a subgroup of $\mathrm{Aut}(X, D)$ so that our main theorems apply.

Corollary 1.4. *Let V be a normal algebraic variety of logarithmic Kodaira dimension $\bar{\kappa}(V) \geq 0$, and let G be a connected linear algebraic subgroup of $\mathrm{Aut}(V)$. Then G is an algebraic torus of dimension at most $\min\{\dim V - \bar{\kappa}(V), \dim V\}$.*

Corollary 1.5. *Let V be a smooth algebraic surface, where (X, D) is a log smooth completion such that $V = X \setminus D$. Suppose that $\bar{\kappa}(V) = 0$ and let G be a connected linear algebraic subgroup of $\mathrm{Aut}(V)$. Then G is an algebraic torus of dimension at most $\bar{q}(X)$. If we assume further that $\bar{p}_g(V) = 0$, then $\dim G \leq q(X)$.*

2. Proof of Theorem 1.1

We first prove that G is a semi-abelian variety (see (1.1) for its definition and related notation) under a slightly weaker condition than that of Theorem 1.1.

We remark that G is a semi-abelian variety if and only if G does not contain any algebraic subgroup isomorphic to the one-dimensional additive algebraic group $\mathbf{G}_a$. In fact, by Chevalley's structure theorem, to show G is a semi-abelian variety, it suffices to show that the affine normal subgroup G_{aff} of G is an algebraic torus. Consider the unipotent radical $R_u(G_{\mathrm{aff}})$ of G_{aff}. If it is not trivial, then it contains $\mathbf{G}_a$. So we may assume that G_{aff} is reductive. Note that any non-trivial semi-simple subgroup of G_{aff} also contains $\mathbf{G}_a$. Thus by the structure theory of reductive groups, $G_{\mathrm{aff}} = R(G_{\mathrm{aff}})$ is an algebraic torus.

Lemma 2.1. *Let (X, D) be a projective log canonical pair, where D is a reduced effective divisor such that $K_X + D$ is pseudo-effective. Let G be a connected algebraic subgroup of $\mathrm{Aut}(X, D)$. Then G is a semi-abelian variety.*

Proof. Take a G-equivariant log resolution $\pi\colon \widetilde{X} \to X$ of the pair (X, D). Then we may write

$$K_{\widetilde{X}} + \widetilde{D} = \pi^*(K_X + D) + \sum a_i E_i,$$

where $\widetilde{D} := \pi_*^{-1} D + E$ with $\pi_*^{-1} D$ the strict transform of D and $E := \sum E_i$ the sum of all π-exceptional divisors. Note that for every E_i, the log discrepancy $a_i := 1 + a(E_i, X, D)$ is non-negative, since (X, D) is log canonical. Thus $K_{\widetilde{X}} + \widetilde{D}$ is also pseudo-effective. Moreover, G is a subgroup of $\mathrm{Aut}(\widetilde{X}, \widetilde{D})$ since π is a G-equivariant log resolution. Therefore, replacing (X, D) by $(\widetilde{X}, \widetilde{D})$, we may assume that (X, D) is log smooth.

Suppose to the contrary that G contains some algebraic subgroup isomorphic to $\mathbf{G}_a$. Consider the faithful action of $\mathbf{G}_a$ on (X, D). It is a generically free action since $\mathbf{G}_a$ admits no non-trivial algebraic subgroup. More precisely, outside the closed subset F of all fixed points of $\mathbf{G}_a$-action, this action is free; i.e., the $\mathbf{G}_a$-orbit of any point $x \in X \setminus F$ is isomorphic to the affine line via the orbit map. Thus we obtain a dominating family of rational curves on X by completing these $\mathbf{G}_a$-orbits. A general rational curve (not contained in D) of this family can only intersect the boundary D in at most one point. Note that if a proper variety is dominated by rational curves, then it is in fact covered by rational curves (cf. [13, Corollary 1.4.4]). Hence by [2, Lemma 2.1], it follows that $K_X + D$ is not pseudo-effective, which contradicts our assumption. □

Next, we give an upper bound of the dimension of a semi-abelian variety acting faithfully on an arbitrary algebraic variety.

Lemma 2.2. *Let G be a semi-abelian variety. Suppose that G acts faithfully on an algebraic variety V of dimension n. Then we have*

$$\dim G \le \min\{n - \bar{\kappa}(V), n\}.$$

In particular, if $\dim G = n$, then V contains a Zariski open orbit with trivial isotropy group.

Proof. Let T_G denote the algebraic torus as in the definition of the semi-abelian variety G. Then T_G acts generically freely on V by [3, §1.6, Corollaire 1]. In other words, there exists a Zariski open subvariety U of V such that the isotropy group $(T_G)_x$ is trivial for any $x \in U$. Note that the isotropy group G_x has a fixed point x and hence is affine by [1, Proposition 2.1.6]. Thus the neutral component of G_x is contained in $(T_G)_x$, so is trivial for any $x \in U$. Therefore, G_x is finite for any $x \in U$. Then we can easily get

$$n = \dim V \ge \dim G \cdot x = \dim G - \dim G_x = \dim G.$$

Suppose that $\dim G = n = \dim V$. Then for any $x \in U$, the orbit $G \cdot x$ is Zariski-dense in V. Equivalently, since every orbit is locally closed, $G \cdot x$ is a Zariski open subvariety of V. Note that the isotropy group G_x acts trivially on $G \cdot x$ because G is commutative; so does G_x on V. This implies that G_x is trivial since the whole G-action is faithful. Thus in this optimal case, we have proved the assertion in the lemma.

On the other hand, by a theorem due to Rosenlicht (cf. [21, Theorem 2]), there exists a Zariski open subvariety V_0 of V such that the geometric quotient V_0/G exists. Consider the natural quotient map $V_0 \to V_0/G$ with a general fibre $F = G \cdot x_0$ for some $x_0 \in U \cap V_0$. By Iitaka's easy addition formula (cf. [10, Theorem 11.9]), we have

$$\bar{\kappa}(V) \le \bar{\kappa}(V_0) \le \bar{\kappa}(F) + \dim(V_0/G).$$

Note that this general fibre F is isomorphic to G/G_{x_0}, where G_{x_0} is finite as $x_0 \in U$. Thus F is also a semi-abelian variety and hence $\bar{\kappa}(F) = 0$. By a dimension formula for quotient varieties, we have

$$\dim(V_0/G) = \dim V - \dim G + \min_{x \in V} \dim G_x = n - \dim G.$$

Combining the last two displayed (in)equalities, we prove that $\dim G \le n - \bar{\kappa}(V)$. Together with $\dim G \le n$, which we just proved, we obtain the desired upper bound of $\dim G$. □

Remark 2.3. Without the condition $\dim G = n$, one can still show that the semi-abelian variety G acts generically freely on V. Actually, the isotropy group G_x is generically finite by the proof of Lemma 2.2. Note that finite subgroups of a semi-abelian variety form a countable family. So one may use [8, Lemma 5] to conclude that G_x is generically trivial.

Proof of Theorem 1.1. Replacing (X,D) by some G-equivariant log resolution as in the proof of Lemma 2.1, we may assume that (X,D) is log smooth with pseudo-effective K_X+D, and $V \coloneqq X \setminus D$ is a smooth quasi-projective subvariety of X. Note that after this replacement, the Iitaka dimension $\kappa(X, K_X+D)$ remains the same equal to $\bar{\kappa}(V)$. In Lemma 2.1 we have proved that G is a semi-abelian variety. We may then regard $G \le \mathrm{Aut}^0(X,D)$ as an algebraic subgroup of $\mathrm{Aut}(V)$. Indeed, the natural restriction map $G \to G|_V$ is an isomorphism. Thus applying Lemma 2.2 to the faithful G-action on V, the assertion (1) follows.

For the assertion (2), by the above lemma again, X (actually V) contains a Zariski open orbit V' with trivial isotropy group (so $V' \simeq G$). Let $D' \coloneqq X \setminus V'$ be the (total) boundary of this almost homogeneous variety. By taking a G-equivariant log resolution, we may assume that D' is a simple normal crossing divisor containing D. Thus by [2, Theorem 1.1], we have $K_X + D' \sim 0$. But K_X+D is already pseudo-effective. So $D = D'$ and hence $K_X + D \sim 0$. Since the push-forward of a linearly trivial divisor is also linearly equivalent to zero, the original pair (X,D) has trivial log canonical divisor. This shows the assertion (2).

Suppose that $\kappa(X, K_X+D) \ge 0$ and $\dim G = n$. Then $\bar{\kappa}(V) = \kappa(X, K_X+D) = 0$. We now show the second equality in assertion (3a), i.e., $\dim A_G = q(X)$. First it follows from [11, Theorem 28] that the quasi-Albanese morphism $\alpha_V \colon V \to \mathcal{A}_V$ is an open algebraic fibre space (i.e., generically surjective with irreducible general fibres). In particular,

$$\bar{q}(V) = \dim \mathcal{A}_V \le \dim V = n.$$

By the universal properties of the Albanese morphism alb_V and the quasi-Albanese morphism α_V, we know that the G-action on V descends uniquely to an action of G on the abelian variety $\mathrm{Alb}(V)$ and the semi-abelian variety $\mathcal{A}_V$, respectively. In other words, we have the following two exact sequences of connected algebraic groups:

$$1 \longrightarrow K_A \longrightarrow G \longrightarrow \mathrm{Aut}^0(\mathrm{Alb}(V)) = \mathrm{Alb}(V),$$

$$1 \longrightarrow K_\alpha \longrightarrow G \longrightarrow \mathrm{Aut}^0(\mathcal{A}_V),$$

where K_A and K_α denote the corresponding kernels. On the other hand, both G and $\mathcal{A}_V$ are semi-abelian, so we also have the following two exact sequences of connected algebraic groups:

$$1 \longrightarrow T_G \longrightarrow G \xrightarrow{\mathrm{alb}_G} A_G \longrightarrow 1,$$

$$1 \longrightarrow T_{\mathcal{A}_V} \longrightarrow \mathcal{A}_V \xrightarrow{\mathrm{alb}_{\mathcal{A}_V}} \mathrm{Alb}(\mathcal{A}_V) = \mathrm{Alb}(V) \longrightarrow 1.$$

By the Nishi–Matsumura theorem (cf. [17]), the induced group homomorphism $G \to \mathrm{Alb}(V)$ factors through A_G such that the group homomorphism $A_G \to \mathrm{Alb}(V)$ has a finite kernel. In particular, we have

$$\dim A_G \le \dim \mathrm{Alb}(V) = q(X).$$

Identify the torus $T_G/T_G \cap K_\alpha$ with its image in $\mathrm{Aut}^0(\mathcal{A}_V)$. Note that the induced action of T_G on the abelian variety $\mathrm{Alb}(V)$ is trivial. Thus $T_G/T_G \cap K_\alpha$ acts faithfully on $T_{\mathcal{A}_V}$. Note that a torus acting faithfully on another torus must act by multiplication. Set $d \coloneqq \dim T_G$. We have

$$d - \dim T_G \cap K_\alpha = \dim T_G/T_G \cap K_\alpha \le \dim T_{\mathcal{A}_V} =: t.$$

Note that the torus $T_G \cap K_\alpha$ acts trivially on $\mathcal{A}_V$, and hence it acts faithfully on the general fibre F of the quasi-Albanese morphism $\alpha_V\colon V \to \mathcal{A}_V$. By [3, §1.6, Corollaire 1] we have

$$\dim T_G \cap K_\alpha \le \dim F = \dim V - \dim \mathcal{A}_V = n - \bar{q}(V).$$

In order to satisfy $\dim G = d + \dim A_G = n = q(X) + t + (n - \bar{q}(V))$, all of the last three displayed inequalities should be equalities. In particular, we have $\dim A_G = q(X)$.

Note that (X, D) is a projective dlt pair and hence X has only rational singularities (cf. [15, Theorem 5.22]). For such X, its irregularity $q(X)$ does not depend on its resolution. Thus $\dim \mathrm{Alb}(V)$ equals $q(X)$ of the original X. This completes the proof of the assertion (3a).

For the last assertion (3b), if the dimension of the torus part T_G of G is maximal (i.e., $d = n$), then we have $\dim G = n$, and hence all statements in the assertion (3a) hold. Moreover, it follows from $G = T_G$ that $A_G = 0$. Thus $q(X) = \dim A_G = 0$. (In particular, $\mathcal{A}_V$ is an algebraic torus of dimension $\bar{q}(V) = t \le n$.) We have completed the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2, which shall be divided into Theorems 3.6 and 3.11. The first theorem considers the logarithmic Iitaka surfaces (i.e., $\bar{\kappa} = 0$ and $\bar{p}_g = 1$), while all other algebraic surfaces with $\bar{\kappa} = \bar{p}_g = 0$ are dealt with by the second one. We first prepare some general results used to prove both theorems.

3.1. **Preliminaries.** We will frequently and implicitly use the following lemma to compare logarithmic invariants of an algebraic variety with an open subvariety.

Lemma 3.1 (cf. [10, Proposition 11.4])**.** *Let X be an algebraic variety, and let U be a non-empty Zariski open subvariety of X. Then we have*

$$\bar{q}(X) \le \bar{q}(U),\ \bar{p}_g(X) \le \bar{p}_g(U),\ \textit{and}\ \bar{\kappa}(X) \le \bar{\kappa}(U).$$

The following lemma is a slight generalization of [9, Lemma 2], which is used to compute $\bar{q}$.

Lemma 3.2. *Let (X, D) be a log smooth pair of dimension n with $V \coloneqq X \setminus D$. Let $\sum D_i$ be the irreducible decomposition of D. Then we have*

$$\bar{q}(V) = q(X) + \operatorname{rank} \operatorname{Ker}(\bigoplus \mathbb{Z}[D_i] \to \mathrm{NS}(X)),$$

where $\mathrm{NS}(X) \coloneqq \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ denotes the Néron–Severi group of X.

Proof. We first have the following long exact sequence of local cohomology:

$$\cdots \to H^1_D(X, \mathbb{Z}) \to H^1(X, \mathbb{Z}) \to H^1(V, \mathbb{Z}) \to H^2_D(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}) \to \cdots,$$

where $H^i_D(X, \mathbb{Z})$ are the cohomology groups of X with support in D. Since D is of codimension 1 in X, we get that $H^i_D(X, \mathbb{Z}) = 0$ for $i = 0, 1$ (cf. [18, Lemma 23.1]). It is also known that

$$H^2_D(X, \mathbb{Z}) \simeq \bigoplus H^2_{D_i}(X, \mathbb{Z}) \simeq \bigoplus H^0(D_i, \mathbb{Z}) = \bigoplus \mathbb{Z}[D_i],$$

as the first isomorphism follows from the Mayer–Vietoris sequence and the second one holds by Poincaré–Lefschetz duality. Note that $\dim_{\mathbb{C}} H^1(V,\mathbb{C}) - \dim_{\mathbb{C}} H^1(X,\mathbb{C}) = (\bar{q}(V) + q(X)) - 2q(X) = \bar{q}(V) - q(X)$. On the other hand, the preceding exact sequence yields

$$H^1(V,\mathbb{Z})/H^1(X,\mathbb{Z}) \simeq \mathrm{Ker}(\bigoplus \mathbb{Z}[D_i] \to H^2(X,\mathbb{Z})) = \mathrm{Ker}(\bigoplus \mathbb{Z}[D_i] \to \mathrm{NS}(X)).$$

So we obtain the lemma. □

As a corollary of the above lemma and the well-known behavior of Néron–Severi groups under a point blowup, we can readily see that the logarithmic irregularity $\bar{q}$ is (somewhat) invariant under any point blowup.

Lemma 3.3. *Let (X, D) be a log smooth pair of dimension n with $V := X \setminus D$. Let $\pi\colon \widetilde{X} \to X$ be the blowup of some point. Let $\widetilde{D}$ be the sum of the strict transform $\pi_*^{-1}D$ and some reduced effective exceptional divisors. Denote $\widetilde{V} := \widetilde{X} \setminus \widetilde{D}$. Then we have $\bar{q}(\widetilde{V}) = \bar{q}(V)$.* □

For a log surface, we also need the following formula to calculate $\bar{p}_g$, which is due to Sakai (cf. [22, Lemma 1.12]). See also [19, Chapter 1, Lemma 2.3.1] for a similar treatment.

Lemma 3.4. *Let (X, D) be a log smooth pair of dimension 2 with $V := X \setminus D$. Then we have*

$$\bar{p}_g(V) = p_g(X) + h^1(\mathcal{O}_D) - q(X) + \gamma(D),$$

where $\gamma(D) := \dim \mathrm{Ker}\{H^1(X,\mathcal{O}_X) \to H^1(D,\mathcal{O}_D)\}$. In particular, if X is a regular surface, then $\bar{p}_g(V) = p_g(X) + h^1(\mathcal{O}_D)$.

Remark 3.5. For a log smooth surface pair (X, D), if D is connected such that $h^1(\mathcal{O}_D) = 1$, then D contains a smooth elliptic curve or a cycle of smooth rational curves. Indeed, recall that the *arithmetic genus* of the divisor D is defined as $p_a(D) := 1 - \chi(\mathcal{O}_D) = h^1(\mathcal{O}_D)$, where $\chi(\mathcal{F}) := \sum(-1)^i h^i(\mathcal{F})$ denotes the *Euler characteristic* of a coherent sheaf $\mathcal{F}$. By the Riemann–Roch theorem, one has the following genus formula:

$$p_a(D) = 1 + \frac{1}{2} D.(K_X + D).$$

Then it is easy to see that the arithmetic genus of a tree of smooth rational curves is zero. Meanwhile, if D contains a curve of genus greater than one, then $p_a(D) \geq 2$.

For a general (not necessarily connected) boundary D, the above genus formula also holds. By the induction on the number of the connected components of D, one can show that

$$h^1(\mathcal{O}_D) = \sum h^1(\mathcal{O}_{D^j}),$$

where each D^j is a connected component of D. Therefore, if we assume that $h^1(\mathcal{O}_D) = 1$, then there exists a unique connected component of D such that its arithmetic genus is one. We know the behavior of this connected component from the discussion above.

3.2. **Logarithmic Iitaka surfaces.** In this subsection, we will prove Theorem 1.2 in the setting of the title, which means $\bar{\kappa} = 0$ and $\bar{p}_g = 1$. We actually prove the following theorem.

Theorem 3.6. *Let V be a logarithmic Iitaka surface, and let (X, D) be a log smooth completion such that $V = X \setminus D$. Then the neutral component of $\mathrm{Aut}(X, D)$ is a semi-abelian variety of dimension at most $\bar{q}(V)$.*

By the classification theory of smooth projective surfaces, we can show that for a logarithmic Iitaka surface V, the ambient surface X can only have the following possibilities. In this paper, a *ruled surface* always means a birationally ruled surface, i.e., birationally equivalent to $C \times \mathbb{P}^1$ for some smooth curve C. The genus of the base curve C is also called the genus of the ruled surface. In particular, an *elliptic ruled surface* is a ruled surface of genus 1.

Lemma 3.7. *Let V be a logarithmic Iitaka surface, and let (X, D) be a log smooth completion such that $V = X \setminus D$. Then X belongs to one of the following cases:*

(1) X is a rational surface; $p_g(X) = q(X) = 0$ and $h^1(\mathcal{O}_D) = 1$.
(2) X is an elliptic ruled surface; $p_g(X) = 0$ and $q(X) = 1$.
(3) X is (birationally) a $K3$ surface or an abelian surface; $p_g(X) = 1$.

Proof. By Lemma 3.1, one has $\kappa(X) \leq \bar{\kappa}(V) = 0$ and $p_g(X) \leq \bar{p}_g(V) = 1$. We first consider the case $\kappa(X) < 0$. If $q(X) = 0$, this is just the case (1). If $q(X) \geq 1$, we will show that X cannot be a ruled surface of genus $q(X) > 1$. Let $\mathrm{alb}_X \colon X \to B \subseteq \mathrm{Alb}(X)$ be the Albanese morphism of X with B a smooth projective curve of genus $q(X) \geq 1$ because $p_g(X) = 0$. For a general point $b \in B$, the fibre $F \coloneqq \mathrm{alb}_X^{-1}(b)$ of b is a smooth rational curve. Define $F_V \coloneqq F \cap V$. Then by Iitaka's easy addition formula (cf. [10, Theorem 11.9]), we have

$$0 = \bar{\kappa}(V) \leq \bar{\kappa}(F_V) + \dim B,$$

which implies that $\bar{\kappa}(F_V) \geq 0$. On the other hand, by Kawamata's addition formula (for morphisms of relative dimension one; cf. [10, Theorem 11.15]), we have

$$0 = \bar{\kappa}(V) \geq \bar{\kappa}(F_V) + \kappa(B) \geq \kappa(B) \geq 0.$$

Thus $\bar{\kappa}(F_V) = \kappa(B) = 0$, and hence B is an elliptic curve. This gives us the case (2).

We next consider the case $\kappa(X) = 0$. To obtain the case (3), we just need to rule out the Enriques surfaces and the hyperelliptic surfaces. If X is (birationally) an Enriques surface, then there exists a finite étale cover $\sigma \colon \widetilde{X} \to X$ for some (birationally) $K3$ surface $\widetilde{X}$. Let $\widetilde{D} \coloneqq \sigma^{-1}D$ and $\widetilde{V} \coloneqq \widetilde{X} \setminus \widetilde{D}$. Then we have $\bar{\kappa}(\widetilde{V}) = \bar{\kappa}(V) = 0$ since $\sigma|_{\widetilde{V}} \colon \widetilde{V} \to V$ is also a finite étale cover (cf. [10, Theorem 11.10]). On the other hand, it follows from Lemma 3.4 that $h^1(\mathcal{O}_D) = 1$ and hence there exists a unique connected component D^1 of D such that $h^1(\mathcal{O}_{D^1}) = 1$ (see also Remark 3.5). Take a connected component $\widetilde{D}^1$ of $\widetilde{D}$ which maps onto D^1. Then consider the induced finite étale cover $\widetilde{D}^1 \to D^1$. Say its degree is d. We obtain the following equalities:

$$1 - h^1(\mathcal{O}_{\widetilde{D}^1}) = \chi(\mathcal{O}_{\widetilde{D}^1}) = d \cdot \chi(\mathcal{O}_{D^1}) = d \cdot (1 - h^1(\mathcal{O}_{D^1})) = 0.$$

Hence by Lemma 3.4 again we have $\bar{p}_g(\widetilde{V}) = p_g(\widetilde{X}) + h^1(\mathcal{O}_{\widetilde{D}}) \geq p_g(\widetilde{X}) + h^1(\mathcal{O}_{\widetilde{D}^1}) = 2$. This contradicts the fact that $\bar{\kappa}(\widetilde{V}) = 0$.

If X is (birationally) a hyperelliptic surface, then there exists a minimal model X_m of X obtained by contracting all (-1)-curves. Let $\mu \colon X \to X_m$ denote the composite contraction morphism and $D_m \coloneqq \mu_* D$. There is an effective divisor R_μ with support the union of all μ-exceptional divisors such that $K_X = \mu^* K_{X_m} + R_\mu$

by the ramification formula (cf. [10, Theorem 5.5]). Then we have

$$\begin{aligned}0 = \bar{\kappa}(V) &= \kappa(X, K_X + D) = \kappa(X, \mu^* K_{X_m} + R_\mu + D)\\ &= \kappa(X, \mu^* K_{X_m} + N R_\mu + D) \text{ for } N \gg 0 \quad \text{by [10, Lemma 10.5]}\\ &\geq \kappa(X, \mu^* K_{X_m} + \mu^* D_m) = \kappa(X_m, K_{X_m} + D_m).\end{aligned}$$

Note that $K_{X_m} \sim_{\mathbb{Q}} 0$ and D_m is nef (see e.g. [6, Lemma 2.11]). So we have

$$0 \geq \kappa(X_m, K_{X_m} + D_m) = \kappa(X_m, D_m) \geq 0,$$

and hence $D_m \sim_{\mathbb{Q}} 0$ by the abundance theorem for surfaces (cf. [4, Theorem 6.2]). Then D_m being effective implies that $D_m = 0$; i.e., D is μ-exceptional. Thus by the projection formula,

$$H^0(X, K_X + D) = H^0(X, \mu^* K_{X_m} + R_\mu + D) = H^0(X_m, K_{X_m}) = 0.$$

This contradicts the assumption $\bar{p}_g(V) = h^0(X, K_X + D) = 1$. $\square$

Remark 3.8. Let V be a logarithmic Iitaka surface, where (X, D) is a log smooth completion such that $V = X \setminus D$. Let $\sum D_i$ be the irreducible decomposition of D. We would like to introduce an associated divisor D_A separately for each case in Lemma 3.7 as follows.

(1) If X is a rational surface, by Remark 3.5 there are two subcases:
   (i) if D_i is a smooth elliptic curve for some i, then let $D_A = D_i$;
   (ii) if $p_a(D_i) = 0$ for all i, then there is a cycle of smooth rational curves $\sum_{i=1}^r D_i =: D_A$.
(2) If X is an elliptic ruled surface, we have seen in the proof of Lemma 3.7 that the general fibre F_V (of the restriction morphism $\mathrm{alb}_X|_V \colon V \to B$) is rational and has logarithmic Kodaira dimension zero. So it is isomorphic to the one-dimensional algebraic torus $\mathbf{G}_m$, and hence $D.F = 2$. We then denote by D_A the sum of all irreducible components of D which are mapped onto B by the Albanese morphism (or ruled fibration) alb_X. Namely, D_A is a sum of two cross-sections or a double section of alb_X.
(3) If $p_g(X) = 1$, then let $D_A = 0$.

We then claim that in each case above, $X \setminus D_A$ is still a logarithmic Iitaka surface. Indeed, it suffices to show that $\bar{p}_g(X \setminus D_A) = 1$ since $\bar{\kappa}(X \setminus D_A) \leq \bar{\kappa}(X \setminus D) = 0$. The case (1) is easy by Lemma 3.4. For the case (2), our $\bar{p}_g$-formula may not be used due to some unknown invariants. However, we note that the general fibre of $X \setminus D_A \to B$ is still $\mathbf{G}_m$ by the definition of D_A. So by Kawamata's addition formula (cf. [10, Theorem 11.15]), $\bar{\kappa}(X \setminus D_A) \geq \bar{\kappa}(\mathbf{G}_m) + \kappa(B) = 0$. Choose $M \in |K_X + D|$ and $N \in |m(K_X + D_A)|$ for some positive integer m. Then by $mM \sim m(K_X + D) \sim N + m(D - D_A) \geq 0$ and $\kappa(X, K_X + D) = 0$, we have $mM = N + m(D - D_A)$. Thus $M - (D - D_A) = N/m$ is an effective divisor so that $|K_X + D_A|$ is non-empty.

We keep using the notation D_A till the end of this subsection. Next, we provide a new and much shorter proof of Iitaka's Theorem III in [9].

Lemma 3.9. *Let V be a logarithmic Iitaka surface, and let (X, D) be a log smooth completion such that $V = X \setminus D$. Suppose that there is no (-1)-curve in $X \setminus D_A$. Then $K_X + D_A \sim 0$.*

Proof. We only need to show that $K_X + D_A$ is nef. Indeed, by the claim in Remark 3.8 we have seen that $\kappa(X, K_X + D_A) = 0$ and $\bar{p}_g(X \setminus D_A) = 1$. Then by the abundance theorem for surfaces, $K_X + D_A \sim_{\mathbb{Q}} 0$ (cf. [4, Theorem 6.2]). In particular, $K_X + D_A \sim 0$ since $\bar{p}_g(X \setminus D_A) = 1$.

Suppose to the contrary that there exists an irreducible curve C such that $(K_X + D_A).C < 0$. Then $C^2 < 0$ because $\kappa(X, K_X + D_A) = 0$. If $C \not\subseteq D_A$, then $D_A.C \geq 0$ and hence $K_X.C < 0$. So by the adjunction formula, we know that C is a (-1)-curve and $K_X.C = -1$. Thus $0 \leq D_A.C < -K_X.C = 1$ implies that $D_A.C = 0$. This means $C \cap D_A = \varnothing$, contradicting our assumption. If $C \subseteq D_A$, write $D_A = C + D'_A$ with $D'_A.C \geq 0$. Then we have

$$(K_X + C).C \leq (K_X + C + D'_A).C = (K_X + D_A).C < 0.$$

So by the adjunction formula again, C is a smooth rational curve, i.e., $p_a(C) = 0$. Hence the cases (1)(i) and (2) in Remark 3.8 cannot happen, since in both cases $p_a(C) \geq 1$. For the case (1)(ii) note that $D'_A.C = 2$, since D_A is a cycle of smooth rational curves. Thus

$$0 > (K_X + D_A).C = (K_X + C + D'_A).C = (K_X + C).C + 2 = 2p_a(C) = 0,$$

which is absurd. For the last case (3), it is obvious that C is a (-1)-curve, which is impossible under our assumption. □

As we mentioned in Remark 1.3, the natural geometric approach may not apply to bound the dimension. Hence the following easy observation could be thought of as a starting point for proving our main theorem.

Lemma 3.10. *Let (X, D) be a log smooth pair of dimension 2 with $V \coloneqq X \setminus D$ such that $K_X + D \sim 0$. Then $\dim \mathrm{Aut}^0(X, D) = \bar{q}(V)$.*

Proof. Let $\Theta_X(-\log D)$ denote the logarithmic tangent sheaf, which is just the dual of the logarithmic differential sheaf $\Omega^1_X(\log D)$. Then $H^0(X, \Theta_X(-\log D))$ is the Lie algebra of the connected algebraic group $\mathrm{Aut}^0(X, D)$. In our situation, $\Omega^2_X(\log D) = \mathcal{O}_X(K_X + D) = \mathcal{O}_X$. Hence $\Theta_X(-\log D) = (\Omega^1_X(\log D))^\vee \simeq \Omega^1_X(\log D)$. □

Proof of Theorem 3.6. We have already seen by Theorem 1.1 that $\mathrm{Aut}^0(X, D)$ is a semi-abelian variety. Let G denote the neutral component of $\mathrm{Aut}(X, D_A)$. Then $\mathrm{Aut}^0(X, D) \leq G$ by the special choice of D_A (see Remark 3.8 for its definition). Let

$$\pi \colon X = X_0 \xrightarrow{\pi_0} X_1 \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_{m-1}} X_m$$

be the composition of morphisms π_i such that for every $0 \leq i \leq m-1$:

(1) π_i is a blowdown of some (-1)-curve E_i in X_i,
(2) $D_{A,i+1} \coloneqq \pi_{i*} D_{A,i}$,
(3) $E_i \subset X_i \setminus D_{A,i}$.

We may assume that $X_m \setminus D_{A,m}$ has no (-1)-curve. Note that $X_m \setminus D_{A,m}$ is still a logarithmic Iitaka surface and the D_A-part of $D_{A,m}$ is itself. Then by Lemma 3.9, we have $K_{X_m} + D_{A,m} \sim 0$, and hence

$$\dim \mathrm{Aut}^0(X_m, D_{A,m}) = \bar{q}(X_m \setminus D_{A,m})$$

by Lemma 3.10. Further, it follows from Lemmas 3.3 and 3.1 that

$$\bar{q}(X_m \setminus D_{A,m}) = \bar{q}(X \setminus D_A) \leq \bar{q}(V).$$

Thus we only need to prove that π is G-equivariant so that $G \leq \mathrm{Aut}^0(X_m, D_{A,m})$. Indeed, the class of each (-1)-curve in $\overline{\mathrm{NE}}(X_i)$ is a $(K_{X_i}+D_{A,i})$-negative extremal ray, so is preserved by the connected group G. Hence each π_i is G-equivariant, and so is the composition π.[2]

From the discussion above, we have

$$\begin{aligned}\dim \mathrm{Aut}^0(X,D) \leq \dim \mathrm{Aut}^0(X,D_A) \leq \dim \mathrm{Aut}^0(X_m,D_{A,m}) &= \bar{q}(X_m \setminus D_{A,m}) \\ &\leq \bar{q}(V).\end{aligned}$$

This completes the proof of Theorem 3.6. □

3.3. **Surfaces with $\bar{\kappa} = \bar{p}_g = 0$.** Parallel to the previous subsection, we are going to prove Theorem 1.2 for smooth algebraic surfaces with both logarithmic Kodaira dimension and logarithmic geometric genus vanishing. Together with logarithmic Iitaka surfaces, they are all smooth algebraic surfaces with vanishing logarithmic Kodaira dimension.

Theorem 3.11. *Let (X,D) be a log smooth pair of dimension 2 with $V := X \setminus D$ such that $\bar{\kappa}(V) = \bar{p}_g(V) = 0$. Then the neutral component of $\mathrm{Aut}(X,D)$ is a semi-abelian variety of dimension at most $q(X)$.*

Given a smooth algebraic surface V with $\bar{\kappa}(V) = \bar{p}_g(V) = 0$, similarly with Lemma 3.7, we also have some restriction on this surface if it further admits a faithful algebraic 1-torus action. Recall that an elliptic ruled surface is a (birationally) ruled surface of genus 1.

Lemma 3.12. *Let (X,D) be a log smooth pair of dimension 2 with $V := X \setminus D$. Suppose that $\bar{\kappa}(V) = \bar{p}_g(V) = 0$ and $\mathbf{G}_m \leq \mathrm{Aut}(X,D)$. Then X is an elliptic ruled surface.*

Proof. By a classical characterization of the $\mathbf{G}_m$-surfaces (i.e., algebraic surfaces admitting algebraic 1-torus action), there exists an invariant Zariski open subvariety $U \subseteq V$ equivariantly isomorphic to $C_0 \times \mathbf{G}_m$ with $\mathbf{G}_m$ acting only on the second factor by translation, where C_0 is a smooth curve (probably non-projective; cf. [20, §1.6, Lemma, and §2.2, Theorem]). Thus there is an equivariant birational map $f\colon X \dashrightarrow C \times \mathbb{P}^1$ with C the smooth completion of C_0. Since X cannot be a ruled surface of genus $q(X) > 1$ by the same argument as in the proof of Lemma 3.7, we only need to rule out the case that X is a rational surface.

Suppose to the contrary that X is a rational surface. Then obviously the curve C can be taken as $\mathbb{P}^1$. So we obtain an equivariant birational map $f\colon X \dashrightarrow Y := \mathbb{P}^1 \times \mathbb{P}^1$. Note that $\bar{\kappa}(U) \geq \bar{\kappa}(V) = 0$, and hence $C_0 = \mathbf{G}_m$ or $\mathbb{P}^1 \setminus \{x_1, \ldots, x_t\}$ with $t \geq 3$.

Case 1. $C_0 = \mathbf{G}_m$. Take an equivariant resolution $\pi\colon \widetilde{X} \to X$ of the indeterminacy points of f and $\mathrm{Sing}(X \setminus U)$ such that $\varphi := f \circ \pi\colon \widetilde{X} \to Y$ is an equivariant birational morphism and $\pi^{-1}(X \setminus U)$ is a simple normal crossing divisor. Since our logarithmic invariants $\bar{\kappa}$, $\bar{p}_g$, and $\bar{q}$ are independent of the choice of the log smooth completion (cf. [10, §11]), it follows that

$$\bar{\kappa}(\widetilde{V}) = \bar{\kappa}(V) = 0,\ \bar{p}_g(\widetilde{V}) = \bar{p}_g(V) = 0, \text{ and } \bar{q}(\widetilde{V}) = \bar{q}(V) \leq 2,$$

[2]The G-equivariance of the morphism π also follows from a general result of Blanchard which asserts that π is G-equivariant as long as $\pi_*\mathcal{O}_X = \mathcal{O}_{X_m}$ (see e.g. [1, Proposition 4.2.1] for the precise statement).

where $\widetilde{V} := \widetilde{X} \setminus \widetilde{D}$ with $\widetilde{D} := \pi^{-1}(D)$. Let $\widetilde{B} := \pi^{-1}(X \setminus U) \setminus \widetilde{D}$. We consider the new pair $(\widetilde{X}, \widetilde{D}+\widetilde{B})$. It is easy to see that $\widetilde{X} \setminus (\widetilde{D}+\widetilde{B}) = \pi^{-1}(U) \simeq U \simeq \mathbf{G}_m \times \mathbf{G}_m$ under the birational morphism φ. We can take a reduced effective divisor $D_Y :=$ 2 sections + 2 fibres on Y such that $\varphi^{-1}(D_Y) = \widetilde{D} + \widetilde{B}$, since the restriction of φ to $\widetilde{X} \setminus (\widetilde{D} + \widetilde{B})$ is an isomorphism.

We have two possibilities according to the dimension of $\varphi(\widetilde{B})$. Suppose that $\dim \varphi(\widetilde{B}) = 0$; i.e., $\widetilde{B}$ is φ-exceptional. Note that φ is the composition of blowups of (fixed) points. Then by Lemma 3.3, we have $\bar{q}(\widetilde{V}) = \bar{q}(Y \setminus D_Y) = 2$. So by [11, Corollary 29], the quasi-Albanese morphism $\alpha_{\widetilde{V}} \colon \widetilde{V} \to \mathcal{A}_{\widetilde{V}}$ is birational, and hence $\bar{p}_g(\widetilde{V}) = 1$ since $\bar{p}_g$ is a birational invariant for smooth varieties. This contradicts our assumption. Suppose that $\dim \varphi(\widetilde{B}) = 1$. Then $\varphi(\widetilde{D})$ is a proper subset of D_Y with some fibre or section taken away so that $\bar{\kappa}(Y \setminus \varphi(\widetilde{D})) = -\infty$. In this case, by the logarithmic ramification formula we have

$$K_{\widetilde{X}} + \varphi^{-1}(\varphi(\widetilde{D})) = \varphi^*(K_Y + \varphi(\widetilde{D})) + E,$$

where E is an effective φ-exceptional divisor (cf. [10, Theorem 11.5]). It follows that

$$\bar{\kappa}(\widetilde{X} \setminus \varphi^{-1}(\varphi(\widetilde{D}))) = \bar{\kappa}(Y \setminus \varphi(\widetilde{D})) = -\infty.$$

Also note that $\widetilde{D} \le \varphi^{-1}(\varphi(\widetilde{D}))$ and hence $\bar{\kappa}(\widetilde{V}) = \bar{\kappa}(\widetilde{X} \setminus \widetilde{D}) \le \bar{\kappa}(\widetilde{X} \setminus \varphi^{-1}(\varphi(\widetilde{D}))) = -\infty$, which is a contradiction.

Case 2. $C_0 = \mathbb{P}^1 \setminus \{x_1, \ldots, x_t\}$ with $t \ge 3$. The proof of this case is quite similar with the first one. We keep using the notation there but take $D_Y :=$ 2 sections $+\, t$ fibres on Y with respect to the first projection $p_1 \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, such that $\varphi^{-1}(D_Y) = \widetilde{D} + \widetilde{B}$. If $\dim \varphi(\widetilde{B}) = 0$, then by Lemma 3.3 we have $\bar{q}(\widetilde{V}) = \bar{q}(Y \setminus D_Y) \ge 3$. This contradicts the fact that $\bar{q}(\widetilde{V}) \le 2$. If $\dim \varphi(\widetilde{B}) = 1$, we also have two possibilities:

- $\varphi(\widetilde{B})$ contains a section: we can derive a contradiction like $\bar{\kappa}(\widetilde{V}) = -\infty$ as in Case 1.
- $\varphi(\widetilde{B})$ consists of fibres: according to the number of the fibres, we can get $\bar{q}(\widetilde{V}) \ge 3$, or $\bar{q}(\widetilde{V}) = 2$ and $\bar{p}_g(\widetilde{V}) = 1$, or $\bar{\kappa}(\widetilde{V}) = -\infty$. All of these cases cannot happen under our assumptions.

Therefore, we have proved that X cannot be a rational surface, and hence this completes the proof of Lemma 3.12. $\square$

Proof of Theorem 3.11. First it follows directly from Theorem 1.1 that $G := \mathrm{Aut}^0(X, D)$ is a semi-abelian variety of dimension at most 2. If $\dim G = 2$, then by Theorem 1.1(2), $K_X + D \sim 0$ and hence $\bar{p}_g(V) = 1$, which is impossible. So we only need to consider the case $\dim G = 1$. If G is complete, then by the Nishi–Matsumura theorem (cf. [17]), the induced group homomorphism $G \to \mathrm{Alb}(V)$ has a finite kernel. In particular, we have $\dim G \le \dim \mathrm{Alb}(V) = q(X)$. The last remaining case is $G = \mathbf{G}_m$. By Lemma 3.12, X is an elliptic ruled surface with $q(X) = 1$ in this case. So $\dim G = 1 = q(X)$. $\square$

3.4. **Proof of Theorem 1.2.** This follows immediately from Theorems 3.6 and 3.11.

Acknowledgments

The author would like to thank his supervisor, Professor De-Qi Zhang, for suggesting this problem and for many inspiring discussions. The author also thanks the referee for numerous valuable comments and corrections on an earlier version of this paper. The author was supported by a research assistantship of the National University of Singapore.

References

[1] Michel Brion, Preena Samuel, and V. Uma, *Lectures on the structure of algebraic groups and geometric applications*, CMI Lecture Series in Mathematics, vol. 1, Hindustan Book Agency, New Delhi; Chennai Mathematical Institute (CMI), Chennai, 2013. MR3088271

[2] Michel Brion and De-Qi Zhang, *Log Kodaira dimension of homogeneous varieties*, Advanced Studies in Pure Mathematics **75** (2017), 1–6.

[3] Michel Demazure, *Sous-groupes algébriques de rang maximum du groupe de Cremona* (French), Ann. Sci. École Norm. Sup. (4) **3** (1970), 507–588. MR0284446

[4] Osamu Fujino, *Minimal model theory for log surfaces*, Publ. Res. Inst. Math. Sci. **48** (2012), no. 2, 339–371, DOI 10.2977/PRIMS/71. MR2928144

[5] Osamu Fujino, *On quasi-albanese maps*, preprint (2015). Available at `https://www.math.kyoto-u.ac.jp/~fujino/quasi-albanese2.pdf`.

[6] Fei Hu, Sheng-Li Tan, and De-Qi Zhang, *Periodic subvarieties of a projective variety under the action of a maximal rank abelian group of positive entropy*, preprint (2016), `arXiv:1604.02684`.

[7] Shigeru Iitaka, *Logarithmic forms of algebraic varieties*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **23** (1976), no. 3, 525–544. MR0429884

[8] S. Iitaka, *On logarithmic Kodaira dimension of algebraic varieties*, Complex analysis and algebraic geometry, Iwanami Shoten, Tokyo, 1977, pp. 175–189. MR0569688

[9] Shigeru Iitaka, *On logarithmic $K3$ surfaces*, Osaka J. Math. **16** (1979), no. 3, 675–705. MR551583

[10] Shigeru Iitaka, *Algebraic geometry, An introduction to birational geometry of algebraic varieties*, Graduate Texts in Mathematics, vol. 76, North-Holland Mathematical Library, 24, Springer-Verlag, New York-Berlin, 1982. MR637060

[11] Yujiro Kawamata, *Characterization of abelian varieties*, Compositio Math. **43** (1981), no. 2, 253–276. MR622451

[12] Yujiro Kawamata, *Minimal models and the Kodaira dimension of algebraic fiber spaces*, J. Reine Angew. Math. **363** (1985), 1–46, DOI 10.1515/crll.1985.363.1. MR814013

[13] János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996. MR1440180

[14] János Kollár, *Lectures on resolution of singularities*, Annals of Mathematics Studies, vol. 166, Princeton University Press, Princeton, NJ, 2007. MR2289519

[15] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, with the collaboration of C. H. Clemens and A. Corti, translated from the 1998 Japanese original, vol. 134, Cambridge University Press, Cambridge, Cambridge Tracts in Mathematics, 1998. MR1658959

[16] Serge Lang, *Abelian varieties*, Springer-Verlag, New York-Berlin, 1983. Reprint of the 1959 original. MR713430

[17] Hideyuki Matsumura, *On algebraic groups of birational transformations*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 151–155. MR0159825

[18] James S. Milne, *Lectures on Étale cohomology (v2.21)*, 2013. Available at `http://www.jmilne.org/math/CourseNotes/lec.html`, p. 202.

[19] Masayoshi Miyanishi, *Open algebraic surfaces*, CRM Monograph Series, vol. 12, American Mathematical Society, Providence, RI, 2001. MR1800276

[20] P. Orlik and P. Wagreich, *Algebraic surfaces with k^*-action*, Acta Math. **138** (1977), no. 1-2, 43–81, DOI 10.1007/BF02392313. MR0460342

[21] Maxwell Rosenlicht, *Some basic theorems on algebraic groups*, Amer. J. Math. **78** (1956), 401–443, DOI 10.2307/2372523. MR0082183

[22] Fumio Sakai, *Semistable curves on algebraic surfaces and logarithmic pluricanonical maps*, Math. Ann. **254** (1980), no. 2, 89–120, DOI 10.1007/BF01467073. MR597076

[23] Jean-Pierre Serre, *Morphismes universels et variétés d'albanese*, Exposés de séminaires (1950-1999), Documents Mathématiques (Paris) [Mathematical Documents (Paris)], vol. 1, Société Mathématique de France, Paris, 2001, pp. 141–160. MR1942136

[24] Hideyasu Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ. **14** (1974), 1–28, DOI 10.1215/kjm/1250523277. MR0337963

Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076

Current address: Department of Mathematics, University of British Columbia, 1984 Mathematics Road, Vancouver, BC, Canada V6T 1Z2

Email address: `hf@u.nus.edu`