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HERMITIAN RANKS OF COMPACT COMPLEX MANIFOLDS

DANIELE ANGELLA AND ADRIANO TOMASSINI

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ABSTRACT. We investigate degenerate special-Hermitian metrics on compact complex manifolds; in particular, degenerate Kähler and locally conformally Kähler metrics on special classes of non-Kähler manifolds.

INTRODUCTION

In this paper, we continue to study special-Hermitian structures on compact complex manifolds, in view of a (far-to-be-obtained) possible classification of compact complex non-Kähler manifolds.

In particular, we focus on the existence of degenerate special-Hermitian metrics. Here we investigate degenerate Kähler metrics and degenerate locally conformally Kähler metrics by introducing the notion of special-Hermitian ranks, as a first development of the Kähler rank by R. Harvey and H. B. Lawson and by I. Chiose in the non-Kähler setting. Here we investigate classes of non-Kähler examples.

R. Harvey and H. B. Lawson provided an intrinsic characterization of the Kähler condition. In [HL83, Theorem (14)], they proved that, on a compact complex manifold X, one and only one of the following facts holds: (i) there is a positive $(1,1)$-form being closed (namely, a Kähler metric); (ii) there is a (nontrivial) positive bidimension-$(1,1)$-current being the component of a boundary.

This led them to introduce a notion of Kähler rank for compact complex surfaces, in terms of the foliated set

$$\mathcal{B}(X) \;:=\; \left\{x \in X \;:\; \exists \varphi \in P^{\infty}_{\mathrm{bdy}}(X) \text{ such that } \varphi_x \neq 0\right\} \;,$$

where $P^{\infty}_{\mathrm{bdy}}(X)$ denotes the subcone of smooth currents in the cone of positive bidimension-$(1,1)$-currents on X being a boundary.

By [CT13, Corollary 4.3], the Kähler rank of a compact complex surface X is equal to the maximal rank that a nonnegative closed $(1,1)$-form may attain at some point of X. (See also [FGV16, Definition 1.2].) This allowed I. Chiose to extend the notion of Kähler rank to higher-dimensional compact complex manifolds

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([Chi13, Definition 1.1]) by setting

$$\mathrm{Kr}(X) \;:=\; \max\left\{k \in \mathbb{N} \;:\; \exists \omega \in \wedge^{1,1}X \text{ s.t. } \omega \geq 0,\; d\omega = 0, \text{ and } \omega^k \neq 0\right\} \;. \tag{0.1}$$

By relaxing the Kähler condition, several notions of special-Hermitian metrics can be defined; e.g., Hermitian-symplectic, balanced in the sense of Michelsohn [Mic82], pluri-closed [Bis89], astheno-Kähler [JY93], Gauduchon [Gau77], strongly-Gauduchon [Pop13], and others. The notion in Equation (0.1) can be restated for some of these metrics; we will consider, e.g., the SKT case (1.2).

In particular, we introduce and study the *Hermitian locally conformally Kähler rank*. It is defined in Equation (1.1). Essentially, we replace the condition of $d\omega = 0$ in Equation (0.1) by $d\omega - \vartheta \wedge \omega = 0$ for some d-closed 1-form $\vartheta = 0$. By the Poincaré Lemma, ϑ is locally d-exact: $\vartheta \stackrel{\text{loc}}{=} dg$. Then $\exp(-g)\omega$ is a local conformal change of ω being Kähler.

Both the Kähler and the lcK conditions are cohomological in nature. Moreover, cohomologies of nilmanifolds (namely, compact quotients of connected simply-connected nilpotent Lie groups,) can often be reduced as Lie algebra invariants. It follows that the Kähler rank and the lcK rank of nilmanifolds is often encoded in the Lie algebra; see Lemma 2.1. This allows us to study explicitly the Kähler rank and the lcK rank of 6-dimensional nilmanifolds. (With the possible exception of nilmanifolds associated to the Lie algebra $\mathfrak{h}_7 = (0, 0, 0, 12, 13, 23)$ in the notation of Salamon [Sal01]; see [Rol11].) This is done in Section 3. Also compare [FGV16, Section 4.2], where the same results have been obtained independently.

As a further example, we consider a non-Kähler manifold obtained as a torus-suspension in [Mag12]. The investigation of these manifolds was suggested by Valentino Tosatti and may deserve further study in non-Kähler geometry.

1. Hermitian ranks

In this section, we recall the definitions of Kähler rank for compact complex surfaces by R. Harvey and H. B. Lawson, and for compact complex manifolds by I. Chiose. Along the same lines, we also introduce the notions of lcK rank and pluri-closed rank.

1.1. Kähler rank.

Let X be a compact complex surfaces. Denote by $P_{\mathrm{bdy}}(X)$ the cone of positive bidimension-$(1,1)$-currents on X being a boundary, and denote by $P^{\infty}_{\mathrm{bdy}}(X)$ the subcone of smooth currents. On a compact complex surface X, the cone $P^{\infty}_{\mathrm{bdy}}(X)$ coincides with the cone $P^{\infty}_{\mathrm{bdy}_{1,1}}(X)$ of positive bidimension-$(1,1)$-currents on X being component of a boundary, and any form $\varphi \in P^{\infty}_{\mathrm{bdy}}(X)$ is simple (i.e., of rank less or equal than one) at every point of X [HL83, Proposition (37)]. Set

$$\mathcal{B}(X) \;:=\; \left\{x \in X \;:\; \exists \varphi \in P^{\infty}_{\mathrm{bdy}}(X) \text{ such that } \varphi_x \neq 0\right\} \;.$$

The open subset $\mathcal{B}(X) \subseteq X$ carries an intrinsically defined complex analytic foliation $\mathcal{F}$, which is characterized by the property that $\varphi\lfloor_{\mathcal{F}} = 0$ for any $\varphi \in P^{\infty}_{\mathrm{bdy}}(X)$ [HL83, Theorem 40]. The *Kähler rank* of the compact complex surface X [HL83, Definition 41] is defined to be: (a) two, when X admits Kähler metrics (that is, the open subset $\mathcal{B}(X)$ in X is empty); (b) one, when the complement of the open subset $\mathcal{B}(X)$ in X is contained in a complex curve and is nonempty; (c) zero, otherwise.

The Kähler rank of compact complex surfaces is a bimeromorphic invariant [CT13, Corollary 4.1]. Surfaces with even first Betti number have Kähler rank two [Lam99, Buc99]. Elliptic non-Kähler surfaces have Kähler rank one [HL83, page 187]. Nonelliptic non-Kähler surfaces are in class VII. Under the GSS conjecture, their minimal model is one of the following: (i) Inoue surfaces: Kähler rank is one [HL83, §10]; (ii) Hopf surface: Kähler rank is one or zero according to the type [HL83, §9]; (iii) Kato surfaces: Kähler rank is zero [CT13].

Now, let X be a compact complex manifold of complex dimension $n \geq 2$. Notice that, when $n = 2$, the Kähler rank, as in [HL83, Definition 41], is equal to the maximal rank that a nonnegative closed $(1,1)$-form may attain at some point of X, thanks to [CT13, Corollary 4.3]. Then, the following definition by I. Chiose is coherent. Compare also [FGV16, Definition 1.2].

Definition 1.1 ([Chi13, Definition 1.1])**.** Let X be a compact complex manifold of complex dimension n. The *Kähler rank* of X is defined to be

$$\mathrm{Kr}(X) := \max\left\{k \in \mathbb{N} : \exists \omega \in \wedge^{1,1}X \text{ s.t. } \omega \geq 0,\ d\omega = 0,\ \text{and } \omega^k \neq 0\right\} \in \{0, \dots, n\}\,.$$

Remark 1.2. Note that, by [Chi13, Theorem 0.2], for compact complex manifolds of complex dimension 3 with maximal Kähler rank, the "tamed-to-compatible" conjecture ([LZ09, page 678], [ST10, Question 1.7]) holds.

1.2. **Locally conformally Kähler rank.** Now, we consider *locally conformally Kähler* structures [DO98]. Such a structure is given by (ϑ, ω), where ϑ is a closed 1-form and ω is a Hermitian metric satisfying $d_\vartheta\omega = 0$, where

$$d_\vartheta \ := \ d - \vartheta \wedge \ .$$

Note that, locally, $\vartheta \stackrel{\text{loc}}{=} df$ for some smooth function f. Therefore $\exp(-f)\omega$ is a local Kähler structure being a local conformal transformation of ω.

We now admit degenerate metrics, and we introduce the following rank. (We add the adjective "Hermitian" in order to avoid confusion with the notion of an lcK rank introduced in [GOPP06], which regards the Lee form ϑ.)

Definition 1.3. Let X be a compact complex manifold of complex dimension n. The *Hermitian locally conformally Kähler rank* of X is defined to be

$$\begin{aligned} \mathrm{HlcKr}(X) := \max\big\{k \in \mathbb{N} \ :\ & \exists \vartheta \in \wedge^1 X \text{ s.t. } d\vartheta = 0, \\ & \exists \omega \in \wedge^{1,1}X \text{ s.t. } \omega \geq 0,\ d_\vartheta\omega = 0,\ \text{and } \omega^k \neq 0\big\} \\ & \in \{0, \dots, n\}\ . \end{aligned} \tag{1.1}$$

Clearly, since $d_0 = d$, it holds that $\mathrm{Kr}(X) \leq \mathrm{HlcKr}(X) \leq \dim X$. Moreover, if X admits a locally conformally Kähler metric, then clearly $\mathrm{HlcKr}(X) = \dim X$.

1.3. **Pluri-closed rank.** Finally, we consider *pluri-closed metrics* [Bis89], namely, Hermitian metrics ω such that $\partial\overline{\partial}\omega = 0$, also called *SKT metrics*. We define the following.

Definition 1.4. Let X be a compact complex manifold of complex dimension n. The *pluri-closed rank* of X is defined to be

$$\begin{aligned} \mathrm{SKTr}(X) := \max\left\{k \in \mathbb{N} \ :\ \exists \omega \in \wedge^{1,1}X \text{ s.t. } \omega \geq 0,\ \partial\overline{\partial}\omega = 0,\ \text{and } \omega^k \neq 0\right\} \\ \in \{0, \dots, n\}\ . \end{aligned} \tag{1.2}$$

Note that $\mathrm{Kr}(X) \leq \mathrm{SKT}(X)$. Moreover, by [Gau77, Théorème 1], any compact Hermitian manifold admits a unique *Gauduchon metric* in the conformal class up to scaling, that is, a metric ω satisfying $\partial\overline{\partial}\omega^{n-1} = 0$, where $n = \dim X$. In particular, it follows that, on compact complex surfaces, the pluri-closed rank is always maximum, equal to 2.

2. Special-Hermitian ranks of homogeneous manifolds of solvable Lie groups

In this section, we investigate the Kähler and the Hermitian lcK ranks of homogeneous manifolds of solvable Lie groups.

Let $X = \Gamma\backslash G$ be a solvmanifold, namely, a compact quotient of a connected simply-connected solvable Lie group G by a co-compact discrete subgroup Γ. Assume that X is endowed with an invariant complex structure (that is, the complex structure is induced by a complex structure on G being invariant with respect to the action of G on itself given by left-translations). Then $\wedge^{\bullet,\bullet}\mathfrak{g}^* \hookrightarrow \wedge^{\bullet,\bullet}X$ is a subcomplex.

In the definition of special-Hermitian ranks, we can restrict ourselves to invariant metrics: set the *invariant Kähler rank* and the *invariant Hermitian lcK rank* to be

$$\mathrm{Kr}(\mathfrak{g}) := \max\left\{k \in \mathbb{N} \;:\; \exists\omega \in \wedge^{1,1}\mathfrak{g}^* \text{ s.t. } \omega \geq 0,\; d\omega = 0,\; \text{and } \omega^k \neq 0\right\} ,$$

$$\mathrm{HlcKr}(\mathfrak{g}) := \max\left\{k \in \mathbb{N} \;:\; \exists\vartheta \in \wedge^{1}\mathfrak{g}^* \text{ s.t. } d\vartheta = 0,\right.$$
$$\left.\exists\omega \in \wedge^{1,1}\mathfrak{g}^* \text{ s.t. } \omega \geq 0,\; d_\vartheta\omega = 0,\; \text{and } \omega^k \neq 0\right\} .$$

They have the advantage of being more easily computed. In general, it holds that

$$\mathrm{Kr}(\mathfrak{g}) \;\leq\; \mathrm{Kr}(X) \qquad \text{and} \qquad \mathrm{HlcKr}(\mathfrak{g}) \;\leq\; \mathrm{HlcKr}(X)\,.$$

In fact, we prove that equalities hold, under the assumption that the map $H^{\bullet,\bullet}_{\overline{\partial}}(\mathfrak{g}) \to H^{\bullet,\bullet}_{\overline{\partial}}(X)$ induced by the inclusion $\wedge^{\bullet,\bullet}\mathfrak{g}^* \to \wedge^{\bullet,\bullet}X$ is an isomorphism. The assumption holds true, e.g., when G is nilpotent and the complex structure is either holomorphically-parallelizable, or Abelian, or nilpotent, or rational [Con07, Rol11]. It also holds true when X is a compact complex surface being diffeomorphic to a solvmanifold [ADT14].

Lemma 2.1. *Let $X = \Gamma\backslash G$ be a solvmanifold. Assume that the map $H^{\bullet,\bullet}_{\overline{\partial}}(\mathfrak{g}) \to H^{\bullet,\bullet}_{\overline{\partial}}(X)$ induced by the inclusion $\wedge^{\bullet,\bullet}\mathfrak{g}^* \to \wedge^{\bullet,\bullet}X$ is an isomorphism. Then the Hermitian locally conformally Kähler rank* $\mathrm{HlcKr}(X)$ *and the invariant Hermitian locally conformally Kähler rank* $\mathrm{HlcKr}(\mathfrak{g})$ *are equal. In particular, the Kähler rank* $\mathrm{Kr}(X)$ *and the invariant Kähler rank* $\mathrm{Kr}(\mathfrak{g})$ *are equal.*

Proof. Clearly, $\mathrm{HlcKr}(\mathfrak{g}) \leq \mathrm{HlcKr}(X)$. Let ϑ be a d-closed 1-form, and let $\omega \geq 0$ be a $(1,1)$-form satisfying $d_\vartheta\omega = 0$ such that $\omega^k \neq 0$. We show that there exist an invariant d-closed 1-form $\hat{\vartheta}$ and an invariant $(1,1)$-form $\hat{\omega} \geq 0$ satisfying $d_{\hat{\vartheta}}\hat{\omega}$ such that $\hat{\omega}^k \neq 0$.

By the assumption and by the Frölicher spectral sequence, the average map

$$\mu\colon\; \wedge^{\bullet,\bullet}X \to \wedge^{\bullet,\bullet}\mathfrak{g}^* \,, \qquad \mu(\alpha) \;:=\; \int_X \alpha\lfloor_m \eta(m)$$

(here, η is a bi-invariant volume form, thanks to Milnor) induces the identity in de Rham cohomology. In particular, $\hat{\vartheta} := \mu(\vartheta)$ is an invariant d-closed 1-form, and there exists an f smooth function such that $\hat{\vartheta} = \vartheta + df$.

Note that $d_{\vartheta+df} = \exp(f) \cdot d_\vartheta(\exp(-f) \cdot \text{-})$. Therefore $\tilde{\omega} := \exp(f) \cdot \omega \geq 0$ is a $(1,1)$-form satisfying $d_{\hat{\vartheta}}\tilde{\omega} = 0$ such that $\tilde{\omega}^k = \exp(kf)\omega^k \neq 0$. In particular, $[\tilde{\omega}] \in H^2_{d_{\hat{\vartheta}}}(X)$. By [Hat60], the average map μ induces the identity also in the cohomology of the twisted differential $d_{\hat{\vartheta}}$. So we get that $\hat{\omega} := \mu(\tilde{\omega}) \geq 0$ is an invariant $(1,1)$-form satisfying $d_{\hat{\vartheta}}\hat{\omega} = 0$, and there exists α as 1-form such that $\hat{\omega} = \tilde{\omega} + d_{\hat{\vartheta}}\alpha$.

Moreover, we have

$$\hat{\omega}^k \;=\; \omega^k + d_{k\hat{\vartheta}}\varphi, \qquad \text{where } \varphi \;:=\; \sum_{\substack{s+t=k\\ t\geq 1}} \binom{k}{s} \cdot \tilde{\omega}^s \wedge \alpha \wedge (d_{(t-1)\hat{\vartheta}}\alpha)^{t-1} \,.$$

That is, $[\hat{\omega}^k] = [\tilde{\omega}^k]$ in $H^{2k}_{d_{k\hat{\vartheta}}}(X)$. Therefore, since $\hat{\omega}^k$ is invariant, it follows that $\mu(\tilde{\omega}^k) = \hat{\omega}^k$. Since $\tilde{\omega}^k \geq 0$ and $\tilde{\omega}^k \neq 0$, then $\hat{\omega}^k \neq 0$.

As for the case of Kähler rank, it suffices to note that $[\vartheta] = 0$ if and only if $[\hat{\vartheta}] = 0$. $\square$

As a direct consequence, we have the following.

Corollary 2.2. *Let $X = \Gamma\backslash G$ be a solvmanifold. Assume that the map $H^{\bullet,\bullet}_{\overline{\partial}}(\mathfrak{g}) \to H^{\bullet,\bullet}_{\overline{\partial}}(X)$ induced by the inclusion $\wedge^{\bullet,\bullet}\mathfrak{g}^* \to \wedge^{\bullet,\bullet}X$ is an isomorphism. Then the Kähler rank* $\mathrm{Kr}(X)$ *(respectively, the Hermitian locally conformally Kähler rank* $\mathrm{HlcKr}(X)$*) is maximum if and only if there exists a Hermitian metric being Kähler (respectively, locally conformally Kähler).*

3. Special-Hermitian ranks of 6-dimensional nilmanifolds

By using Lemma 2.1, we can compute the Kähler and Hermitian lcK ranks of 6-dimensional nilmanifolds with invariant complex structures, except possibly for the nilmanifolds associated to the Lie algebra $\mathfrak{h}_7 = (0,0,0,12,13,23)$ in the notation of Salamon [Sal01]. In fact, the assumption of the map $H^{\bullet,\bullet}_{\overline{\partial}}(\mathfrak{g}) \to H^{\bullet,\bullet}_{\overline{\partial}}(X)$ induced by the inclusion being an isomorphism is satisfied; see [Con07, Rol11].

It is well known [Uga07, COUV] that, up to (linear-)equivalence, the invariant complex structures on 6-dimensional nilmanifolds are parametrized into the following families: there exists a global co-frame $\{\varphi^1, \varphi^2, \varphi^3\}$ of invariant $(1,0)$-forms such that the structure equations are

- **(P):** $d\varphi^1 = 0$, $d\varphi^2 = 0$, $d\varphi^3 = \rho\varphi^1 \wedge \varphi^2$, where $\rho \in \{0,1\}$;
- **(I):** $d\varphi^1 = 0$, $d\varphi^2 = 0$, $d\varphi^3 = \rho\varphi^1 \wedge \varphi^2 + \varphi^1 \wedge \bar{\varphi}^1 + \lambda\varphi^1 \wedge \bar{\varphi}^2 + D\varphi^2 \wedge \bar{\varphi}^2$, where $\rho \in \{0,1\}$, $\lambda \in \mathbb{R}^{\geq 0}$, $D \in \mathbb{C}$ with $\Im D \geq 0$;
- **(II):** $d\varphi^1 = 0$, $d\varphi^2 = \varphi^1 \wedge \bar{\varphi}^1$, $d\varphi^3 = \rho\varphi^1 \wedge \varphi^2 + B\varphi^1 \wedge \bar{\varphi}^2 + c\varphi^2 \wedge \bar{\varphi}^1$, where $\rho \in \{0,1\}$, $B \in \mathbb{C}$, $c \in \mathbb{R}^{\geq 0}$, with $(\rho, B, c) \neq (0,0,0)$;
- **(III):** $d\varphi^1 = 0$, $d\varphi^2 = \varphi^1 \wedge \varphi^3 + \varphi^1 \wedge \bar{\varphi}^3$, $d\varphi^3 = \varepsilon\varphi^1 \wedge \bar{\varphi}^1 \pm i\left(\varphi^1 \wedge \bar{\varphi}^2 - \varphi^2 \wedge \bar{\varphi}^1\right)$, where $\varepsilon \in \{0,1\}$.

3.1. **Kähler rank of 6-dimensional nilmanifolds.** As for the Kähler rank, we have the following.

Proposition 3.1. *On 6-dimensional nilmanifolds endowed with invariant complex structures (except possibly for the nilmanifolds associated to the Lie algebra $\mathfrak{h}_7$), the Kähler rank takes the following values:*

$$\begin{array}{rrcll} \textbf{(P):} & \mathrm{Kr}(X) & = & 3 & \text{if } \rho = 0\,, \\ & \mathrm{Kr}(X) & = & 2 & \text{if } \rho = 1\,; \\ \textbf{(I):} & \mathrm{Kr}(X) & = & 2\,; & \\ \textbf{(II):} & \mathrm{Kr}(X) & = & 1 & \text{if } (\rho, B, c) \neq (1,1,0)\,, \\ & \mathrm{Kr}(X) & \geq & 1 & \text{if } (\rho, B, c) = (1,1,0)\,; \\ \textbf{(III):} & \mathrm{Kr}(X) & = & 1\,. & \end{array}$$

Proof. Thanks to Lemma 2.1, we are reduced to computing the invariant ranks. The arbitrary invariant $(1,1)$-form ω such that $\omega \geq 0$ is

$$\begin{aligned} \omega &= ir^2\varphi^1 \wedge \bar\varphi^1 + is^2\varphi^2 \wedge \bar\varphi^2 + it^2\varphi^3 \wedge \bar\varphi^3 \\ &\quad + \left(u\varphi^1 \wedge \bar\varphi^2 - \bar u \varphi^2 \wedge \bar\varphi^1\right) + \left(v\varphi^2 \wedge \bar\varphi^3 - \bar v \varphi^3 \wedge \bar\varphi^2\right) + \left(z\varphi^1 \wedge \bar\varphi^3 - \bar z \varphi^3 \wedge \bar\varphi^1\right)\,, \end{aligned} \tag{3.1}$$

where $r, s, t \in \mathbb{R}$, $u, v, z \in \mathbb{C}$ satisfy

$$\begin{aligned} r^2 \,&\geq\, 0\,, \qquad s^2 \,\geq\, 0\,, \qquad t^2 \,\geq\, 0\,, \\ r^2s^2 \,&\geq\, |u|^2\,, \qquad s^2t^2 \,\geq\, |v|^2\,, \qquad r^2t^2 \,\geq\, |z|^2\,, \\ r^2s^2t^2 + 2\Re(i\bar u \bar v z) \,&\geq\, t^2|u|^2 + r^2|v|^2 + s^2|z|^2\,. \end{aligned} \tag{3.2}$$

We have

$$\begin{aligned} \frac{1}{2}\omega^2 \;=\;& \left(r^2s^2 - |u|^2\right)\varphi^{12\bar1\bar2} + \left(-ir^2v - \bar u z\right)\varphi^{12\bar1\bar3} + \left(is^2z - uv\right)\varphi^{12\bar2\bar3} \\ & + \left(ir^2\bar v - u\bar z\right)\varphi^{13\bar1\bar2} + \left(r^2t^2 - |z|^2\right)\varphi^{13\bar1\bar3} + \left(-it^2u - \bar v z\right)\varphi^{13\bar2\bar3} \\ & + \left(-is^2\bar z - \bar u \bar v\right)\varphi^{23\bar1\bar2} + \left(it^2\bar u - v\bar z\right)\varphi^{23\bar1\bar3} + \left(s^2t^2 - |v|^2\right)\varphi^{23\bar2\bar3} \end{aligned}$$

(for simplicity of notation, we shorten, e.g., $\varphi^{12\bar2\bar3} := \varphi^1 \wedge \varphi^2 \wedge \bar\varphi^2 \wedge \bar\varphi^3$) and

$$\frac{1}{6}\omega^3 \;=\; \left(ir^2s^2t^2 - ir^2|v|^2 - is^2|z|^2 - it^2|u|^2 + uv\bar z - \bar u \bar v z\right)\varphi^{123\bar1\bar2\bar3}\,.$$

We compute:

$$
\begin{array}{rrcl}
\textbf{(P):} & \partial\omega & = & -\bar{z}\rho\,\varphi^{12\bar{1}} - \bar{v}\rho\,\varphi^{12\bar{2}} + it^2\rho\,\varphi^{12\bar{3}} \;; \\
\textbf{(I):} & \partial\omega & = & (-v + \lambda z - \rho\bar{z})\,\varphi^{12\bar{1}} + (-\bar{v}\rho + z\bar{D})\,\varphi^{12\bar{2}} + (it^2\rho)\,\varphi^{12\bar{3}} \\
& & & +(-it^2)\,\varphi^{13\bar{1}} + (-it^2\lambda)\,\varphi^{23\bar{1}} + (-it^2\bar{D})\,\varphi^{23\bar{2}} \;; \\
\textbf{(II):} & \partial\omega & = & \left(-is^2 + z\bar{B} - \bar{z}\rho\right)\,\varphi^{12\bar{1}} + (-cv - \bar{v}\rho)\,\varphi^{12\bar{2}} + \left(it^2\rho\right)\,\varphi^{12\bar{3}} \\
& & & + (\bar{v})\,\varphi^{13\bar{1}} + \left(-it^2 c\right)\,\varphi^{13\bar{2}} + \left(-it^2\bar{B}\right)\,\varphi^{23\bar{1}} \;; \\
\textbf{(III):} & \partial\omega & = & (\mp iz - \varepsilon v)\,\varphi^{12\bar{1}} + (\mp iv)\,\varphi^{12\bar{2}} + (u - \bar{u} - it^2\varepsilon)\,\varphi^{13\bar{1}} \\
& & & +(is^2 \pm t^2)\,\varphi^{13\bar{2}} + (v)\,\varphi^{13\bar{3}} + (is^2 \mp t^2)\,\varphi^{23\bar{1}} \;.
\end{array}
$$

The statement follows. □

Remark 3.2. The results in Proposition 3.1 have been obtained independently in [FGV16, Section 4.2].

3.2. **Hermitian locally conformally Kähler rank of 6-dimensional nilmanifolds.** As for the Hermitian lcK rank, we have the following.

Proposition 3.3. *On 6-dimensional nilmanifolds endowed with invariant complex structures (except possibly for the nilmanifolds associated to the Lie algebra $\mathfrak{h}_7$), the Hermitian locally conformally Kähler rank takes the following values:*

$$
\begin{array}{rlcll}
\textbf{(P):} & \mathrm{HlcKr}(X) & = & 3 & \textit{if } \rho = 0 \,, \\
& \mathrm{HlcKr}(X) & = & 2 & \textit{if } \rho = 1 \,; \\
\textbf{(I):} & \mathrm{HlcKr}(X) & = & 3 & \textit{if } (\rho, \lambda, D) = (0, 0, -1) \,, \\
& \mathrm{HlcKr}(X) & = & 2 & \textit{if } (\rho, \lambda, D) \neq (0, 0, -1) \,; \\
\textbf{(II):} & \mathrm{HlcKr}(X) & = & 2 \,; & \\
\textbf{(III):} & \mathrm{HlcKr}(X) & = & 1. &
\end{array}
$$

Proof. By [Saw07, Main Theorem], a nontoral compact nilmanifold with a left-invariant complex structure has a locally conformally Kähler structure if and only if it is biholomorphic to a quotient of the Heisenberg group times $\mathbb{R}$. In particular, the only 6-dimensional non-Abelian nilpotent Lie algebra admitting lcK structures is $\mathfrak{h}_3$, which appears in family **(I)** with parameters $\rho = 0$, $\lambda = 0$, $D = -1$.

In case **(II)**, consider the d-closed 1-form $\vartheta := \varphi^2 + \bar{\varphi}^2$ and the d_ϑ-closed 2-form, $\Omega := i\,\varphi^1 \wedge \bar{\varphi}^1 + i\,\varphi^2 \wedge \bar{\varphi}^2 \geq 0$.

In case **(III)**, the arbitrary d-closed 1-form is $\vartheta = \vartheta_1\varphi^1 + \vartheta_3\varphi^3 + \bar{\vartheta}_1\bar{\varphi}^1 + \vartheta_3\bar{\varphi}^3$, where $\vartheta_1 \in \mathbb{C}$ and $\vartheta_3 \in \mathbb{R}$. By straightforward computations, which we performed with the aid of Sage [S$^+$09], we get that the arbitrary form ω in (3.1) with conditions (3.2) is d_ϑ-closed if and only if both $r^2 = 0$ and $s^2 = 0$. □

3.3. **Hermitian pluri-closed rank of 6-dimensional nilmanifolds.** Finally, we consider the pluri-closed rank $\mathrm{SKTr}(X)$ of X as defined in (1.2) in Definition 1.4.

In the case of solvmanifolds X with associated Lie algebra $\mathfrak{g}$, a notion of *invariant pluri-closed rank* $\mathrm{SKTr}(\mathfrak{g})$ can be defined. Clearly, $\mathrm{SKTr}(\mathfrak{g}) \leq \mathrm{SKTr}(X)$. Note that, in this case, the argument in the proof of Lemma 2.1 does not apply. Indeed, there we make use of the map induced by the wedge product in the Morse–Novikov cohomology, $H^2_{d_\vartheta}(X) \times H^2_{d_\vartheta}(X) \to H^4_{d_{2\vartheta}}(X)$, which in turn is a consequence of the Leibniz rule for the twisted differential operator, namely $d_{k\vartheta}(\alpha \wedge \beta) = d_{h\vartheta}\alpha \wedge \beta + (-1)^{\deg \alpha}\, \alpha \wedge d_{(k-h)\vartheta}\beta$. But the $\partial\overline{\partial}$-operator, and the corresponding Aeppli cohomology, do not share these properties.

In Table 1, we show the invariant pluri-closed rank of 6-dimensional nilmanifolds and also summarizing the ranks computed in the previous sections. The results follow by computing:

$$\begin{array}{rrcl}
\textbf{(P):} & \partial\overline{\partial}\omega & = & -it^2\rho\,\varphi^{12\bar{1}\bar{2}}\ ; \\
\textbf{(I):} & \partial\overline{\partial}\omega & = & \left(it^2(-\rho + D + \bar{D} - \lambda^2)\right)\,\varphi^{12\bar{1}\bar{2}}\ ; \\
\textbf{(II):} & \partial\overline{\partial}\omega & = & \left(-it^2(\rho + c^2 + |B|^2)\right)\,\varphi^{12\bar{1}\bar{2}}\ ; \\
\textbf{(III):} & \partial\overline{\partial}\omega & = & (-2it^2)\,\varphi^{12\bar{1}\bar{2}} + (-2is^2)\,\varphi^{13\bar{1}\bar{3}}\ .
\end{array}$$

Remark 3.4. From the results in Table 1, we note in particular the upper-semicontinuity of the Hermitian ranks. We wonder whether this property holds in general.

TABLE 1. Special-Hermitian ranks for 6-dimensional nilmanifolds endowed with invariant complex structures.

	class	$\mathrm{Kr}(\mathbf{X})$	$\mathrm{HlcKr}(\mathbf{X})$	$\mathrm{SKTr}(\mathfrak{g})$
(P)	$\rho = 0$	3	3	3
	$\rho = 1$	2	2	2
(I)	$-\rho + D + \bar{D} - \lambda^2 = 0$	2	2	3
	$-\rho + D + \bar{D} - \lambda^2 \neq 0,\ (\rho, \lambda, D) \neq (0, 0, -1)$	2	2	2
	$(\rho, \lambda, D) = (0, 0, -1)$	2	3	2
(II)	$(\rho, B, c) \neq (1, 1, 0)$	1	2	2
	$(\rho, B, c) = (1, 1, 0)$	≥ 1	2	2
(III)		1	1	1

4. Kähler rank of a non-Kähler manifold constructed as suspension

As another example, we consider here a non-Kähler manifold constructed as suspension over a torus. We consider an explicit case of a more general construction which has been investigated by G. Þ. Magnusson [Mag12] to disprove the abundance and Iitaka conjectures for complex non-Kähler manifolds. See also [Tos15, Example 3.1] (and the references therein), where V. Tosatti uses the same construction to get a complex non-Kähler manifold with vanishing first Bott–Chern class, whose canonical bundle is not holomorphically torsion.

We first recall the construction by Yoshihara [Yos80, Example 4.1] of a complex 2-torus X with an automorphism f such that the induced automorphism on $H^0(X;K_X) \simeq \mathbb{C}$ has infinite order.

Consider the roots $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$ of the equation

$$x^2 - (1+\sqrt{-1})x + 1 \;=\; 0 \;.$$

The minimal polynomial over $\mathbb{Q}$ of α and $\bar{\beta}$ is

$$x^4 - 2x^3 + 4x^2 - 2x + 1 \;=\; 0 \;.$$

In particular,

$$\begin{pmatrix} \alpha^4 \\ \bar{\beta}^4 \end{pmatrix} = 2\begin{pmatrix} \alpha^3 \\ \bar{\beta}^3 \end{pmatrix} - 4\begin{pmatrix} \alpha^2 \\ \bar{\beta}^2 \end{pmatrix} + 2\begin{pmatrix} \alpha \\ \bar{\beta} \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \;.$$

Consider the following lattice in $\mathbb{C}^2$:

$$\Gamma \;:=\; \mathrm{span}_{\mathbb{Z}} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha \\ \bar{\beta} \end{pmatrix}, \begin{pmatrix} \alpha^2 \\ \bar{\beta}^2 \end{pmatrix}, \begin{pmatrix} \alpha^3 \\ \bar{\beta}^3 \end{pmatrix} \right\} \;.$$

Consider the torus

$$X \;:=\; \mathbb{C}^2/\Gamma \;.$$

The automorphism

$$f\colon \mathbb{C}^2 \to \mathbb{C}^2 \;, \qquad f\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \;:=\; \begin{pmatrix} \alpha & \\ & \bar{\beta} \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

induces an automorphism of X.

Now, we recall the construction by G. Þ. Magnusson [Mag12] of the non-Kähler manifold M. Let

$$C \;:=\; \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$$

be an elliptic curve. Then M is the total space of a holomorphic fibre bundle $M \to C$ with fibre X as follows:

$$M \;:=\; X \times \mathbb{C}/\mathbb{Z}^2 \;,$$

where $\mathbb{Z}^2 \circlearrowleft X \times \mathbb{C}$ acts as

$$(\ell, m) \cdot (z, w) \;:=\; (f^m(z), w + \ell + m\,\tau) \;.$$

Note that M is not Kähler [Mag12, Proposition 1.2] because of [Fuj78, Corollary 4.10].

We claim that the Kähler rank of M is equal to 1.

In fact, note that the form

$$dw \wedge d\bar{w}$$

on $\mathbb{C}$ yields a d-closed $(1,1)$-form of rank 1 on M. Whence the Kähler rank of M is greater than or equal to 1. On the other side, assume that there exists ω a d-closed $(1,1)$-form of rank at least 2 on M. It corresponds to a d-closed $\mathbb{Z}^2$-invariant $(1,1)$-form of rank at least 2 on $X \times \mathbb{C}$. By the inclusion $\iota\colon X \ni x \mapsto (x, 0) \in X \times \mathbb{C}$, it yields a d-closed f-invariant $(1,1)$-form of rank at least 1 on X—say ω again. Notice that f sends invariant forms (with respect to the action of $\mathbb{C}^2$ on X) to invariant forms. We have $\omega = \omega_{\mathrm{inv}} + d\eta$, where ω_{inv} is invariant and η is a 1-form. Then $f^*\omega = f^*\omega_{\mathrm{inv}} + df^*\eta$, where $f^*\omega_{\mathrm{inv}}$ is invariant. We get that $f^*\omega_{\mathrm{inv}} = \omega_{\mathrm{inv}}$ mod $d(\wedge^1\mathfrak{g}^*)$. Since the Lie algebra $\mathfrak{g}$ of X is Abelian, we get $f^*\omega_{\mathrm{inv}} = \omega_{\mathrm{inv}}$. We get that ω_{inv} is a d-closed invariant f-invariant $(1,1)$-form of rank at least 1 on X.

But this is not possible, since the only invariant f-invariant $(1,1)$-forms on X are generated over $\mathbb{C}$ by $dz^1 \wedge d\bar{z}^2$ and $dz^2 \wedge d\bar{z}^1$.

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Dipartimento di Matematica e Informatica "Ulisse Dini", Università degli Studi di Firenze, viale Morgagni 67/a, 50134 Firenze, Italy

Email address: `daniele.angella@gmail.com`

Email address: `daniele.angella@unifi.it`

Dipartimento di Scienze Matematiche, Fisiche e Informatiche, Unità di Matematica e Informatica, Università di Parma, Parco Area delle Scienze 53/A, 43124 Parma, Italy

Email address: `adriano.tomassini@unipr.it`