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DISTORTION OF LIPSCHITZ FUNCTIONS ON $c_0(\Gamma)$

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Abstract. Let Γ be an uncountable cardinal. We construct a real symmetric 1-Lipschitz function on the unit sphere of $c_0(\Gamma)$ whose restriction to any nonseparable subspace is a distortion.

1. Introduction

Let us start by recalling the classical definitions of oscillation stability and distortion. Let X be a real infinite dimensional Banach space, and let $f : S_X \to \mathbb{R}$ be a real valued function. The function f is said to be oscillation stable if for every infinite dimensional subspace $Z \subset X$ and $\varepsilon > 0$ there exists a further subspace $Y \subset Z$ such that the oscillation of f on S_Y is at most ε, i.e., $|f(x) - f(y)| \leq \varepsilon$, $x, y \in S_Y$.

The function f is said to be a distortion if there exists an $\varepsilon > 0$ such that for every infinite dimensional subspace Y of X there exist $x, y \in S_Y$ such that $|f(x) - f(y)| \geq \varepsilon$.

It is clear that oscillation stability and distortion are in a sense opposite properties. More precisely, any function f on S_X is either oscillation stable, or it is a distortion on $S_X \cap Y$ for some subspace $Y \subset X$. On the other hand, the distortion passes to subspaces and so a distorting function is not oscillation stable on any subspace of X.

It is a classical result of James [7] that every equivalent norm on the Banach space c_0, resp. ℓ_1, is oscillation stable. On the other hand, the spaces $\ell_p, 1 < p < \infty$, admit a distorting renorming by the results of Odell and Schlumprecht [11]. It turns out, by combining the result of [11] with the work of Milman [10] that every equivalent norm on a Banach space is oscillation stable if and only if the space in question is saturated by copies of c_0, or ℓ_1.

The supply of Lipschitz functions on a Banach space is much larger than that of renormings, so one would expect that distorting Lipschitz functions are more abundant. Using the concepts of asymptotic set ([15], [4], [11]) and the Mazur map, one can transfer the distorting norm from the unit sphere of ℓ_2 into a distorting Lipschitz function on the unit sphere of ℓ_1. So while all equivalent norms on the space ℓ_1 are oscillation stable, Lipschitz functions may be distorting. The details of this procedure are described, e.g., in the article of Odell-Schlumprecht in [12].

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It is important to note that the transfer preserves the symmetry of the mappings involved.

As regards the remaining relevant space c_0, there is the following result by Gowers [3].

Theorem 1 (Gowers). *Every Lipschitz function $f : S_{c_0} \to \mathbb{R}$ is oscillation stable.*

Putting the above-mentioned results together, one can conclude that every Lipschitz function on a Banach space is oscillation stable if and only if the space in question is saturated by copies of c_0.

Our interest in the present note lies in the nonseparable oscillation stabilization (resp. distortion) of Lipschitz functions. More precisely, let X be a nonseparable real Banach space with density character Γ, and $f : S_X \to \mathbb{R}$ be a Lipschitz function. Given a nonseparable subspace Y of X (say of the same density character Γ), and $\varepsilon > 0$ is there a further infinite dimensional subspace Z (of the density character Γ) of Y such that f on S_Z has oscillation at most ε? In the special case of $\ell_p(\Gamma)$ spaces this problem can be resolved by using the separable results combined with their symmetry. Indeed, let $(X, \|\cdot\|)$ be a Banach space with a symmetric (possibly uncountable) Schauder basis $\{e_\gamma\}_{\gamma\in\Gamma}$, where Γ is any nonempty set. We say that a function $f : X \to \mathbb{R}$ is symmetric if the value $f(x)$ is preserved under any permutation of the coordinates of x. It is clear that a symmetric function on X is uniquely determined by its values on the span of any countably infinite set $\{e_{\gamma_i}\}_{i=1}^{\infty}$. Thanks to the construction of Maurey [9] of a distorting and symmetric norm on every $\ell_p, 1 < p < \infty$, it is easy to formally extend the distorting (and symmetric) norms onto $\ell_p(\Gamma), 1 < p < \infty$, for every infinite set Γ. It is immediate to check that the extensions will preserve the distortion.

Similarly, one can extend the distorting Lipschitz and symmetric function from ℓ_1 (constructed using the symmetric distorting norm on ℓ_2), onto arbitrary $\ell_1(\Gamma)$. The distortion property will again be preserved.

It is natural to ask if there exists any nonseparable Banach space X such that all norms (resp. Lipschitz functions) on X are oscillation stable (resp. distorted) in the nonseparable sense. The obvious remaining test space is of course the space $c_0(\Gamma)$.

Our main result, solving the Problem 199 in the recent book of Guirao, Montesinos and Zizler [5], is that there exists a nonseparably distorting Lipschitz function on $c_0(\Gamma)$. More precisely, we have the next result.

Theorem 2. *There is a 1-Lipschitz symmetric function $F : S_{c_0(\Gamma)} \to \mathbb{R}$, such that for every nonseparable subspace $Y \subseteq c_0(\Gamma)$ there are points $x, y \in S_Y$ such that $|F(x) - F(y)| > \frac{1}{4}$.*

On the other hand, the nonseparable oscillation stability of equivalent norms on $c_0(\Gamma)$, resp. $\ell_1(\Gamma)$ still holds. This folklore result is apparently well-known to experts in the field. We would like to thank Tomasz Kania for bringing this fact to our attention. The case of $\ell_1(\Gamma)$ was dealt with in the paper of Giesy [2]. The case of $c_0(\Gamma)$ seems not to have been published in the refereed journal, although there exists a short note of Granero containing the proof. For the convenience of the reader, we have included in this note the formal statement and the proof, which goes along the lines of the classical James theorem.

Theorem 3. *Let Γ be an uncountable cardinal, $X = c_0(\Gamma)$ (resp. $\ell_1(\Gamma)$). For every equivalent norm $|||\cdot|||$ on X, $\varepsilon > 0$ and a subspace $Z \subset X$ there exist a constant $c > 0$ and a subspace $Y \subseteq Z$ with $\operatorname{dens} Y = \operatorname{dens} Z$ such that $c - \varepsilon < |||x||| \leq c$ for every point $x \in Y$, $\|x\| = 1$.*

In the sequel we will need the following well-known fact [1, p. 12]. Suppose (M, d) is a metric space and $g : S \to \mathbb{R}$ a K-Lipschitz function on some $S \subseteq M$. Then the following formula defines a K-Lipschitz function $\hat{g} : M \to \mathbb{R}$ such that $\hat{g}|_S = g$:

$$\hat{g}(x) = \inf_{y \in S} \{g(y) + Kd(x, y)\}. \tag{1}$$

In the construction of F, we will use a simple modification of the formula (1), which will ensure that the range of F is contained in $[0, 1]$. We omit the completely straightforward proof.

Lemma 4 (Modified extension formula)**.** *Suppose (M, d) is a metric space and $g : S \to \mathbb{R}$ a K-Lipschitz function on some $S \subseteq M$, taking values only in the interval $[0, 1]$. Then the following formula defines a K-Lipschitz function $\overline{g} : M \to \mathbb{R}$, taking values only in $[0, 1]$ such that $\overline{g}|_S = g$:*

$$\overline{g}(x) = \min\{\inf_{y \in S} \{g(y) + Kd(x, y)\}, 1\}. \tag{2}$$

2. Proofs of the results

Proof of Theorem 2. To prove the theorem, it suffices to construct (as we will do) the symmetric 1-Lipschitz function $F : c_0(\omega_1) \to \mathbb{R}$ and show it does not stabilize on the sphere of any subspace $Y \subseteq c_0(\omega_1)$ with $\operatorname{dens} Y = \omega_1$. Indeed, in the general case we use the symmetric extension of F to $c_0(\Gamma)$, and we check easily that any nonseparable space $Y \subset c_0(\Gamma)$ contains a further nonseparable subspace of density ω_1, which is contained in some $c_0(\Lambda)$, $\Lambda \subset \Gamma$, $|\Lambda| = \omega_1$.

The meaning of symmetry is that the function value $F(x)$ does not depend on the particular distribution of the coordinates of the vector x in the domain, but only on the set of the coordinate values of x. We define an equivalence relation $\sim$ on $c_{00}(\omega_1)$ in the following way: $x \sim y$ whenever $|\operatorname{supp} x| = |\operatorname{supp} y|$ and there exists a bijection f from $\operatorname{supp} x$ to $\operatorname{supp} y$ (both understood as finite sets of ordinal numbers) such that $x(\gamma) = y(f(\gamma))$. We will call every equivalence class $[x] \in X := c_{00}(\omega_1)/\sim$ a shape.

Note that if $x \sim y$, $x, y \in c_{00}(\omega_1)$, then $\|x\| = \|y\|$. Next, let us denote by $L = \{S_i\}_{i=1}^{\infty}$ the sequence of all shapes of norm one with finite support and rational coordinates.

Lemma 5. *Let $x, y, \in S_j, j \in \mathbb{N}$. Then for any shape $S \in L$, $d(x, S) = d(y, S)$ holds.*

Proof. The statement of the lemma follows readily from the fact that for every $w \in S$ there exists a $z \in S$ such that $\|x - w\| = \|y - z\|$. To prove the latter statement, if $x \sim y$, then there is a bijection $\varphi : \operatorname{supp} x \to \operatorname{supp} y$ such that for every $\gamma \in \omega_1$ we have $x(\gamma) = y(\varphi(\gamma))$. Set $s_y = \inf\{\alpha |\ \forall \beta \in \operatorname{supp} y : \beta < \alpha\}$. Define a mapping $\psi : \operatorname{supp} w \to \omega_1$ by

$$\psi(\gamma) = \begin{cases} \varphi(\gamma)\,, & \gamma \in \operatorname{supp} x \cap \operatorname{supp} w\,, \\ \alpha + \gamma\,, & \gamma \in \operatorname{supp} w \setminus \operatorname{supp} x\,. \end{cases}$$

Clearly ψ is a bijection onto its image. If we define z as $z(\psi(\gamma)) = w(\gamma)$ for $\gamma \in \operatorname{supp} w$ and 0 elsewhere, then $z \in [w] = S$. It follows from the definition of z that $\|x - w\| = \|y - z\|$. Therefore $d(x, S) = d(y, S)$ for any shape $S \in L$. □

We define inductively a function $\pi : L \to L$, which is going to "clone" every shape to an identical shape repeated several times in a row. Suppose $x \in S_1$ with $\operatorname{supp} x = \{1, ..., k\}$ for some $k \in \mathbb{N}$. Define $\pi(S_1) = [y]$, where $y(i) = y(k+i) = y(2k+i) = ... = y(k^2 + i) = x(i)$ for $i \in \{1, ..., k\}$ and $y(\gamma) = 0$ for $\gamma \in \omega_1 \setminus \{1, ..., k(k+1)\}$.

Suppose π has been defined for all S_i, $i < j$, and

$$k = \max\{\max_{i \le j} |\operatorname{supp} S_i|, \max_{i<j} |\operatorname{supp} \pi(S_i)|\}.$$

If $x \in S_j$ is such that $\operatorname{supp} x = \{1, \dots, l\}$ for some $l \in \mathbb{N}$, then we set $\pi(S_j) = [y]$, where $y(i) = y(l+i) = y(2l+i) = \cdots = y(kl+i) = x(i)$ for $i \in \{1, \dots, l\}$ and $y(\gamma) = 0$ for $\gamma \in \omega_1 \setminus \{1, \dots, (k+1)l\}$.

Note that for all $i < j \in \mathbb{N}$ the distance between any $x \in \pi(S_j)$ and $y \in S_i \cup \pi(S_i) \cup S_j$ is equal to 1. Indeed, every element x with S_j has some coordinate which equals 1 or -1 and therefore $\pi(S_j)$ has at least $k{+}1$ such coordinates, where k is the maximum "length" of a support of previously treated shapes (S_i or $\pi(S_i)$ for $i \le j$ or $i < j$ respectively). Therefore, there exists a point $\gamma \in \omega_1$ where $|x(\gamma)| = 1$ and $y(\gamma) = 0$.

We construct our function F on the set $S = \bigcup\{x : x \in [x], [x] \in L\}$ by an inductive repetition of the extension operation. The extension onto the unit sphere $S_{c_0(\omega_1)}$ is then unique and 1-Lipschitz as the set S is dense in $S_{c_0(\omega_1)}$. Moreover, as the values of F will depend only on the shape $[x] \in L$, it follows that F is symmetric.

Set $F(x) = 0$ for all $x \in S_1$ and $F(y) = 1$ for all $y \in \pi(S_1)$. Such a function is clearly 1-Lipschitz. After having defined F on the set $S_1 \cup \pi(S_1)$, we extend F to the set $S_1 \cup \pi(S_1) \cup S_2$ via the extension formula (2). Of course, if $\pi(S_1) = S_2$, the extension is trivial (as the domain has not increased) and we move to the definition of $F(\pi(S_2))$ described below. Note that $F(S_2) \subseteq [0, 1]$. We will check below in the general inductive step that F is constant on the set S_2. Furthermore, we set $F(\pi(S_2)) = 1$ if $F(S_2) \le \frac{1}{2}$ and $F(\pi(S_2)) = 0$ if $F(S_2) > \frac{1}{2}$. Thus F is still 1-Lipschitz, as $\pi(S_2)$ has distance one from each of the sets S_1, $\pi(S_1)$ and S_2.

Let us describe the general inductive step. Suppose F has been defined on the sets $S_1, \dots, S_{j-1}$, $\pi(S_1), \dots, \pi(S_{j-1})$ and it is constant on every such a set. We use the formula (2) to extend F to the set S_j if it hasn't been defined there yet. Note that again $F(S_j) \subseteq [0, 1]$. Let us check that F is constant on S_j (or S_l). Pick two points $x, y, \in S_j$. Using (1)

$$\hat{F}(y) = \inf_{w \in \bigcup_{i=1}^{j-1} S_i \cup \pi(S_i)} \{F(w) + \|y - w\|\}. \tag{3}$$

Since by our inductive assumption F is constant on every set S_i, $\pi(S_i)$, $i \in \{1, ..., j-1\}$, replacing y with x in the formula (3) and using Lemma 5 gives the same value. As the formula gives the same values for all $x \in S_j$, so does the formula (2). We conclude F is constant on S_j.

Finally, having defined F on the sets $S_1, ..., S_j$ and $\pi(S_1), ..., \pi(S_{j-1})$, we set $F(\pi(S_j)) = 1$ if $F(S_j) \le \frac{1}{2}$ and $F(\pi(S_j)) = 0$ if $F(S_j) > \frac{1}{2}$. We finish the definition of F by extending it continuously onto $S_{c_0(\omega_1)}$. Clearly, F is 1-Lipschitz and symmetric.

Next we are going to show that for every subspace $Y \subseteq c_0(\omega_1)$ with $\operatorname{dens} Y = \omega_1$ there exist two points $x, y \in S_Y$ with $|F(x) - F(y)| > \frac{1}{4}$.

Our next lemma, which is probably a folklore result, is a variant to some results of Rodriquez-Salinas [13].

Lemma 6. *Let $Y \subseteq c_0(\omega_1)$ be a subspace with $\operatorname{dens} Y = \omega_1$. Then there exists a transfinite sequence $\{x_\gamma\}_{\gamma=1}^{\omega_1}$ of norm one vectors from Y with pairwise disjoint supports, i.e.,*

$$\operatorname{supp}(x_\alpha) \cap \operatorname{supp}(x_\beta) = \emptyset, \quad \alpha \neq \beta.$$

In particular, Y contains a subspace isomorphic to $c_0(\omega_1)$.

Proof. We proceed by transfinite induction. Choose a norm one vector $x_1 \in S_Y$. After having chosen $\{x_\gamma : 1 \leq \gamma < \Omega\}$, for some $\Omega < \omega_1$, we consider the countable set $\Lambda = \bigcup_{\gamma<\Omega} \operatorname{supp}(x_\gamma) \subset [1, \omega_1)$. Since $Y \subset c_0(\omega)$ is a nonseparable Weakly Compactly Generated (WCG) space ([6, p. 211]), it is also a Weakly Lindeloff Determined (WLD) space, and so $w^* - \operatorname{dens} Y^* = \omega_1$ ([6, p. 181]). Hence $V = \overline{\{\delta_\gamma : \gamma \in \Lambda\}}^{w^*}$ is the proper w^*-closed subspace of Y^*. Hence $Z = \{y \in Y : y(\gamma) = 0, \ \gamma \in \Lambda\} = V_\perp$ is a nontrivial subspace of Y, and we may find the next element of the sequence $x_\Omega \in S_Z$. This procedure yields the desired long sequence $\{x_\gamma : 1 \leq \gamma < \omega_1\}$, which is equivalent to the long Schauder basis of $c_0(\omega_1)$. $\square$

Choose a sequence $\{y_\gamma\}_{\gamma\in\omega_1}$ of norm one vectors in $c_0(\omega_1)$ with finite support and rational coordinates such that $\operatorname{supp} y_\gamma \subseteq \operatorname{supp} x_\gamma$ and $\|x_\gamma - y_\gamma\| < \frac{1}{8}$. As $\{y_\gamma\}_{\gamma\in\omega_1}$ is an uncountable sequence and L is countable, it follows that there exists a shape $S \in L$ which corresponds to infinitely many y_γ. So there is an infinite sequence of distinct indices $\{\gamma_i\}_{i=1}^{\infty}$ from the set ω_1 such that $y_{\gamma_i} \in S$ for each $i \in \mathbb{N}$. Let d be the number of times S is cloned in $\pi(S)$. Then set

$$x = \sum_{i=1}^{d} x_{\gamma_i}, \ y = \sum_{i=1}^{d} y_{\gamma_i}$$

and observe $x \in Y$, $\|x - y\| < \frac{1}{8}$. Indeed,

$$\begin{aligned} \|x - y\| = \sup_{\alpha\in\omega_1} \left| \sum_{i=1}^{d} x_{\gamma_i}(\alpha) - \sum_{i=1}^{d} y_{\gamma_i}(\alpha) \right| &= \max_{i\in\{1,\dots,d\}} \sup\nolimits_{\alpha\in\operatorname{supp} x_{\gamma_i}} |x_{\gamma_i}(\alpha) - y_{\gamma_i}(\alpha)| \\ &= \max_{i\in\{1,\dots,d\}} \|x_{\gamma_i} - y_{\gamma_i}\| < \frac{1}{8} \end{aligned}$$

as all the x_{γ_i} have disjoint supports and $\operatorname{supp} y_{\gamma_i} \subseteq \operatorname{supp} x_{\gamma_i}$, $i \in \{1, ..., d\}$. Therefore we get

$$|F(x) - F(x_{\gamma_1})| \geq |F(y) - F(y_{\gamma_1})| - |F(x) - F(y)| - \|x_{\gamma_1} - y_{\gamma_1}\| \geq \frac{1}{2} - \frac{1}{8} - \frac{1}{8} = \frac{1}{4}.$$

$\square$

The strategy of the proof of Theorem 3 is similar to that of the classical James proof in the separable case. Namely, we are constructing a (long) sequence of disjointly supported vectors in Z (equivalent to the canonical basis of X) so that the one-sided estimate of the norm on their linear span nearly satisfies either the supremum (resp. the summable) norm. To this end we need to find a biorthogonal system of functionals such that their supports are disjoint with those of the orthogonal vectors, thus guaranteeing the desired one-sided estimates. This is the main technical step in the proof. The estimates going in the opposite direction are then satisfied automatically thanks to the extremal properties of the canonical norms on X.

Proof of Theorem 3. Let $Z \subset X$ be a closed subspace of density character Λ.

By Lemma 6 (for $c_0(\Gamma)$), resp. a result of Rosenthal [14] (for $\ell_1(\Gamma)$) there exist a subspace $Y_1 \subset Z$ which is isomorphic to $c_0(\Lambda)$, resp. $\ell_1(\Lambda)$. So we may assume without loss of generality that $Z = X$.

We start with the case $X = c_0(\Gamma)$. Let Γ be an uncountable cardinal, and let $|||\cdot|||$ be an equivalent norm on $c_0(\Gamma)$. For $\Lambda \subset \Gamma$, denote

$$S_\Lambda = \sup\{|||x||| : x \in c_0(\Gamma), \operatorname{supp}(x) \subset \Lambda, \|x\| \le 1\}.$$

Let $S = \inf_{\Lambda\subset\Gamma,|\Lambda|=|\Gamma|} S_\Lambda$, $\varepsilon > 0$. Choose $\Lambda \subset \Gamma$ such that $S_\Lambda < S + \varepsilon$. For simplicity of notation, we may assume without loss of generality that $\Lambda = \Gamma$. This means, in particular, that $S \le S_\Lambda < S+\varepsilon$, for every $\Lambda \subset \Gamma$. By a simple transfinite induction, choose a transfinite sequence $\{u_\alpha\}_{\alpha=1}^\Gamma$ of disjointly and finitely supported vectors from $c_0(\Gamma)$, $\|u_\alpha\| \le 1$, such that $S \le |||u_\alpha||| < S + \varepsilon$. Indeed, once the initial segment $\{u_\alpha : 1 \le \alpha < \Delta\}$ has been constructed for some $\Delta < \Gamma$, set $\Lambda = \Gamma \setminus \bigcup_{1\le\alpha<\Delta} \operatorname{supp}(u_\alpha)$, and use the property $S \le S_\Lambda < S + \varepsilon$ to find u_Δ.

Choose a sequence $\{f_\alpha\}_{\alpha=1}^\Gamma \subset B_{(\ell_1,|||\cdot|||^*)}$ finitely supported and such that

$$f_\alpha(u_\alpha) > S - \varepsilon.$$

For the rest of the proof we distinguish two cases.

Case 1. Suppose that $\operatorname{cof}(\Gamma) > \omega_0$ (i.e., the cofinality of Γ is an uncountable cardinal). Then, by passing to a suitable subset and reindexing, we may assume in addition that $|\operatorname{supp}(f_\alpha)| = n$ for some fixed $n \in \mathbb{N}$. So we have

$$|\{u_\alpha : u_\alpha \ne u_\beta,\ \operatorname{supp}(f_\beta) \cap \operatorname{supp}(u_\alpha) \ne \emptyset\}| \le n, \ \text{ whenever } \ \beta < \Gamma.$$

Our next objective is to pass to a biorthogonal system $\{(u_\alpha, f_\alpha)\}_{\alpha\in\Lambda}$ indexed by a set $\Lambda \subset \Gamma$ of cardinality Γ.

To this end, we first partition the set $\{u_\alpha\}_{\alpha=1}^\Gamma$ using transfinite induction as follows. Let

$$U_1 = \{u_1\} \cup \{u_\alpha : \operatorname{supp}(f_\alpha) \cap \operatorname{supp}(u_1) \ne \emptyset\}.$$

Having found the sets $U_\alpha, \alpha < \Omega < \Gamma$, we let $U_\Omega = \emptyset$ provided $u_\Omega \in \bigcup_{\gamma<\Omega} U_\gamma$, and otherwise we let

$$U_\Omega = \{u_\Omega\} \cup \{u_\alpha : \alpha \in \Gamma \setminus \{\beta : u_\beta \in \bigcup_{\gamma<\Omega} U_\gamma\},\ \operatorname{supp}(f_\alpha) \cap \operatorname{supp}(u_\Omega) \ne \emptyset\}.$$

If the set $\Xi = \{\beta : |U_\beta| = 1\}$ has cardinality $|\Gamma|$, then it is clear that $\operatorname{supp}(u_\alpha) \cap \operatorname{supp}(f_\beta) = \emptyset$ for every distinct $\alpha, \beta \in \Xi$. In this case we are done choosing $\Lambda = \Xi$. Otherwise, we discard the elements $u_\alpha, \alpha \in \Xi$ from future consideration by assuming for simplicity of notation that $\Xi = \emptyset$. Consider now the set

$$W_1 = \bigcup_{\alpha=1}^{\Gamma} (U_\alpha \setminus \{u_\alpha\}).$$

It is clear that $W_1 \subset \{u_\alpha\}_{\alpha=1}^\Gamma$ has cardinality $|\Gamma|$. Moreover,

$$|\{u_\alpha \in W_1 \setminus \{u_\beta\} : \operatorname{supp}(f_\beta) \cap \operatorname{supp}(u_\alpha) \ne \emptyset\}| \le n - 1, \ \text{ whenever } \ u_\beta \in W_1.$$

Repeating the previous argument inductively at most n-times, we arrive at the finite sequence of sets $W_n \subset W_{n-1} \subset \cdots \subset W_1$ so that

$$|\{u_\alpha \in W_n \setminus \{u_\beta\} : \operatorname{supp}(f_\beta) \cap \operatorname{supp}(u_\alpha) \neq \emptyset\}| = 0, \quad \text{whenever } \ u_\beta \in W_n.$$

We have found the biorthogonal system by letting $\Lambda = W_n$.

Case 2. Suppose that $\operatorname{cof}(\Gamma) = \omega_0$. We partition the set $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$ where $\Gamma_n \nearrow \Gamma$ is an increasing sequence of uncountable regular cardinals [8, pp. 27, 40]. We reindex the original sequence $\{u_\alpha\}_{\alpha=1}^{\Gamma}$ as a collection $\{u_\alpha^n\}_{\alpha\in\Gamma_n}$, $n \in \mathbb{N}$. Since Γ_n are uncountable and regular, their cofinality $\operatorname{cof}(\Gamma_n)$ is larger than ω_0. By the previous Case 1 we may assume without loss of generality that $|\operatorname{supp}(f_\alpha^n)| = k_n$ is constant for $\alpha \in \Gamma_n$, and $\operatorname{supp}(f_\alpha^n) \cap \operatorname{supp}(u_\beta^n) = \emptyset$ for distinct $\alpha, \beta \in \Gamma_n$. Clearly, by removing a suitable subset of cardinality at most Γ_{m-1} from $\{u_\alpha^m\}_{\alpha\in\Gamma_m}$ we may also assume $\operatorname{supp}(f_\alpha^n) \cap \operatorname{supp}(u_\beta^m) = \emptyset$ for $\alpha \in \Gamma_n$, $\beta \in \Gamma_m$, $m > n$. It remains to deal with the case $m < n$. We will proceed by induction in n, with replacing the original index sets $\Gamma_1, \dots, \Gamma_n$ with suitable subsets of the same cardinality and such that the condition $\operatorname{supp}(f_\alpha^n) \cap \operatorname{supp}(u_\beta^m) = \emptyset$ for all $\alpha \in \Gamma_n, \beta \in \Gamma_m$, $m < n$ will be achieved.

Let us describe the general inductive step for n. We distinguish the following cases. Either there is $m < n$ and a some $u_\beta^m, \beta \in \Gamma_m$ such that the set

$$Q_\beta = \{\alpha \in \Gamma_n : \operatorname{supp}(u_\beta^m) \cap \operatorname{supp}(f_\alpha^n) \neq \emptyset\}$$

has cardinality Γ_n. In this case, remove β from Γ_m, and replace Γ_n by the set Q_β. For n still fixed, this can be repeated at most k_n-times and results in the relation $\operatorname{supp}(f_\alpha^n) \cap \operatorname{supp}(u_\beta^m) = \emptyset$ for all $\alpha \in \Gamma_n, \beta \in \Gamma_m$, $m < n$. Note that in this case we have removed at most k_n elements from the original set $\bigcup_{m<n}\{u_\beta^m\}_{\beta\in\Gamma_m}$, and the reduced set Γ_n has the same cardinality as the original one.

Alternatively, during one of the previous finitely many inductive steps, all Q_β have cardinality less than Γ_n. Then we replace Γ_n by $\Gamma_n \setminus \bigcup_{\beta\in\bigcup_{i=1}^{n-1}\Gamma_i} Q_\beta$, which is a set of cardinality Γ_n and leads again to the relation $\operatorname{supp}(f_\alpha^n) \cap \operatorname{supp}(u_\beta^m) = \emptyset$ for all $\alpha \in \Gamma_n, \beta \in \Gamma_m$, $m < n$. Clearly, an inductive step n affects the index sets $\Gamma_m, m < n$ by removing at most finitely many terms, and the cardinality of the reduced Γ_n remains the same. Hence, upon completing the whole induction in n, the final sets Γ_m will have the same cardinality as the original ones. This ends the argument in the case $\operatorname{cof}(\Gamma) = \omega_0$.

Once our system is biorthogonal the result for $c_0(\Gamma)$ follows easily. Indeed, whenever $a_i \in \mathbb{R}, \alpha_i \in \Lambda, i = 1, \dots, k$,

$$\max_{j\in\{1,\dots,k\}} f_{\alpha_j}\Big(\sum_{i=1}^{k} a_i u_{\alpha_i}\Big) \le (S-\varepsilon)\max_i |a_i| \le |||\sum_{i=1}^{k} a_i u_{\alpha_i}||| \le (S+\varepsilon) \max_{i\in\{1,\dots,k\}} |a_i|.$$

The argument for $X = \ell_1(\Gamma)$ is easier. Let Γ be an uncountable cardinal, and let $|||\cdot|||$ be an equivalent norm on $\ell_1(\Gamma)$. For $\Lambda \subset \Gamma$ denote

$$S_\Lambda = \inf\{|||x||| : x \in \ell_1(\Gamma), \operatorname{supp}(x) \subset \Lambda, \|x\| \le 1\}.$$

Let $S = \sup_{\Lambda\subset\Gamma,|\Lambda|=|\Gamma|} S_\Lambda$. Choose $\Lambda \subset \Gamma, |\Lambda| = |\Gamma|$ such that $S_\Lambda > S - \frac{\varepsilon}{4}$. For simplicity of notation, we may assume without loos of generality that $\Lambda = \Gamma$. This means, in particular, that $S \ge S_\Lambda \ge S - \frac{\varepsilon}{4}$, for every $\Lambda \subset \Gamma$. By a simple transfinite induction, choose a transfinite sequence $\{u_\alpha\}_{\alpha=1}^{\Gamma}$ of disjointly and finitely supported vectors from $\ell_1(\Gamma)$, $\|u_\alpha\| \le 1$, such that $S \ge |||u_\alpha||| \ge S - \frac{\varepsilon}{4}$. Indeed, once the initial segment $\{u_\alpha : 1 \leq \alpha < \Delta\}$ has been constructed for some $\Delta < \Gamma$, set $\Lambda = \Gamma \setminus \bigcup_{1\leq\alpha<\Delta} \operatorname{supp}(u_\alpha)$, and use the property $S \geq S_\Lambda \geq S - \frac{\varepsilon}{4}$ to find u_Δ. It is now easy to verify the property

$$S \geq ||| \sum a_i u_{\alpha_i} ||| \geq S - \frac{\varepsilon}{4}$$

whenever $\sum |a_i| = 1$. □

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