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FISHER-KOLMOGOROV TYPE PERTURBATIONS OF THE RELATIVISTIC OPERATOR: DIFFERENTIAL VS. DIFFERENCE

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Dedicated to Jean Mawhin for his 75th anniversary

Abstract. We are concerned with the existence of multiple periodic solutions for differential equations involving Fisher-Kolmogorov perturbations of the relativistic operator of the form

$$-\left[\phi(u')\right]' = \lambda u(1-|u|^q),$$

as well as for difference equations, of type

$$-\Delta\left[\phi(\Delta u(n-1))\right] = \lambda u(n)(1-|u(n)|^q);$$

here $q > 0$ is fixed, Δ is the forward difference operator, $\lambda > 0$ is a real parameter and

$$\phi(y) = \frac{y}{\sqrt{1-y^2}} \quad (y \in (-1,1)).$$

The approach is variational and relies on critical point theory for convex, lower semicontinuous perturbations of C^1-functionals.

1. Introduction

This paper is concerned with problems involving Fisher-Kolmogorov nonlinearities of the type:

$$-\left[\phi(u')\right]' = \lambda u(1-|u|^q), \quad u(0)-u(T) = 0 = u'(0)-u'(T), \tag{1.1}$$

respectively,

$$-\Delta\left[\phi(\Delta u(n-1))\right] = \lambda u(n)(1-|u(n)|^q), \quad u(n) = u(n+T) \quad (n \in \mathbb{Z}), \tag{1.2}$$

where $q > 0$ is fixed, $\Delta u(n) = u(n+1) - u(n)$ is the usual forward difference operator, $\lambda > 0$ is a real parameter and

$$\phi(y) = \frac{y}{\sqrt{1-y^2}} \quad (y \in (-1,1)).$$

Notice that besides the trivial solution, problems (1.1) and (1.2) always have the pair of constant solutions $u \equiv \pm 1$ and these are the only constant nontrivial solutions of (1.1) and (1.2). Here we are interested in the multiplicity of pairs of nonconstant solutions of (1.1) and (1.2).

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The typical example which involves the above type of nonlinearities was originally motivated by models in biological population dynamics and led to the scalar reaction-diffusion equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = u(1 - u^2),$$

referred to as *the classical Fisher-Kolmogorov* (FK) *equation* ([14], [15], [18]). In the last years interest has turned to higher-order equations of type

$$u^{iv} - pu'' = u(1 - u^2),$$

which corresponds, if $p > 0$, to *the extended Fisher-Kolmogorov* (EFK) *equations*; these are models for phase transitions and other bistable phenomena. In this direction we refer the reader to [11], [12], [21] - [24], [28] where existence of solutions was studied by a variety of methods such as topological shooting methods, phase-plane analysis and variational methods. Also, a difference equation related to the FK equation was considered in [1], [10].

A multiplicity result as the one in this paper (Theorem 2.1) was obtained in [11], [12], [28] for EFK equations. We notice also the paper [8], where a multiplicity result is given for periodic problems involving the discrete p-Laplacian operator.

On the other hand, in recent years special attention was paid to various qualitative aspects for boundary value problems involving the so-called *relativistic operator*: $u \mapsto [\phi(u')]'$. Among others and far from being exhausted, related to existence and multiplicity of periodic solutions for such problems, we refer the reader to [3] - [5], [7], [9], [16], respectively to [2], [17], [19], [20] for discrete versions.

It is the aim of this paper to obtain multiplicity of nonconstant solutions for problems (1.1) and (1.2). First, let us note that both of them can be seen as eigenvalue problems. In this view, we prove in Theorem 2.1 (resp. Theorem 3.1) that (1.1) (resp. (1.2)) has a prescribed number of distinct pairs of nonconstant solutions for large enough values of the parameter λ. On the other hand, for any $\lambda > 0$ fixed, we obtain that a prescribed number of distinct pairs of nonconstant solutions can be obtained for (1.1), provided that the period T is sufficiently large (Theorem 2.1). Our approach is a variational one and relies on a generalization of a result for smooth functionals due to Clark [13] to convex, lower semicontinuous perturbations of C^1-functionals.

Before concluding this introductory part, we briefly recall some topics in the frame of Szulkin's critical point theory [27], which will be needed in the sequel.

Let $(Y, \|\cdot\|)$ be a real Banach space and let $\mathcal{I} : Y \to (-\infty, +\infty]$ be a functional of the type

$$\mathcal{I} = \mathcal{F} + \psi, \tag{1.3}$$

where $\mathcal{F} \in C^1(Y, \mathbb{R})$ and $\psi : Y \to (-\infty, +\infty]$ is convex, lower semicontinuous and proper (i.e., $D(\psi) := \{u \in Y : \psi(u) < +\infty\} \neq \emptyset$). A point $u \in Y$ is said to be *a critical point* of $\mathcal{I}$ if $u \in D(\psi)$ and satisfies the inequality

$$\langle \mathcal{F}'(u), v - u\rangle + \psi(v) - \psi(u) \geq 0 \quad \forall\ v \in D(\psi).$$

A sequence $\{u_n\} \subset D(\psi)$ is called a (PS)-*sequence* if $\mathcal{I}(u_n) \to c \in \mathbb{R}$ and

$$\langle \mathcal{F}'(u_n), v - u_n\rangle + \psi(v) - \psi(u_n) \geq -\varepsilon_n \|v - u_n\| \quad \forall\ v \in D(\psi),$$

where $\varepsilon_n \to 0$. The functional $\mathcal{I}$ is said *to satisfy the* (PS) *condition* if any (PS)-sequence has a convergent subsequence in Y.

Let Σ be the collection of all symmetric subsets of $Y \setminus \{0\}$ which are closed in Y. A nonempty set $A \in \Sigma$ is said to have *genus* k (denoted $\gamma(A) = k$) if k is the smallest integer with the property that there exists an odd continuous mapping $h : A \to \mathbb{R}^k \setminus \{0\}$. If such an integer does not exist, $\gamma(A) = +\infty$. For properties and more details of the notion of genus we refer the reader to [25, 26]. Let $\Gamma \subset 2^Y$ be the collection of all nonempty compact symmetric subsets of Y, considered with the Hausdorff-Pompeiu distance and

$$\Gamma_j := cl\{A \in \Gamma : \ 0 \notin A, \ \gamma(A) \geq j\}$$

(cl is the closure in Γ). The following is a generalization of the result for smooth functions in [25, Theorem 5.19] to functionals of type (1.3) and it is proved in [27, Theorem 4.3].

Theorem 1.1. *Let $\mathcal{I}$ be of type* (1.3) *with $\mathcal{F}$ and ψ even. Also, suppose that $\mathcal{I}$ is bounded from below, satisfies the (PS) condition and $\mathcal{I}(0) = 0$. If*

$$c_m := \inf_{A \in \Gamma_m} \sup_{v \in A} \mathcal{I}(v) < 0,$$

then the functional $\mathcal{I}$ has at least m distinct pairs of nontrivial critical points.

2. The differential problem (1.1)

Using the ideas from [4], we introduce a variational formulation for problem (1.1). With this aim let $C := C[0,T]$ be endowed with the usual supremum norm $\|\cdot\|_\infty$ and $W^{1,\infty} := W^{1,\infty}(0,T)$. For each $v \in C$ we set $\overline{v} := \frac{1}{T}\int_0^T v(t)dt$ and we write $v(t) = \overline{v} + \tilde{v}(t)$, where $\overline{\tilde{v}} = 0$. If $v \in W^{1,\infty}$, then $\tilde{v}$ vanishes at some $t_0 \in (0,T)$ and so

$$|\tilde{v}(t)| = |\tilde{v}(t) - \tilde{v}(t_0)| \leq \int_0^T |v'(s)|ds \leq T\|v'\|_\infty.$$

Next, denoting

$$K := \{v \in W^{1,\infty} \ : \ \|v'\|_\infty \leq 1, \ v(0) = v(T)\},$$

it is clear that

$$\|\tilde{v}\|_\infty \leq T \quad \forall\, v \in K. \tag{2.1}$$

Also, it is not difficult to show that (see [4, Lemma 4])

$$|v(t)|^p \geq |\overline{v}|^p - pT|\overline{v}|^{p-1} \quad \forall\, v \in K, \ \forall\, t \in [0,T] \text{ and } p \geq 1. \tag{2.2}$$

From [6] we know that the even functional $\Psi : C \to (-\infty, +\infty]$,

$$\Psi(v) = \begin{cases} \displaystyle\int_0^T [1 - \sqrt{1 - v'^2}] & \text{if } v \in K, \\ +\infty & \text{otherwise,} \end{cases}$$

is proper, convex and lower semicontinuous on C and it is easy to see that

$$\Psi(v) \leq \int_0^T |v'|^2 \quad \forall\, v \in K. \tag{2.3}$$

Next, let $\mathcal{G}_\lambda : C \to \mathbb{R}$ be defined by

$$\mathcal{G}_\lambda(u) = \lambda \int_0^T \left[\frac{|u|^{q+2}}{q+2} - \frac{u^2}{2}\right].$$

Notice that $\mathcal{G}_\lambda$ is even, of class C^1 on C and its derivative is given by

$$\langle \mathcal{G}'_\lambda(u), v\rangle = \lambda \int_0^T (|u|^q - 1)uv, \quad u, v \in C.$$

Then the energy functional $I_\lambda : C \to (-\infty, +\infty]$ associated to problem (1.1) is given by

$$I_\lambda = \Psi + \mathcal{G}_\lambda$$

and it has the structure required by Szulkin's critical point theory.

Recall, by *a solution* of (1.1) we mean a function $u \in C^1[0, T]$, with $\|u'\|_\infty < 1$ and $\phi(u')$ differentiable, which satisfies (1.1).

From Proposition 2 in [4], one has the following:

Proposition 2.1. *If $u \in K$ is a critical point of I_λ, then u is a solution of problem* (1.1).

Lemma 2.1. *I_λ is bounded from below and satisfies the (PS) condition.*

Proof. Let $u \in K = D(\Psi)$. From (2.1) we have

$$\int_0^T \frac{u^2}{2} = \int_0^T \frac{(\overline{u} + \tilde{u})^2}{2} \le \frac{T^3}{2} + \frac{T}{2}|\overline{u}|^2.$$

Also, on account of (2.2) we obtain

$$\mathcal{G}_\lambda(u) \ge \frac{\lambda T}{q+2}|\overline{u}|^{q+2} - \lambda T^2 |\overline{u}|^{q+1} - \lambda \int_0^T \frac{u^2}{2}.$$

It follows

$$I_\lambda(u) \ge \mathcal{G}_\lambda(u) \ge \frac{\lambda T}{q+2}|\overline{u}|^{q+2} - \lambda T^2|\overline{u}|^{q+1} - \frac{\lambda T}{2}|\overline{u}|^2 - \frac{\lambda T^3}{2} \tag{2.4}$$

which clearly shows that I_λ is bounded from below. To see that I_λ satisfies the (PS) condition, let $\{u_n\} \subset K = D(\Psi)$ be a (PS)-sequence. We write (2.4) with u_n instead of u and, from the fact that $\{I_\lambda(u_n)\}$ is bounded, we get that $\{\overline{u}_n\}$ is bounded. Then, Lemma 3 ii) in [4] ensures that $\{u_n\}$ has a convergent subsequence in C. □

Theorem 2.1. *If $\lambda > 4\pi^2 m^3/T^2$ for some $m \in \mathbb{N}$, $m \ge 2$, then problem* (1.1) *has at least $m - 1$ distinct pairs of nonconstant solutions.*

Proof. Using that $u = \pm 1$ is the only pair of nontrivial constant solutions for (1.1), it suffices to prove that (1.1) has at least m distinct pairs of nontrivial solutions. From Theorem 1.1, Proposition 2.1 and Lemma 2.1 this can be reduced to showing that there is some $A_m \in \Gamma_m \subset 2^C$ such that

$$\sup_{v \in A_m} I_\lambda(v) < 0. \tag{2.5}$$

With this aim, we consider the finite dimensional space

$$X_m := span\left\{\sin\frac{\pi x}{T}, \sin\frac{2\pi x}{T}, \ldots, \sin\frac{m\pi x}{T}\right\}$$

equipped with the norm

$$\left\|\alpha_1 \sin\frac{\pi x}{T} + \cdots + \alpha_m \sin\frac{m\pi x}{T}\right\|_{X_m}^2 = \alpha_1^2 + \cdots + \alpha_m^2.$$

Since the norms $\|\cdot\|_{X_m}$ and $\|\cdot\|_{L^{q+2}}$ are equivalent on X_m, there exists a positive constant $c(m)$ such that

$$\|v\|_{L^{q+2}} \le c(m)\|v\|_{X_m}. \tag{2.6}$$

Next, as in e.g. [24], [28], we introduce the subset A_m of C by

$$A_m = \left\{ \sum_{k=1}^{m} \alpha_k \sin\frac{k\pi x}{T} \;:\; \alpha_1^2 + \cdots + \alpha_m^2 = \rho^2 \right\} \;(\subset X_m),$$

where, since $\lambda > 4\pi^2 m^3/T^2$, the positive number ρ can be chosen $\le 2/\sqrt{\lambda}$ and such that

$$\frac{m^3\pi^2}{T} - \frac{\lambda T}{4} + \frac{\lambda (c(m))^{q+2}}{q+2}\rho^q < 0.$$

It is easy to see that the odd mapping $H : A_m \to S^{m-1}$ ($m-1$ dimension unit sphere in the Euclidean space $\mathbb{R}^m$) defined by

$$H\left(\sum_{k=1}^{m} \alpha_k \sin\frac{k\pi x}{T}\right) = \left(\frac{\alpha_1}{\rho}, \ldots, \frac{\alpha_m}{\rho}\right)$$

is a homeomorphism between A_m and S^{m-1}. According to [26, Corrolary 5.5], $\gamma(A_m) = m$ and so, $A_m \in \Gamma_m$.

Let $v \in A_m$. Clearly, $v(0) = v(T)$ and we have

$$\begin{aligned} |v'| &\le \sum_{k=1}^{m} \left|\alpha_k \frac{k\pi}{T}\cos\frac{k\pi x}{T}\right| \le \frac{m\pi}{T}\sum_{k=1}^{m}|\alpha_k| \\ &\le \frac{m^{3/2}\pi}{T}\left(\sum_{k=1}^{m}\alpha_k^2\right)^{1/2} = \frac{m^{3/2}\pi}{T}\rho. \end{aligned} \tag{2.7}$$

Therefore, as $T > 2\pi m\sqrt{m/\lambda} \ge \pi m^{3/2}\rho$, one has $\|v'\|_\infty < 1$, meaning that $v \in K$. On the other hand, we compute

$$\begin{aligned} \int_0^T v^2 &= \int_0^T \left(\sum_{k=1}^{m}\alpha_k \sin\frac{k\pi x}{T}\right)^2 = \sum_{k=1}^{m}\alpha_k^2 \int_0^T \sin^2\frac{k\pi x}{T} \\ &= \frac{1}{2}\sum_{k=1}^{m}\alpha_k^2 \int_0^T \left(1 - \cos\frac{2k\pi x}{T}\right) = \frac{T}{2}\rho^2. \end{aligned} \tag{2.8}$$

Then, using (2.3), (2.6) - (2.8), we estimate I_λ as follows:

$$\begin{aligned} I_\lambda(v) &= \Psi(v) + \lambda\int_0^T \frac{|v|^{q+2}}{q+2} - \lambda\int_0^T \frac{v^2}{2} \le \int_0^T |v'|^2 + \frac{\lambda (c(m)\rho)^{q+2}}{q+2} - \frac{\lambda T}{4}\rho^2 \\ &\le \rho^2\left[\frac{m^3\pi^2}{T} - \frac{\lambda T}{4} + \frac{\lambda(c(m))^{q+2}\rho^q}{q+2}\right] \;(< 0 - \text{ from the choice of } \rho), \end{aligned}$$

which shows that (2.5) holds true. □

3. The difference problem (1.2)

Analogously to the previous section, we first give the variational formulation for problem (1.2). Let H_T be the space of all *T-periodic $\mathbb{Z}$-sequences* in $\mathbb{R}$, i.e., of mappings $u : \mathbb{Z} \to \mathbb{R}$ such that $u(n) = u(n+T)$ for all $n \in \mathbb{Z}$. On H_T we shall refer to the following two (equivalent) norms:

$$\|u\| = \left(\sum_{j=1}^{T} |u(j)|^2 \right)^{1/2} \quad \text{and} \quad \|u\|_{q+2} = \left(\sum_{j=1}^{T} |u(j)|^{q+2} \right)^{\frac{1}{q+2}}.$$

Also, for each $u \in H_T$ we set

$$\overline{u} := \frac{1}{T} \sum_{j=1}^{T} u(j), \quad \tilde{u} := u - \overline{u}.$$

Let the closed convex subset $\mathbf{K}$ of H_T be defined by

$$\mathbf{K} := \{u \in H_T : \ |\Delta u|_\infty \le 1\},$$

where $|\Delta u|_\infty := \max_{j=1,\dots,T} |\Delta u(j)|$. We introduce the (even) functions

$$\mathbf{\Psi}(u) = \begin{cases} \displaystyle\sum_{j=1}^{T} \Phi[\Delta u(j)] & \text{if } u \in \mathbf{K}, \\ +\infty & \text{otherwise}, \end{cases}$$

where $\Phi(y) = 1 - \sqrt{1-y^2}$ $(y \in [-1,1])$, respectively

$$\mathbf{G}_\lambda(u) = \lambda \sum_{j=1}^{T} \left[\frac{|u(j)|^{q+2}}{q+2} - \frac{u(j)^2}{2} \right].$$

Then the functional $\mathbf{I}_\lambda : H_T \to (-\infty, +\infty]$ associated to problem (1.2) will be given by

$$\mathbf{I}_\lambda = \mathbf{\Psi} + \mathbf{G}_\lambda$$

and it is not difficult to see that it has the structure required by Szulkin's critical point theory, the derivative of $\mathbf{G}_\lambda$ being given by

$$\langle \mathbf{G}_\lambda'(u), v \rangle = \lambda \sum_{j=1}^{T} (|u(j)|^q - 1) u(j) v(j), \quad (u, v \in H_T).$$

A solution of problem (1.2) is an element $u \in H_T$ such that $|\Delta u(n)| < 1$, for all $n \in \mathbb{Z}$, which satisfies the equation in (1.2).

Proposition 3.1. *Any critical point of* $\mathbf{I}_\lambda$ *is a solution of problem* (1.2).

Proof. For any $e \in H_T$, on account of Lemmas 5 and 6 in [19], problem

$$\Delta \left[\phi(\Delta u(n-1)) \right] = \overline{u} + e(n), \quad u(n) = u(n+T) \quad (n \in \mathbb{Z}) \tag{3.1}$$

has a unique solution u_e, which is also a solution of the variational inequality

$$\sum_{j=1}^{T} \{ \Phi[\Delta v(j)] - \Phi[\Delta u(j)] + \overline{u}(\overline{v} - \overline{u}) + e(j)(v(j) - u(j)) \} \ge 0 \ \forall \ v \in \mathbf{K}. \tag{3.2}$$

We show that u_e is actually the unique solution of (3.2). With this aim, let $J : \mathbf{K} \to \mathbb{R}$ be defined by

$$J(u) = \sum_{j=1}^{T} \left\{ \Phi[\Delta u(j)] + \frac{\overline{u}^2}{2} + e(j)u(j) \right\}.$$

If u is a solution of (3.2), then the inequality $\frac{\overline{v}^2}{2} - \frac{\overline{u}^2}{2} \geq \overline{u}(\overline{v} - \overline{u})$ plugged in (3.2) implies that

$$\sum_{j=1}^{T} \left\{ \Phi[\Delta v(j)] - \Phi[\Delta u(j)] + \frac{\overline{v}^2}{2} + e(j)v(j) - \frac{\overline{u}^2}{2} - e(j)u(j) \right\} \geq 0 \ \forall \ v \in \mathbf{K},$$

which shows that J has a minimum at u. Then the uniqueness of u_e as a solution of (3.2) follows by the strict convexity of J.

Next, let w be a critical point of $\mathbf{I}_\lambda$. Then, for all $v \in \mathbf{K}$, one has

$$\sum_{j=1}^{T} \{\Phi[\Delta v(j)] - \Phi[\Delta w(j)] + \lambda\left(|w(j)|^q - 1\right) w(j)(v(j) - w(j))\} \geq 0,$$

which can be written

$$\sum_{j=1}^{T} \{\Phi[\Delta v(j)] - \Phi[\Delta w(j)] + \overline{w}(v(j) - w(j))\}$$

$$+ \sum_{j=1}^{T} \left[\lambda\left(|w(j)|^q - 1\right) w(j) - \overline{w}\right] (v(j) - w(j)) \geq 0,$$

for all $v \in \mathbf{K}$. Hence, w is a solution of the variational inequality

$$\sum_{j=1}^{T} \{\Phi[\Delta v(j)] - \Phi[\Delta w(j)] + \overline{w}(\overline{v} - \overline{w}) + e_w(j)(v(j) - w(j))\} \geq 0 \quad \forall \ v \in \mathbf{K}, \tag{3.3}$$

with $e_w \in H_T$ given by $e_w(n) = \lambda\left(|w(n)|^q - 1\right) w(n) - \overline{w}$ $(n \in \mathbb{Z})$.

Therefore, by (3.3) and the uniqueness of the solution of (3.2), one can see that actually w solves problem (1.2). □

Lemma 3.1. $\mathbf{I}_\lambda$ *is bounded from below and satisfies the (PS) condition.*

Proof. Let $u \in \mathbf{K} = D(\mathbf{\Psi})$. By the equivalence of the norms in H_T, there exists a positive constant C_1 such that

$$\|u\|_{q+2}^{q+2} \geq C_1 \|u\|^{q+2}.$$

Then, we have

$$\mathbf{I}_\lambda(u) \geq \mathbf{G}_\lambda(u) \geq \frac{\lambda C_1}{q+2}\|u\|^{q+2} - \frac{\lambda}{2}\|u\|^2 \tag{3.4}$$

which clearly shows that $\mathbf{I}_\lambda$ is bounded from below. Now, if $\{u_k\}$ is a sequence in $\mathbf{K}$ such that $\{\mathbf{I}_\lambda(u_k)\}$ is bounded, then one has from (3.4) that $\{u_k\}$ is bounded in H_T and hence it contains a convergent subsequence.

Theorem 3.1. *If* $\lambda > 8mT$ *for some* $m \in \mathbb{N}$ *with* $2 \leq m \leq T$, *then problem* (1.2) *has at least* $m - 1$ *distinct pairs of nonconstant solutions.*

Proof. Similar to the proof of Theorem 2.1, since $u = \pm 1$ is the only pair of nontrivial constant solutions for (1.2), it is sufficient to show that (1.2) has at least m distinct pairs of nontrivial solutions. By virtue of Theorem 1.1, Proposition 3.1 and Lemma 3.1 we have to prove that there is some $A_m \in \Gamma_m \subset 2^{H_T}$ such that

$$\sup_{v \in A_m} \mathbf{I}_\lambda(v) < 0. \tag{3.5}$$

For this, let $w^1, w^2, \ldots, w^T$ be an orthonormal basis in the space H_T endowed with the Euclidean norm $\|\cdot\|$. We consider the set

$$A_m := \left\{ \sum_{k=1}^m \alpha_k w^k : \ \alpha_1^2 + \cdots + \alpha_m^2 = \rho^2 \right\}$$

where, since $\lambda > 8mT$, the positive number ρ can be chosen $\leq 1/(2\sqrt{m})$ and such that

$$4mT - \frac{\lambda}{2} + \frac{\lambda T \sqrt{m^{q+2}}}{q+2} \rho^q < 0.$$

Since the mapping $\mathbf{H} : A_m \to S^{m-1}$ defined by

$$\mathbf{H}\left(\sum_{k=1}^m \alpha_k w^k \right) = \left(\frac{\alpha_1}{\rho}, \ldots, \frac{\alpha_m}{\rho} \right)$$

is an odd homeomorphism between A_m and S^{m-1}, then we have $\gamma(A_m) = m$. Hence, $A_m \in \Gamma_m$.

Now, let $v = \sum_{k=1}^m \alpha_k w^k \in A_m$. Then, for each $j = 1, \ldots, T$, we obtain

$$\begin{aligned} |\Delta v(j)| &\leq \sum_{k=1}^m \left|\alpha_k w^k(j+1)\right| + \sum_{k=1}^m \left|\alpha_k w^k(j)\right| \leq 2 \sum_{k=1}^m |\alpha_k| \\ &\leq 2\sqrt{m} \left(\sum_{k=1}^m \alpha_k^2 \right)^{1/2} = 2\rho\sqrt{m} \end{aligned} \tag{3.6}$$

and since $\rho \leq 1/(2\sqrt{m})$, one has $|\Delta v|_\infty \leq 1$, which shows that $v \in \mathbf{K}$. On the other hand, we have

$$\sum_{j=1}^T v(j)^2 = \|v\|^2 = \sum_{k=1}^m \alpha_k^2 = \rho^2 \tag{3.7}$$

and

$$\begin{aligned} \sum_{j=1}^T |v(j)|^{q+2} &\leq \sum_{j=1}^T \left(\sum_{k=1}^m |\alpha_k| |w^k(j)| \right)^{q+2} \leq T \left(\sum_{k=1}^m |\alpha_k| \|w^k\| \right)^{q+2} \\ &= T \left(\sum_{k=1}^m |\alpha_k| \right)^{q+2} \leq T \left(\rho\sqrt{m} \right)^{q+2}. \end{aligned} \tag{3.8}$$

Then, using (3.6) - (3.8), it follows

$$\begin{aligned}\mathbf{I}_\lambda(v) &= \boldsymbol{\Psi}(v) + \frac{\lambda}{q+2}\sum_{j=1}^{T}|v(j)|^{q+2} - \frac{\lambda}{2}\sum_{j=1}^{T}v(j)^2 \\ &\le \sum_{j=1}^{T}|\Delta v(j)|^2 + \frac{\lambda T\left(\rho\sqrt{m}\right)^{q+2}}{q+2} - \frac{\lambda\rho^2}{2} \\ &\le \rho^2\left[4mT - \frac{\lambda}{2} + \frac{\lambda T\sqrt{m^{q+2}}}{q+2}\rho^q\right] \ (< 0 - \text{ from the choice of } \rho).\end{aligned}$$

Therefore, (3.5) holds true and the proof is complete. □

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