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RIESZ BASES OF EXPONENTIALS ON UNBOUNDED MULTI-TILES

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Abstract. We prove the existence of Riesz bases of exponentials of $L^2(\Omega)$, provided that $\Omega \subset \mathbb{R}^d$ is a measurable set of finite and positive measure, not necessarily bounded, that satisfies a multi-tiling condition and an arithmetic property that we call *admissibility.* This property is satisfied for any bounded domain, so our results extend the known case of bounded multi-tiles. We also extend known results for submulti-tiles and frames of exponentials to the unbounded case.

1. Introduction

The main goal of this paper is to study the existence of Riesz basis of exponentials in $L^2(\Omega)$ for domains $\Omega \subset \mathbb{R}^d$ of finite and positive measure, not necessarily bounded.

The existence of bases of exponentials is a very well-studied problem. For orthonormal bases, the question of existence is related to the famous Fuglede's conjecture [9] (also known as the spectral set conjecture). It states that if Ω is a domain of positive and finite measure, an orthogonal basis of exponentials $\left\{e^{2\pi i\gamma.\omega} : \gamma \in \Gamma\right\}$ for $L^2(\Omega)$ exists if and only if the set Ω tiles $\mathbb{R}^d$ by translations along some discrete set Λ. This latter means that

$$\sum_{\lambda\in\Lambda} \chi_\Omega(\omega+\lambda) = 1, \quad \text{a.e. } \omega \in \mathbb{R}^d.$$

Fuglede's conjecture is false in dimensions greater than or equal to 3, and it is open for $d = 1, 2$. (See [7,8,17,23].) However, it has been proved for a great number of special cases. For example, it is always true for lattices [9] (see also [12]). That is, if H is a full lattice in $\mathbb{R}^d$, the system $\left\{e^{2\pi ih.\omega} : h \in H\right\}$ is an orthogonal basis of $L^2(\Omega)$ if and only if Ω tiles $\mathbb{R}^d$ with translations by Λ, the dual lattice of H. It is also true for convex bodies [13].

On the other hand, it has also been proved that there are sets Ω that do not possess an orthonormal basis of exponentials, as it is the case, for example, of the unit ball of $\mathbb{R}^d$ when $d > 1$ and the case of non-symmetric convex bodies [18]. Since orthogonality imposes a very severe restriction, it is natural to look at Riesz bases instead.

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The system $\{e^{2\pi i\gamma.\omega} : \gamma \in \Gamma\}$ is a **Riesz basis** of $L^2(\Omega)$ if it is complete and satisfies that

$$A\sum_{\gamma\in\Gamma} |\, c_\gamma\,|^2 \le \big\|\sum_{\gamma\in\Gamma} c_\gamma e^{2\pi i\gamma.\omega}\big\|^2 \le B\sum_{\gamma\in\Gamma} |\, c_\gamma\,|^2 \quad \forall\{c_\gamma\}\in\ell^2(\Gamma),$$

for some positive constants $A, B > 0$.

The more general problem of the existence of Riesz bases of exponentials is of a different nature and brings new challenges. Again the relevant question here is which domains Ω admit a Riesz basis of exponentials, and which discrete sets Γ give rise to Riesz basis of exponentials for some domain. There are few cases of sets where it is possible to prove the existence of such bases. However, as far as we know, there is no example of a set Ω of finite measure (even in the line) that do not support a basis of this type.

One of the reasons that makes the problem significant and relevant is that the existence of a Riesz basis of exponentials for a set Ω is equivalent to the existence of a set of stable sampling and interpolation for the associated Paley-Wiener space PW_Ω (see, for example, [22, 25]).

Recently, G. Kozma and S. Nitzan made a significative advance for this problem. They proved that any finite union of rectangles in $\mathbb{R}^d$ admits a Riesz basis of exponentials [14, 15].

Morover, S. Grepstad and N. Lev [10], discovered that bounded measurable sets $\Omega \subset \mathbb{R}^d$ that satisfy a multi-tiling condition, support a Riesz basis of exponentials. The proof uses the theory of quasi-crystals developed in [19, 20], and requires the condition that the boundary of the domain Ω has Lebesgue measure zero. Later on, Kolountzakis [16] found a much simpler proof and was able to remove the zero measure boundary condition. More precisely, they proved that if a bounded measurable set k-tiles $\mathbb{R}^d$ by translations on a lattice Λ (see Definition 2.3), then there exist vectors $a_1, \dots, a_k \in \mathbb{R}^d$ such that $E(H, a_1, \dots, a_k)$ is a Riesz basis of $L^2(\Omega)$. Here,

$$E(H\,;\, a_1, \dots, a_k) := \{e^{2\pi i(a_j+h).\omega} \,:\, h \in H,\, j = 1,\, \dots, k\}, \tag{1.1}$$

where H is the dual lattice of Λ. That is, bounded multi-tile sets with respect to a lattice, always support a basis of exponentials with the set of frequencies being a finite union of translations on the dual lattice.

This result was extended in [1] to locally compact abelian groups. They used fiberization techniques from the theory of shift-invariant spaces [6]. They also proved, in this general setting, a converse of this result. That is, if a set $\Omega \subset \mathbb{R}^d$ is such that there exist a lattice H and vectors $a_1, \dots, a_k \in \mathbb{R}^d$ with $E(H, a_1, \dots, a_k)$ a Riesz basis of $L^2(\Omega)$, then Ω must multi-tile $\mathbb{R}^d$ at level k for Λ, the dual lattice of H. This can be seen as an extension of Fuglede's Theorem for lattices, for the case of multi-tiles and Riesz bases.

A natural question raised by Kolountzakis in [16] was if this result was still valid for *unbounded* multi-tile sets of finite measure. In [1] the authors answered this question in the negative. They constructed a counterexample of an unbounded multi-tile set of level 2 in the line, that does not possess a Riesz basis of exponentials with the special structure (1.1).

In this paper, we prove that *unbounded* multi-tile sets of $\mathbb{R}^d$ of finite measure do support a Riesz basis of exponentials if they satisfy an extra arithmetic condition that we call *admissibility* (see Definition 2.4 for a precise definition). Our main result is:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^d$ be a measurable set such that $0 < |\Omega| < +\infty$ and let $\Lambda \subset \mathbb{R}^d$ be a full lattice. If*

(i) *Ω multi-tiles $\mathbb{R}^d$ at level k by translations on Λ,*

(ii) *Ω is admissible for Λ,*

then there exists $a_1, \dots, a_k \in \mathbb{R}^d$ such that the set $E(H\,;\, a_1, \dots, a_k)$ is a Riesz basis of $L^2(\Omega)$.

In the last section we apply our results to obtain relationships between submulti-tiles (see Definition 4.1) and frames of exponentials.

The paper is organized as follows: In Section 2 we set the notation and introduce the definition of admissibility. We also review the results from the theory of shift-invariant spaces that we will need later. Section 3 is devoted to the proofs of our results on multi-tiles and the existence of Riesz bases of exponentials. Finally, in Section 4 we explore the relation between submulti-tiles and frames of exponentials.

2. Preliminaries

Let $\Lambda \subset \mathbb{R}^d$ be a **full lattice**. This means that there is a $d \times d$ invertible matrix M such that $\Lambda = M\mathbb{Z}^d$. Recall that the **fundamental domain** with respect to the lattice Λ is the set $D = M\mathbb{T}^d$, which is a set of representatives of the quotient $\mathbb{R}^d/\Lambda$.

Let $H \subset \mathbb{R}^d$ be the **dual lattice** of Λ. This is the set

$$H = \{\, h \in \mathbb{R}^d \,:\, \langle h, \lambda \rangle \in \mathbb{Z} \text{ for all } \lambda \in \Lambda \}.$$

It is easy to see that $H = (M^t)^{-1}\mathbb{Z}^d$.

From now on, when working with a lattice Λ, we will always denote by D its fundamental domain and by H its dual lattice. For notational simplicity, we will denote by e_α the function $e_\alpha(\omega) = e^{2\pi i \alpha \cdot \omega}$, $\alpha, \omega \in \mathbb{R}^d$, and $\#A$ will be the cardinal of the set A.

We will also need the following definition.

Definition 2.1. We will say that a system of exponentials is **structured** if it is of the form $E(H, a_1, \dots, a_k)$ as in (1.1) with $H \subset \mathbb{R}^d$ a lattice and $a_1, \dots, a_k \in \mathbb{R}^d$.

2.1. **Multi-tiles.**

Hereafter, given a set $\Omega \subset \mathbb{R}^d$ and a lattice $\Lambda \subset \mathbb{R}^d$, for every $\omega \in D$ we will denote $\Lambda_\omega(\Omega) = \Lambda_\omega := \{\, \lambda \in \Lambda \,:\, \omega + \lambda \in \Omega \,\}$.

Remark 2.2. Observe that if $\Omega \subset \mathbb{R}^d$ is a measurable set of finite measure, then Λ_ω must be finite for almost every $\omega \in D$. This is because

$$\int_D \sum_{\lambda \in \Lambda} \chi_\Omega(\omega + \lambda)\, d\omega = \int_{\mathbb{R}^d} \chi_\Omega(\omega)\, d\omega = |\Omega| < +\infty.$$

Definition 2.3. Let k be a positive integer. We say that a measurable set $\Omega \subset \mathbb{R}^d$ **multi-tiles $\mathbb{R}^d$ at level** k by translations on a lattice Λ (or that Ω k**-tiles** $\mathbb{R}^d$) if for almost every $\omega \in D$,

$$\sum_{\lambda \in \Lambda} \chi_\Omega(\omega + \lambda) = k.$$

Notice that if Ω is a k-tile by translations on Λ, then $\#\Lambda_\omega = k$ for almost every $\omega \in D$.

2.2. **Admissible sets.** In this subsection we introduce the concept of admissible sets.

Definition 2.4. Let $\Omega \subset \mathbb{R}^d$ be a finite measure set and let Λ be a full lattice in $\mathbb{R}^d$. We will say that Ω is **admissible for** Λ if there exist a vector $v \in H$ and a number $n \in \mathbb{N}$, such that for almost every $\omega \in D$, the numbers $\{\langle v, \lambda \rangle : \lambda \in \Lambda_\omega\}$ are distinct elements (mod n). We will also say in that case that Ω is (n, v)-admissible for Λ, if we want to emphasize the dependance on n and v.

When $d = 1$ and $\Lambda = \mathbb{Z}$, this is equivalent to saying that for almost every $\omega \in D$, the elements of $\Lambda_\omega \subset \mathbb{Z}$ are all distinct (mod n).

A graphical way to describe admissibility is the following: Let Ω be admissible with respect to Λ for some $n \in \mathbb{N}$ and some vector $v \in H$. Assume that we pick a different color for each of the elements of $\mathbb{Z}_n$, and we colored $\mathbb{R}^d$ painting the set $D + \lambda$ with the color assigned to the remainder (mod n) of $\langle v, \lambda \rangle$. Then the admissibility says that for almost all $\omega \in D$ the elements of the form $\omega + \lambda$, with $\lambda \in \Lambda$, that belongs to Ω have different colors!

Remark 2.5. Every bounded set $\Omega \subset \mathbb{R}^d$ is admissible. This is because, in this case, the set $\bigcup_{\omega \in D} \Lambda_\omega$ must be finite, so for any $v \in H$, one can just choose a number $n \in \mathbb{N}$ large enough for which all the numbers of $\{\langle v, \lambda \rangle : \lambda \in \Lambda_\omega\}$ are all distinct (mod n).

The following example shows that there exist multi-tiles that are not admissible.

Example 2.6. Consider the partition of $[0, 1)$ in intervals $I_j := \left[\frac{2^j - 2}{2^j}, \frac{2^j - 1}{2^j}\right)$, $j \geq 1$. The set

$$\Omega = [0, 1) \cup \bigcup_{j=1}^{\infty} (I_j + j)$$

is an unbounded subset of $\mathbb{R}$ that 2-tiles by translations on $\mathbb{Z}$ and which is not admissible for $\mathbb{Z}$ (See Figure 1). In order to see that the admissibility fails, note that if n is any fixed natural number and $\omega \in I_n$, then $\Lambda_\omega = \{0, n\}$, which are not distinct (mod n). This example is also interesting as this set does not admit a *structured* Riesz basis of exponentials for any lattice; see [1] for more details.

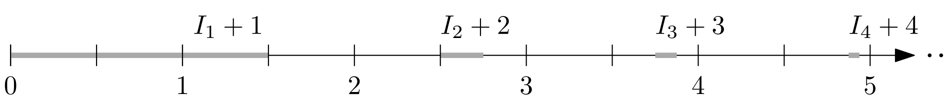

FIGURE 1. The set Ω.

On the other hand, unbounded admissible multi-tiles do exist:

Example 2.7. If in the previous example one translate the intervals I_j only by odd numbers, then

$$\Omega = [0, 1) \cup \bigcup_{j=1}^{\infty} (I_j + 2j + 1)$$

is a 2-tile unbounded set of $\mathbb{R}$ by translations on $\mathbb{Z}$ that is admissible for $\mathbb{Z}$ taking $n = 2$.

2.3. **Shift-invariant spaces.** For the proof of our results we will need to recall some facts from the theory of shift-invariant spaces. The reader is referred to [3] and [6], where a more complete treatment of the general case of shift-invariant spaces in locally compact abelian groups can be found.

Definition 2.8. We say that a closed subspace $V \subset L^2(\mathbb{R}^d)$ is H**-invariant** if

$$f \in V, \quad \text{then} \quad \tau_h f \in V \ \ \forall h \in H,$$

where $\tau_h f(x) = f(x-h)$.

Paley-Wiener spaces are a family of shift-invariant spaces in which we are especially interested. These spaces are defined by

$$PW_\Omega = \{ \, f \in L^2(\mathbb{R}^d) \, : \, \widehat{f} \in L^2(\Omega) \, \},$$

where $\Omega \subset \mathbb{R}^d$ is a measurable set of finite measure. It is easy to see that, in fact, they are invariant by any translation.

An essential tool in the development of shift-invariant theory is the technique known as **fiberization** that we will introduce now.

Proposition 2.9. *The map* $\mathcal{T} : L^2(\mathbb{R}^d) \to L^2(D, \ell^2(\Lambda))$ *defined by*

$$\mathcal{T} f(\omega) = \{\hat{f}(\omega + \lambda)\}_{\lambda \in \Lambda}$$

is an isometric isomorphism.

The evaluation of elements of $L^2(\mathbb{R}^d)$ could not make sense a priori, however, $\mathcal{T}$ is a well-defined mapping by virtue of the next remark.

Remark 2.10. If $\hat{f}$ and $\hat{g}$ are equal almost everywhere, then for almost every $\omega \in D$,

$$\{\hat{f}(\omega + \lambda)\}_{\lambda \in \Lambda} = \{\hat{g}(\omega + \lambda)\}_{\lambda \in \Lambda}.$$

In [11], Helson proved the existence of measurable range functions of an H-invariant space $V \subset L^2(\mathbb{R}^d)$. A **range function** is a mapping

$$\begin{aligned} J_V : D &\to \{ \, \text{closed subspaces of } \ell^2(\Lambda) \, \} \\ \omega &\mapsto J_V(\omega), \end{aligned}$$

which has the property that $f \in V$ if and only if for almost every $\omega \in D$,

$$\mathcal{T} f(\omega) \in J_V(\omega).$$

Furthermore, if $V = \overline{\operatorname{span}}\{ \, \tau_h f \ : \ h \in H, \, f \in \mathcal{A} \, \}$ for some countable set $\mathcal{A} \subset L^2(\mathbb{R}^d)$, then

$$J_V(\omega) = \overline{\operatorname{span}}\{ \, \mathcal{T} f(\omega) \, : \, f \in \mathcal{A} \, \}.$$

We say J_V is measurable in the following sense: for every $v, w \in \ell^2(\Lambda)$, the scalar function $\omega \mapsto \langle P_{J_V(\omega)} v, w \rangle$ is measurable, where $P_{J_V(\omega)}$ is the orthogonal projection onto $J_V(\omega)$. Moreover, a measurable range function of V is essentially unique, i.e., if V has two measurable range functions J_V and J_V', then $J_V = J_V'$ for almost every $\omega \in D$.

Another remarkable result regarding range functions is the characterization of frames and Riesz bases of a shift-invariant space V in terms of the properties of fibers.

Theorem 2.11. *Let $\mathcal{A} \subset L^2(\mathbb{R}^d)$ be a countable set. Then,*

(i) *the system $\{\tau_h f : h \in H,\ f \in \mathcal{A}\}$ is a frame of V with constants $A, B > 0$ if and only if $\{\mathcal{T}f(\omega) : f \in \mathcal{A}\} \subset \ell^2(\Lambda)$ is a frame of $J_V(\omega)$ with constants $A, B > 0$ for almost every $\omega \in D$,*

(ii) *the system $\{\tau_h f : h \in H,\ f \in \mathcal{A}\}$ is a Riesz basis of V with constants $A, B > 0$ if and only if $\{\mathcal{T}f(\omega) : f \in \mathcal{A}\} \subset \ell^2(\Lambda)$ is a Riesz basis of $J_V(\omega)$ with constants $A, B > 0$ for almost every $\omega \in D$.*

In particular, when $\mathcal{A} \subset L^2(\mathbb{R}^d)$ is a finite set, this allows us to translate problems in infinite dimensional H-invariant spaces, into problems of finite dimension that can be treated with linear algebra.

When working with the shift-invariant space $V = PW_\Omega$, we denote its range function as J_Ω. Considering Remark 2.2, we are able to characterize J_Ω as follows.

Proposition 2.12. *Let $\Omega \subset \mathbb{R}^d$ be a measurable set of finite measure. Then for almost every $\omega \in D$ we have*

$$J_\Omega(\omega) \simeq \ell^2(\Lambda_\omega).$$

Proof. Let us fix $\omega \in D \setminus E$ where $E \subset D$ is the zero measure set of exceptions where Λ_ω is not finite and define $S_\omega := \{a \in \ell^2(\Lambda) : \mathrm{Supp}(a) \subseteq \Lambda_\omega\}$, which is isomorphic to $\ell^2(\Lambda_\omega)$. Let $C_b(\Omega)$ be the space of bounded continuous functions defined on Ω; we have that $C_b(\Omega) \subset L^2(\Omega)$.

Let $\widetilde{a} \in S_\omega$; then there is a sequence $a \in \ell^2(\Lambda_\omega)$ such that

$$\widetilde{a}_\lambda = \begin{cases} a_\lambda & \text{if } \lambda \in \Lambda_\omega, \\ 0 & \text{otherwise.} \end{cases}$$

By Tietze's Extension Theorem, there exists $f_a \in C_b(\Omega)$ such that $f_a(\omega + \lambda) = a_\lambda$. If we define $\widetilde{f_a}$ as f_a in Ω and zero in $\mathbb{R}^d \setminus \Omega$, then $(\widetilde{f_a})^\vee \in PW_\Omega$, thus $\mathcal{T}(\widetilde{f_a})^\vee(\omega) = \widetilde{a} \in J_\Omega(\omega)$. This proves that $S_\omega \subseteq J_\Omega(\omega)$. It is easy to prove the other inclusion. We conclude that $J_\Omega(\omega) = S_\omega$. $\square$

As a consequence, we see that Ω is a k-tile if and only if $J_\Omega(\omega)$ are k dimensional for almost every $\omega \in D$.

All these previous results lead to the following theorem whose proof can be found in [1].

Theorem 2.13. *Let Ω be a k-tile measurable subset of $\mathbb{R}^d$. Given $\phi_1, \ldots, \phi_k \in PW_\Omega$ we define*

$$T_\omega = \begin{pmatrix} \widehat{\phi}_1(\omega + \lambda_1) & \ldots & \widehat{\phi}_k(\omega + \lambda_1) \\ \vdots & \ddots & \vdots \\ \widehat{\phi}_1(\omega + \lambda_k) & \ldots & \widehat{\phi}_k(\omega + \lambda_k) \end{pmatrix},$$

where the $\lambda_j = \lambda_j(\omega)$ for $j = 1, \ldots, k$ are the k values of Λ that belong to Λ_ω. Then, the subsequent statements are equivalent:

(i) *The set $\Phi_H = \{\tau_h \phi_j : h \in H,\ j = 1, \ldots, k\}$ is a Riesz basis for PW_Ω.*

(ii) *There exist $A, B > 0$ such that for almost every $\omega \in D$,*

$$A\|x\|^2 \le \|T_\omega x\|^2 \le B\|x\|^2, \tag{2.1}$$

for every $x \in \mathbb{C}^k$.

Moreover, in this case the constants of the Riesz basis are

$$A = \inf_{\omega \in D} \|T_\omega^{-1}\|^{-1} \quad \text{and} \quad B = \sup_{\omega \in D} \|T_\omega\|.$$

We will now show the connection between this theorem and the problem of the existence of Riesz bases of exponentials.

Let $\Omega \subset \mathbb{R}^d$ be a measurable k-tile by translations on a lattice Λ. We want to find $a_1, \ldots, a_k \in \mathbb{R}^d$ such that $E(H\,; a_1, \ldots, a_k) = \{e_{a_j+h} \,:\, h \in H,\, j = 1, \ldots, k\}$ is a Riesz basis of $L^2(\Omega)$.

Define $\phi_1, \ldots, \phi_k$ by their Fourier transform as follows:

$$\hat{\phi}_j := e_{a_j}\, \chi_\Omega, \quad j = 1, \ldots, k. \tag{2.2}$$

Hence, we are looking for $a_1, \ldots, a_k \in \mathbb{R}^d$ such that $\{\hat{\phi}_j e_h \,:\, h \in H,\, j = 1, \ldots, k\}$ is a Riesz basis of $L^2(\Omega)$, which is equivalent to $\{\tau_h \phi_j \,:\, h \in H,\, j = 1, \ldots, k\}$ being a Riesz basis for PW_Ω.

Theorem 2.13 states that this will happen if and only if the matrices

$$T_\omega = \begin{pmatrix} \widehat{\phi}_1(\omega+\lambda_1) & \ldots & \widehat{\phi}_k(\omega+\lambda_1) \\ \vdots & \ddots & \vdots \\ \widehat{\phi}_1(\omega+\lambda_k) & \ldots & \widehat{\phi}_k(\omega+\lambda_k) \end{pmatrix} = \begin{pmatrix} e_{a_1}(\omega+\lambda_1) & \ldots & e_{a_k}(\omega+\lambda_1) \\ \vdots & \ddots & \vdots \\ e_{a_1}(\omega+\lambda_k) & \ldots & e_{a_k}(\omega+\lambda_k) \end{pmatrix} \tag{2.3}$$

are uniformly bounded for almost every $\omega \in D$. Note that, in this case, the columns of T_ω form a Riesz basis of $\mathbb{C}^k$ for almost every $\omega \in D$ with uniform bounds.

To clarify the relation between (i) and (ii) in Theorem 2.13 we sketch the proof from [1] adapted to our setting.

The collection $\{e_{a_j+h} \,:\, h \in H,\, j = 1, \ldots, k\}$ is a Riesz sequence in $L^2(\Omega)$ if there exist positive constants A and B such that for any sequence of complex numbers $\{c_{j,h}\}$ with finitely many non-zero terms,

$$A \sum_{j=1}^{k} \sum_{h \in H} |c_{j,h}|^2 \leq \|P\|^2_{L^2(\Omega)} \leq B \sum_{j=1}^{k} \sum_{h \in H} |c_{j,h}|^2,$$

where P is the exponential polynomial

$$P(\omega) = \sum_{j=1}^{k} \sum_{h \in H} c_{j,h}\, e_{a_j+h}(\omega).$$

We see that

$$\begin{aligned} \|P\|^2_{L^2(\Omega)} &= \int_{\mathbb{R}^d} \left| \sum_{j=1}^{k} \sum_{h \in H} c_{j,h}\, e_{a_j+h}(\omega) \right|^2 \chi_\Omega(\omega)\, d\omega \\ &= \int_{\mathbb{R}^d} \left| \sum_{j=1}^{k} m_j(\omega)\, e_{a_j}(\omega) \right|^2 \chi_\Omega(\omega)\, d\omega, \end{aligned}$$

where

$$m_j(\omega) := \sum_{h \in H} c_{j,h}\, e_h(\omega), \quad j = 1, \ldots, k.$$

By a Λ-periodization argument, this is equal to

$$\sum_{\lambda\in\Lambda}\int_D \left|\sum_{j=1}^{k} m_j(\omega)e_{a_j}(\omega+\lambda)\right|^2 \chi_\Omega(\omega+\lambda)\,d\omega.$$

Since Ω k-tiles $\mathbb{R}^d$ by translations on Λ, we have that for almost every $\omega \in D$, $\Lambda_\omega = \{\lambda_1(\omega),\dots,\lambda_k(\omega)\}$. Therefore we get

$$\|P\|^2_{L^2(\Omega)} = \int_D \sum_{l=1}^{k}\left|\sum_{j=1}^{k} m_j(\omega)e_{a_j}(\omega+\lambda_l)\right|^2 d\omega = \int_D \|T_\omega m(\omega)\|^2_{\mathbb{C}^k}\,d\omega, \tag{2.4}$$

where $m(\omega) = (m_1(\omega),\dots,m_k(\omega))$ and T_ω is the matrix defined before.

On the other hand, using that $\left\{\frac{1}{\sqrt{|D|}}\, e_h \;:\; h\in H\right\}$ is an orthonormal basis of $L^2(D)$, we have

$$\int_D \|m(\omega)\|^2_{\mathbb{C}^k}\,d\omega = \sum_{j=1}^{k}\int_D |m_j(\omega)|^2\,d\omega = |D|\sum_{j=1}^{k}\sum_{h\in H}|c_{j,h}|^2. \tag{2.5}$$

Combining (2.4) and (2.5) and using standard arguments of measure theory, one may check that $E(H\,;a_1,\dots,a_k)$ is a Riesz sequence of $L^2(\Omega)$ if and only if there exist $A, B > 0$ such that for almost every $\omega\in D$, the inequalities in (2.1) hold for every $x\in\mathbb{C}^k$. Actually, inequality (2.1) implies the completeness in $L^2(\Omega)$ of the system $\{e_{a_j+h} \;:\; h\in H,\, j=1,\dots,k\}$ (see [1]).

3. Multi-tiles and Riesz bases

The proof of Theorem 1.1 is based on the techniques used in [1]. Without the assumption that Ω is a bounded domain, we need admissibility as an extra condition.

Proof of Theorem 1.1. As we discussed before, by Theorem 2.13, it suffices to find vectors $a_1,\dots,a_k\in\mathbb{R}^d$ for which there exist $A, B>0$ such that for almost every $\omega\in D$, the inequalities in (2.1) hold for every $x\in\mathbb{C}^k$, where T_ω are the matrices (2.3).

Let us note that for every $\omega\in D$, the matrix T_ω can be decomposed as

$$\begin{pmatrix} e_{a_1}(\lambda_1) & \dots & e_{a_k}(\lambda_1)\\ \vdots & \ddots & \vdots\\ e_{a_1}(\lambda_k) & \dots & e_{a_k}(\lambda_k)\end{pmatrix}\begin{pmatrix} e_{a_1}(\omega) & 0 & \dots & 0 & 0\\ 0 & e_{a_2}(\omega) & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & e_{a_{k-1}}(\omega) & 0\\ 0 & 0 & \dots & 0 & e_{a_k}(\omega)\end{pmatrix} = E_\omega\, U_\omega\,,$$

where U_ω is a unitary matrix. Then, in order to see the inequalities in (2.1), it is enough to prove that for almost every $\omega\in D$,

$$A||x||^2 \le \|E_\omega\, x\|^2 \le B||x||^2, \tag{3.1}$$

for every $x\in\mathbb{C}^k$.

Since Ω is admissible for Λ, there exist $v\in H$ and a number $n\in\mathbb{N}$ such that for almost every $\omega\in D$, the elements in $\{\langle v,\lambda\rangle \;:\; \lambda\in\Lambda_\omega\}$ are distinct (mod n).

Set $\mathcal{F}_n = \left\{e^{2\pi i rs/n}\right\}_{0\le r,s\le n-1}$ to be the Fourier matrix of order n. Any $k\times k$ submatrix of $\mathcal{F}_n$, formed by choosing k consecutive columns and any k rows, is an invertible matrix since it is a Vandermonde matrix.

Now, we define $a_j := \frac{j-1}{n}v$, $j = 1, \dots, k$. We obtain that for almost every $\omega \in D$,

$$E_\omega = \left\{e^{2\pi i(j-1)\langle v,\lambda_l\rangle/n}\right\}_{1\le l,j\le k}$$

is one of those submatrices of $\mathcal{F}_n$ except by a permutation of its rows, and hence invertible.

Moreover, there are finitely many different matrices E_ω because there are finitely many $k\times k$ submatrices of $\mathcal{F}_n$. Thus, there exist $A, B > 0$ such that the inequalities in (3.1) hold for every $x \in \mathbb{C}^k$ and for almost every $\omega \in D$. □

Remark 3.1. The vectors $a_1, \dots, a_k$ defined in the proof of Theorem 1.1, depend only on the vector $v \in H$ and $n \in \mathbb{N}$ from the admissibility condition. Hence, the same structured system of exponentials is a Riesz basis for any k-tile Ω which is (n, v)-admissible for Λ .

Remark 3.2. If n is a prime number, any selection of k columns and k rows from $\mathcal{F}_n$ forms an invertible matrix (see [24] and the references therein). Then, in the proof of Theorem 1.1, if n is a prime number we could also define $a_j := \frac{s_j}{n}v$, $j = 1, \dots, k$ where $s_1, \dots, s_k$ are distinct integers (mod n).

In a more general setting, if n is a power of a prime number, any submatrix of $\mathcal{F}_n$, formed by any k rows and k columns satisfying that their index set $\{s_1, \dots, s_k\}$ is *uniformly distributed over the divisors of* n, is invertible (see [2] for a definition of uniformly distributed). Thus, in the proof of Theorem 1.1, if $n = p^l$ with p prime and l a positive integer, we might as well define $a_j := \frac{s_j}{n}v$, $j = 1, \dots, k$ where $\{s_1, \dots, s_k\}$ is uniformly distributed over the divisors of n.

It is important to remark that there exist multi-tile sets that admit a *structured* Riesz basis of exponentials without being admissible. In the next example we will construct a multi-tile set which is not admissible for $\mathbb{Z}$ but admits a Riesz basis like in (1.1).

Example 3.3. Let $\{1, a_1, a_2\}$ be linearly independent numbers over $\mathbb{Q}$. Take the partition of $[0, 1)$ as in Example 2.6 and consider the following 2-tile set of $\mathbb{R}$ by translations on $\mathbb{Z}$:

$$\Omega = [0, 1) \cup \bigcup_{j=1}^{\infty} (I_j + n_j), \tag{3.2}$$

where the infinite sequence $\{n_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ will be adequately chosen to fit our purpose (see Figure 2).

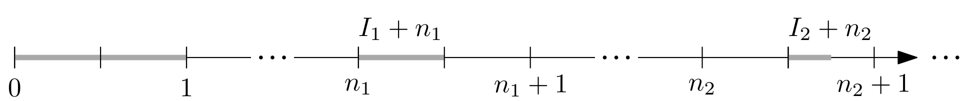

Figure 2. The set Ω.

Consider the functions ϕ_1 and ϕ_2 as defined in (2.2). Recall that in Theorem 1.1 we saw that the integer translations of ϕ_1 and ϕ_2 form a Riesz basis for PW_Ω if and only if the matrices

$$E_\omega = \begin{pmatrix} e_{a_1}(\lambda_1) & e_{a_2}(\lambda_1) \\ e_{a_1}(\lambda_2) & e_{a_2}(\lambda_2) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ e_{a_1}(\lambda_2) & e_{a_2}(\lambda_2) \end{pmatrix}$$

satisfy that there exist $A, B > 0$ such that (3.1) hold.

Let $\beta_1, \beta_2 \in [0,1)$ be two distinct numbers. The matrix,

$$R := \begin{pmatrix} 1 & 1 \\ e^{2\pi i\beta_1} & e^{2\pi i\beta_2} \end{pmatrix}$$

is invertible, and satisfies that,

$$\gamma_{min}\,||x||^2 \leq \|Rx\|^2 \leq \gamma_{max}\,||x||^2, \qquad x \in \mathbb{R}^2,$$

where γ_{min} and γ_{max} are the minimum and maximum eigenvalues of RR^* respectively.

For every $j \in \mathbb{N}$, we have that $\{1, a_1 j, a_2 j\}$ are also linearly independent over $\mathbb{Q}$. By Kronecker's Approximation Theorem there exists $m_j \in \mathbb{Z}$ for which

$$\left\| \left(e^{2\pi i a_1 j m_j}\,,\, e^{2\pi i a_2 j m_j}\right) - \left(e^{2\pi i\beta_1}\,,\, e^{2\pi i\beta_2}\right) \right\|_2 < \varepsilon.$$

Hence, for every $j \in \mathbb{N}$, take $n_j = jm_j$ as the sequence needed in (3.2).

Therefore, for almost every $\omega \in [0,1)$, the matrices $E_\omega E_\omega^*$ and RR^* are close to each other. Thus, the eigenvalues of these matrices must be close too. Then, for a small enough ε we get uniform bounds for (3.1) and consequently $E(\mathbb{Z}\,; a_1, a_2)$ is a Riesz basis of $L^2(\Omega)$.

However, this set is not admissible for $\mathbb{Z}$ because for every $j \in \mathbb{N}$, if $\omega \in I_j$, then $\Lambda_\omega = \{0, jm_j\}$ which are not distinct (mod j).

Remark 3.4. A similar argument can be done to extend the previous example to a k-tile. If $\{1, a_1, \ldots, a_k\}$ are linearly independent numbers over $\mathbb{Q}$, take the multi-tile at level k by translations on $\mathbb{Z}$ set

$$\Omega = [0, k-1) \cup \bigcup_{j=1}^{\infty} (I_j + n_j),$$

and choose $\{n_j\}_{j\in\mathbb{N}} \subset \mathbb{N}_{\geq k}$ in order to adequately approximate E_ω to an invertible matrix for almost every $\omega \in [0,1)$.

Hence, a natural question to ask is which sets Ω support a structured Riesz basis of exponentials. For the *bounded* case, it was proved in [1] that a set $\Omega \subset \mathbb{R}^d$ which admits a Riesz basis of exponentials $E(H\,; a_1, \ldots, a_k)$, for some $a_1, \ldots, a_k \in \mathbb{R}^d$, must be a k-tile of $\mathbb{R}^d$ by translations on Λ. This result holds true in the case of finite measure sets.

Theorem 3.5. *Let $H \subset \mathbb{R}^d$ be a full lattice and $\Lambda \subset \mathbb{R}^d$ its dual lattice. Given a measurable set of finite measure $\Omega \subset \mathbb{R}^d$, if $L^2(\Omega)$ admits a Riesz basis of the form $E(H\,; a_1, \ldots, a_k)$ for some $a_1, \ldots, a_k \in \mathbb{R}^d$, then Ω k-tiles $\mathbb{R}^d$ by translations on Λ.*

Proof. Defining the functions ϕ_j, $j = 1, \ldots, k$ as in (2.2), we get that $\{\tau_h \phi_j :\ h \in H\,,\ j = 1, \ldots, k\}$ is a Riesz basis for PW_Ω. This implies that for almost every $\omega \in D$, $\{\mathcal{T}\phi_1(\omega), \ldots, \mathcal{T}\phi_k(\omega)\}$ is a Riesz basis of $J_\Omega(\omega)$, and hence $\dim J_\Omega(\omega) = k$ for almost every $\omega \in D$. By Proposition 2.12, we conclude that Ω is a k-tile by translations on Λ. ∎

The results obtained so far can be summarized in Figure 3. Note that all the inclusions in the picture are proper.

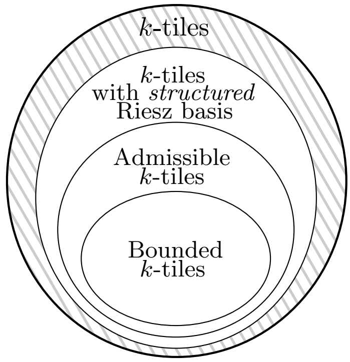

FIGURE 3. k-tile sets of $\mathbb{R}^d$.

4. SUBMULTI-TILES AND FRAMES

In this section we turn our attention to frames of exponentials. Recall that the system $\{e^{2\pi i\gamma.\omega} : \gamma \in \Gamma\}$ is a **frame** of $L^2(\Omega)$ if it satisfies that

$$A\|f\|^2 \leq \sum_{\gamma\in\Gamma} |\langle f, e^{2\pi i\gamma.\omega}\rangle|^2 \leq B\|f\|^2, \quad \forall\, f \in L^2(\Omega)$$

for some positive constants $A, B > 0$.

It is not difficult to see that any bounded measurable set $\Omega \subset \mathbb{R}^d$ supports a frame of exponentials. (This is an easy consequence of these two facts: *i*) for any cube Q in $\mathbb{R}^d$ there exists an orthonormal basis of exponentials for $L^2(Q)$. *ii*) If $\Omega \subseteq Q$, the restriction of an orthonormal basis of exponentials of $L^2(Q)$ to $L^2(\Omega)$ is a frame for the latter.)

Recently S. Nitzan, A. Olevskii and A. Ulanovskii [21] extended the result for any unbounded measurable set of finite measure. We want to note that the proof in [21] used the recently proved Kadison-Singer conjecture and it is not constructive. The goal of this section is to explore the relationship between unbounded *submulti-tiles* and frames and construct concrete examples of frames of exponentials on unbounded sets.

Definition 4.1. Let k be a positive integer. We say that a measurable set $\Omega \subset \mathbb{R}^d$ of finite measure, **submulti-tiles** $\mathbb{R}^d$ **at level** k by translations on a lattice Λ (or that Ω, k**-subtiles** $\mathbb{R}^d$) if for almost every $\omega \in D$,

$$\sum_{\lambda\in\Lambda} \chi_\Omega(x+\lambda) \leq k \quad \text{for almost all } x \in D.$$

When Ω is a k-subtile that is admissible, we can no longer claim that $L^2(\Omega)$ has a structured Riesz basis of exponentials, but instead we can see that it supports a structured *frame* of exponentials. Frames of exponentials are important since they give sets of sampling for the corresponding Paley-Wiener spaces.

The relation between k-subtiles and frames of exponentials was first studied in [5] for the case when Ω is a 1-subtile of finite measure in the context of locally compact abelian groups. Later on, it was proved in [4] that if Ω is a bounded

k-subtile, then it admits a structured frame of exponentials. In this section we adapt this last result to the case where Ω is a finite measure set (not necessarily bounded) with the extra hypothesis of the admissibility. More precisely, we prove the following theorem.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^d$ be a measurable set such that $0 < |\Omega| < +\infty$ and let $\Lambda \subset \mathbb{R}^d$ be a full lattice. If*

(i) *Ω submulti-tiles $\mathbb{R}^d$ at level k by translations on Λ,*
(ii) *Ω is admissible for Λ,*

then there exist $a_1, \dots, a_k \in \mathbb{R}^d$ such that the set $E(H\,; a_1, \dots, a_k)$ is a frame of $L^2(\Omega)$.

The strategy of the proof in [4] consists of giving a bounded k-subtile Ω, enlarging it to obtain a k-tile Δ, and then selecting a structured Riesz basis of $L^2(\Delta)$ (that always exists in the bounded case for k-tiles). This basis, when restricted to Ω is a structured frame for $L^2(\Omega)$.

In our case, since the k-subtile Ω is not necessarily bounded, we need to enlarge it to an admissible k-tile to guarantee the existence of the Riesz basis. This requires an adaptation of the proof in [4]. This is done in the next proposition:

Proposition 4.3. *Let Ω be a measurable set of finite measure that k-subtiles $\mathbb{R}^d$ and is admissible for a lattice $\Lambda \subset \mathbb{R}^d$. Then there exists a measurable set Δ of finite measure which is a k-tile of $\mathbb{R}^d$ and admissible for Λ such that $\Omega \subset \Delta$.*

Proof. We start by giving a characterization of sets that k-subtile $\mathbb{R}^d$ and are admissibles. Let Λ be a full lattice in $\mathbb{R}^d$, let v be a non-zero vector in the dual lattice H and let n be a natural number. Consider the sublattice of Λ defined by

$$\Lambda^{(0)} := \{\lambda \in \Lambda : \langle v, \lambda\rangle \equiv 0 \pmod n\},$$

and let $\Lambda^{(r)}$, $r = 0, \dots, n-1$, be the different cosets of the quotient $\Lambda/\Lambda^{(0)}$. Let $k \geq 1$ be an integer and $\Omega \subset \mathbb{R}^d$ a k-subtile that is $(n, v)-$admissible for Λ.

Define

$$\mathcal{R} := \{R \subset \Lambda : \#R \leq k \text{ and } \lambda - \lambda' \notin \Lambda^{(0)} \text{ if } \lambda, \lambda' \in R,\ \lambda \neq \lambda'\}.$$

The properties imposed on Ω imply that $\Lambda_\omega \in \mathcal{R}$ for almost every $\omega \in D$.

Now, for $R \in \mathcal{R}$ set $D_R := \{\omega \in D : \Lambda_\omega = R\}$. (Note that if $R \neq R'$, then $D_R \cap D_{R'} = \emptyset$ and that D_R could be empty for some $R \in \mathcal{R}$.)

We have $D_R + R \subseteq \Omega$ and we obtain (up to measure zero) the decomposition:

$$\Omega = \bigcup_{R \in \mathcal{R}} D_R + R. \tag{4.1}$$

We will see now that the sets D_R are measurables. Consider the functions

$$\psi_r(\omega) = \sum_{\lambda \in \Lambda^{(r)}} \chi_\Omega(\omega + \lambda),\ \ \omega \in D,\ \ r = 0, \dots, n-1,$$

and let $[R] := \big\{r \in \{0, \dots, n-1\} : r \equiv \langle v, \lambda\rangle \pmod n,\ \text{for some } \lambda \in R\big\}$.

Thus,

$$D_R = \bigcap_{r \in [R]} \psi_r^{-1}(1)\ \cap \bigcap_{r \notin [R]} \psi_r^{-1}(0),$$

which is an intersection of measurable sets.

Conversely, for each partition $\{D_R : R \in \mathcal{R}\}$ of D, in measurable sets (we allow here some of the partition elements to have measure zero), the set Ω defined by (4.1), necessarily k-subtiles $\mathbb{R}^d$ and is $(n, v)-$admissible for Λ.

Now that we obtained the desired decomposition, the proposition follows defining

$$\Delta = \bigcup_{R \in \mathcal{R}} D_R + (R \cup R'),$$

where for each $R \in \mathcal{R}$ we have chosen a set $R' \subseteq \Lambda$ complementary to R, in the sense that $[R] \cap [R'] = \emptyset$ and $\#([R] \cup [R']) = k$. $\square$

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2. By Proposition 4.3, there exists a measurable set of finite measure Δ, such that k-tiles $\mathbb{R}^d$ and is admissible for Λ, which contains Ω. Then, by Theorem 1.1 we know that there exist vectors $a_1, \ldots, a_k \in \mathbb{R}^d$ such that $E(H\,;\, a_1, \ldots, a_k)$ is a Riesz basis of $L^2(\Delta)$. Hence, $E(H\,;\, a_1, \ldots, a_k)$ is a frame of $L^2(\Omega)$. $\square$

As we saw in the previous section, Theorem 3.5 states that if Ω supports a Riesz basis of exponentials $E(H\,;\, a_1, \ldots, a_k)$, for some $a_1, \ldots, a_k \in \mathbb{R}^d$, then it must k-tile $\mathbb{R}^d$ by translations on Λ. When $E(H\,;\, a_1, \ldots, a_k)$ is a frame instead, a similar result can be proved.

Theorem 4.4. *Let $H \subset \mathbb{R}^d$ be a full lattice and let $\Lambda \subset \mathbb{R}^d$ be its dual lattice. Given a measurable set of finite measure $\Omega \subset \mathbb{R}^d$, if $L^2(\Omega)$ admits a frame of the form $E(H\,;\, a_1, \ldots, a_k)$ for some $a_1, \ldots, a_k \in \mathbb{R}^d$, then there exists $\ell \leq k$, such that Ω ℓ-subtiles $\mathbb{R}^d$ by translations on Λ.*

Proof. Proceeding analogously as in Theorem 3.5, we see that $\{\tau_h \phi_j : \ h \in H\,,\ j = 1, \ldots, k\}$ is a frame for PW_Ω. Which implies that $\{\mathcal{T}\phi_1(\omega), \ldots, \mathcal{T}\phi_k(\omega)\}$ is a frame of $J_\Omega(\omega)$ for almost every $\omega \in D$, and thus $\dim(J_\Omega(\omega)) \leq k$ for almost every $\omega \in D$. By Proposition 2.12, we get that $\#\Lambda_w \leq k$. Hence, if we take

$$\ell := \sup_{\omega \in D} \text{ess} \sum_{\lambda \in \Lambda} \chi_\Omega(\omega + \lambda),$$

Ω is an ℓ-subtile of $\mathbb{R}^d$ by translations on Λ. $\square$

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