PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 7, July 2018, Pages 2839–2851
http://dx.doi.org/10.1090/proc/14001
Article electronically published on February 21, 2018

HASSE PRINCIPLE VIOLATIONS FOR ATKIN-LEHNER TWISTS OF SHIMURA CURVES

PETE L. CLARK AND JAMES STANKEWICZ

(Communicated by Romyar T. Sharifi)

Abstract. Let $D > 546$ be the discriminant of an indefinite rational quaternion algebra. We show that there are infinitely many imaginary quadratic fields $l/\mathbb{Q}$ such that the twist of the Shimura curve X^D by the main Atkin-Lehner involution w_D and $l/\mathbb{Q}$ violates the Hasse Principle over $\mathbb{Q}$. More precisely, the number of squarefree d with $|d| \leq X$ such that the quadratic twist of (X^D, w_D) by $\mathbb{Q}(\sqrt{d})$ violates the Hasse Principle is $\gg X/\log^{\alpha_D} X$ and $\ll X/\log^{\beta_D} X$ for explicitly given $0 < \beta_D < \alpha_D < 1$ such that $\alpha_D - \beta_D \to 0$ as $D \to \infty$.

1. Introduction

1.1. The Main Theorem. First we fix some notation and terminology. Throughout, we will let $\ll$ and $\ll_\varepsilon$ denote the classical Vinogradov notation. For $N \in \mathbb{Z}^+$, let $\omega(N)$ be the number of distinct prime factors of N, and let h_N be the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-N})$. For a number field k, we denote by $\mathbb{A}_k$ the adele ring of k.

Let l/k be a quadratic field extension, let $X_{/k}$ be a curve, and let $\iota_{/k}$ be an involution of X, both defined over $\operatorname{Spec}(k)$. If $Y_{/k}$ is also a curve, we say $X \cong_l Y$ if there is an isomorphism of base changed curves $X_l \cong Y_l$. Equivalently, Y is a quadratic twist of X. We denote by $\mathcal{T}(X, \iota, l/k)$ the quadratic twist of X by ι and the quadratic extension l/k. We view $X_{/k}$ itself as the "trivial quadratic twist" of $X_{/k}$ corresponding to the "trivial quadratic extension k/k". We denote by X/ι the quotient under the action of the group $\{1, \iota\}$.

Let $D > 1$ be a squarefree integer which is a product of an even number of primes. Let $B/\mathbb{Q}$ be the (unique, up to isomorphism) nonsplit indefinite quaternion algebra with reduced discriminant D. Let $X^D_{/\mathbb{Q}}$ be the associated Shimura curve, and let w_D be the main Atkin-Lehner involution of $X^D_{/\mathbb{Q}}$ (see, e.g., [Cl03, §0.3.1]).

We can now state our main result.

Main Theorem. *Suppose that the genus of X^D/w_D is at least 2.*[1] *Then:*

a) *Infinitely many quadratic twists of (X^D, w_D) violate the Hasse Principle. More precisely:*

- *Let $\overline{D} = \frac{D}{\gcd(D,2)}$.*
- *Let $e_D = \omega(\overline{D}) + 2$.*

Received by the editors December 5, 2016, and, in revised form, January 20, 2017, September 14, 2017, October 4, 2017, and October 16, 2017.

2010 *Mathematics Subject Classification.* Primary 11G18, 11G30.

[1]The set of all such D is given in Lemma 7 and includes all $D > 546$.

• *Let* $\mathfrak{H}_D = \{$*squarefree* $d \in \mathbb{Z} \mid \mathcal{T}(X^D, w_D, \mathbb{Q}(\sqrt{d})/\mathbb{Q})$ *violates the Hasse Principle*$\}$*, and for* $X \geq 1$*, let* $\mathfrak{H}_D(X) = \#\mathfrak{H}_D \cap [-X, X]$*. Then:*

b) *We have* $\mathfrak{H}_D(X) \gg \dfrac{X}{\log^{1-2^{-e_D}}(X)}$.

c) *We have* $\mathfrak{H}_D(X) = O\left(\dfrac{X}{\log^{1-2^{-e_D}-(2h_D)^{-1}}(X)}\right) = O\left(\frac{X}{\log^{5/8}(X)}\right)$.

1.2. **Background and related work.** It is a fundamental problem to find varieties[2] V defined over a number field k which violate the Hasse Principle – i.e., $V(k) = \varnothing$, while for every completion k_v of k we have $V(k_v) \neq \varnothing$ (equivalently, $V(\mathbb{A}_k) \neq \varnothing$). It is also desirable to understand when these violations are explained by the Brauer-Manin obstruction, as is conjectured to hold whenever V is a curve [Po06, Conjecture 5.1]. There is a large literature on Hasse Principle violations for curves $V_{/k}$. Most examples are sporadic in nature: they apply to one curve at a time. A result of Poonen [Po10] gives an algorithm which takes as input a number field and outputs a curve $V_{/k}$ which violates the Hasse Principle.

Recently Bhargava, Gross, and Wang have shown that when genus $g \geq 1$ hyperelliptic curves over $\mathbb{Q}$ are ordered by height, a positive proportion violate the Hasse Principle and this violation is explained by Brauer-Manin [BGW, Thm. 1.1].

Past work of the first author [Cl08, Thm. 2] and [Cl, Thm. 1] gives two versions of a **Twist Anti-Hasse Principle (TAHP)**. Each gives hypotheses for a curve over a number field $X_{/k}$ endowed with a k-rational involution $\iota : X \to X$ to have infinitely many quadratic field extensions l/k such that $\mathcal{T}(X, \iota, l/k)$ violates the Hasse Principle. One of the hypotheses of TAHP is that $(X/\iota)(k)$ is finite, so ι cannot be a hyperelliptic involution, though the curve X may still be hyperelliptic. TAHP was used to show a refinement of Poonen's result: for every number field k and every $g \geq 2$ there is a bielliptic[3] curve $V_{/k}$ of genus g which violates the Hasse Principle [Cl, Main Theorem 2].

TAHP also applies to certain families of modular curves. In [Cl08, Thm. 1] it was shown that for all squarefree $N > 163$, there are infinitely many primes $p \equiv 1 \pmod 4$ such that $\mathcal{T}(X_0(N), w_N, \mathbb{Q}(\sqrt{p})/\mathbb{Q})$ violates the Hasse Principle. In fact this application came first, and TAHP arose by abstracting the properties of $(X_0(N), w_N)$ that were used in the proof. Note also that this gives a sequence of Hasse Principle violations in which the gonality tends to infinity. This work, along with the work of this paper, therefore adds to a growing body of literature relating the gonality of algebraic curves to their arithmetic.

Atkin-Lehner twists of modular curves arise naturally in terms of moduli spaces of elliptic $\mathbb{Q}$-curves.[4] Let $N \in \mathbb{Z}^+$ be squarefree. Ellenberg called for a study of local and global points on $\mathcal{T}(X_0(N), w_N, \mathbb{Q}(\sqrt{d})/\mathbb{Q})$ in [El04] and asked in particular [El04, Problem A] when these curves have adelic points. Thus [Cl08, Thm. 1] addresses many cases of Ellenberg's Problem A. A more penetrating and systematic analysis was given by Ozman [Oz12]. Ozman only considers twists by quadratic fields l such that no prime ramifies in both l and $\mathbb{Q}(\sqrt{-N})$: let us call these **coprime twists**. For any such coprime twist and each prime number p, Ozman

[2] All curves and varieties will be assumed to be smooth, projective, and geometrically integral.

[3] Every curve of genus 2 is hyperelliptic and no bielliptic curve of genus $g \geq 4$ is hyperelliptic.

[4] Technically, the moduli space in question is the quotient of $X_0(N)$ by a group of Atkin-Lehner involutions. The connection between the two will appear in §2.

gives necessary and sufficient conditions for $\mathcal{T}(X_0(N), w_N, \mathbb{Q}(\sqrt{d})/\mathbb{Q})(\mathbb{Q}_p) \neq \varnothing$. Combining her analysis with TAHP, she obtained the following result.

Theorem 1 (Ozman [Oz12, Thm. 5.4]). *Let $N > 131$ be a prime that is $1 \mod 4$. Let G be the ideal class group of $\mathbb{Q}(\sqrt{-N})$, and let $\alpha = \frac{\#\{g^2 | g \in G\}+1}{2\#G}$. There is a positive constant C_N then such that the number of coprime twists $\mathcal{T}(X_0(N), w_N, \mathbb{Q}(\sqrt{d})/\mathbb{Q})$ with $|d| \leq X$ violating the Hasse Principle is*

$$C_N \frac{X}{\log^{1-\alpha}(X)} + O\left(\frac{X}{\log^{2-\alpha}(X)}\right)$$

.

The family of classical modular curves $X_0(N)_{/\mathbb{Q}}$ is naturally viewed as the $D = 1$ case of the family of Shimura curves $X_0^D(N)_{/\mathbb{Q}}$. A systematic analysis of local points on Atkin-Lehner twists of these curves was given by the second author: for any Atkin-Lehner involution w_m of $X_0^D(N)$ and any quadratic field $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ with discriminant prime to $2DN$ (**coprime twists**) and all prime numbers p, [St14] gives necessary and sufficient conditions for $\mathcal{T}(X_0^D(N), w_m, \mathbb{Q}(\sqrt{d})/\mathbb{Q})(\mathbb{Q}_p) \neq \varnothing$.

The Main Theorem of the present work is thus roughly an analogue of Theorem 1 for the pair (X^D, w_D). Let us call attention to some differences between the results and some new difficulties arising in the Shimura curve case.

- We get local results for *all* $D > 1$ and global results for *all but finitely many* D.
- Our Main Theorem concerns *all* quadratic twists, not just coprime twists.
- In the versions of TAHP of [Cl08], [Cl], one of the hypotheses on $(X, \iota)_{/k}$ is that $X(\mathbb{A}_k) \neq \varnothing$. However, for all $D > 1$ we have $X^D(\mathbb{R}) = \varnothing$.
- In the case of $(X_0(N), w_N)$ the local conditions for a coprime twist by l to have adelic points require every prime divisor of the discriminant of l to lie in a certain Chebotarev set. For such sets, the asymptotic count is a result of Serre [Se76, Thm. 2.8]. However, when $D > 1$ even before twisting, X^D need not have p-adic points at primes $p \nmid D$, and the conditions for this are given as a finite sum coming from the Eichler-Selberg trace formula. So we cannot just apply [Se76, Thm. 2.8]. For each D and coprime d, we can determine with finite calculation whether $\mathcal{T}(X^D, w_D, \mathbb{Q}(\sqrt{d})/\mathbb{Q})(\mathbb{A}_\mathbb{Q}) \neq \varnothing$, but this does not allow us to handle all d's at once.

Let us say a bit about how these difficulties will be overcome. First, we give in §2 a third version of TAHP, Theorem 3, which weakens the hypothesis $X(\mathbb{A}_k) \neq \varnothing$ to the existence of *one* quadratic extension l_0/k such that $\mathcal{T}(X, \iota, l_0/k)(\mathbb{A}_k) \neq \varnothing$. To apply Theorem 3 to (X^D, w_D), we must find quadratic twists with adelic points. By TAHP, the existence of one such twist establishes part a). In §3 we produce an explicit family of such twists, based on a set $\mathcal{S}_D$ of primes defined by congruence conditions as in [St14]. For $d \in \eta_D$, all its prime factors must lie in $\mathcal{S}_D$, but furthermore d must satisfy its own congruence conditions, parity conditions, and positivity conditions. It is nonetheless possible to modify Serre's methods to give asymptotics for η_D. We do so in §4. The full set of discriminants which give Hasse Principle violations is not quite so explicit as to be given by congruence conditions, but the unmodified method of Serre still works to give an upper bound. It turns out that the coprimality condition does not affect the asymptotics.[5]

[5] Similar considerations show that in Theorem 1, the word "coprime" can be omitted, with the effect of changing only the constant C_N.

Because of the above difficulties, the upper and lower bounds on $\mathfrak{H}_D(X)$ obtained in the Main Theorem have different exponents in the logarithm: the discrepancy is $\frac{1}{2h_D}$. Since the class number of an imaginary quadratic field tends to infinity with the absolute value of the discriminant [Co89, §7.D], the discrepancy goes to 0 as D increases.

Example 2. Let ℓ be an odd prime, and let $D = 2\ell$. For all $\varepsilon > 0$, there is $L = L(\epsilon)$ such that for all $\ell > L$ we have

$$X/\log^{7/8}(X) \ll \mathfrak{H}_{2\ell}(X) \ll_\varepsilon X/\log^{7/8-\varepsilon}(X).$$

1.3. **Structure of the paper.** In §2 we prove Theorem 3 and recall results on Shimura curves needed in its application. We give the proof of part a) of the Main Theorem in §3. In §4 we state and prove a moderately sharpened version of parts b) and c) of the Main Theorem. In §5 we discuss relations between our results and Hasse Principle violations over quadratic extensions, both in general and with particular attention to $X^D_{/\mathbb{Q}}$.

2. Preliminaries

2.1. Another Twist Anti-Hasse Principle.

Theorem 3 (Twist Anti-Hasse Principle, v. III)**.** *Let k be a number field. Let $X_{/k}$ be a smooth, projective, geometrically integral curve, and let $\iota : X \to X$ be a k-rational involution. We suppose*

(i) *We have $\{P \in X(k) \mid \iota(P) = P\} = \varnothing$.*
(ii) *We have $\{P \in X(\overline{k}) \mid \iota(P) = P\} \neq \varnothing$.*
(iii) *We have $\mathcal{T}(X, \iota, l_0/k)(\mathbb{A}_k) \neq \varnothing$ for some quadratic extension l_0/k.*
(iv) *The set $(X/\iota)(k)$ is finite.*

Then

a) *The twisted curve $\mathcal{T}(X, \iota, l/k)_{/k}$ has no k-rational points for all but finitely many quadratic extensions l/k.*

b) *There are infinitely many quadratic extensions l/k such that the twisted curve $\mathcal{T}(X, \iota, l/k)$ violates the Hasse Principle over k.*

c) *When $k = \mathbb{Q}$, as $B \to \infty$, the number of squarefree integers d with $|d| \leq B$ such that $\mathcal{T}(X, \iota, \mathbb{Q}(\sqrt{d})/\mathbb{Q})$ violates the Hasse Principle is $\gg \frac{B}{\log B}$.*

Proof. Let l/k be a quadratic extension, and put $Y = \mathcal{T}(X, \iota, l/k)$. Then ι defines a k-rational involution on Y: indeed, $Y_{/l} \cong X_{/l}$, and if σ is the nontrivial field automorphism of l/k, then σ acts on $Y(l) = X(l)$ by $\sigma^* P = \iota(\sigma(P))$, so for all $P \in Y(A)$ and for all k-algebras A we have

$$\sigma^* \iota (\sigma^*)^{-1} = \iota(\sigma \iota \sigma^{-1})\iota^{-1} = \iota\iota\iota^{-1} = \iota.$$

The curve Y/ι is canonically isomorphic to X/ι. Thus for each quadratic extension l/k – including the trivial quadratic extension k/k – there is a map

$$\psi_l : \mathcal{T}(X, \iota, l/k)(k) \to (X/\iota)(k).$$

We have

$$(X/\iota)(k) = \bigcup_{l/k} \psi_l(\mathcal{T}(X, \iota, l/k))(l). \tag{1}$$

Indeed, if $P \in (X/\iota)(k)$ and $q : X \to X/\iota$ is the quotient map, then $q^*(P) = [Q_1] + [Q_2]$ is an effective k-rational divisor of degree 2. If $Q_1 = Q_2$, then Q_1 is a k-rational ι-fixed point. In this case Q_1 is also a k-rational point of $\mathcal{T}(X, \iota, l/k)$ for all quadratic extensions l/k. Otherwise, $Q_1 \neq Q_2 = \iota(Q_1)$, and the Galois action on $\{Q_1, Q_2\}$ determines a unique quadratic extension l/k (possibly the trivial one) such that $Q_1, Q_2 \in \mathcal{T}(X, \iota, l/k)$. This shows (1) and also shows that under our hypothesis (i) the union in (1) is disjoint.

a) For this part we need only assume hypotheses (i) and (iv). Let $(X/\iota)(k) = \{P_1, \ldots, P_n\}$. By hypothesis (i), each P_i is either of the form $\psi_k(Q)$ for $Q \in X(k)$ or $\psi_{\ell_i}(Q)$ for a unique quadratic extension l_i/k. Thus there are at most n quadratic extensions l/k such that $\mathcal{T}(X, \iota, l/k)(k) \neq \varnothing$.

b) If we replace (iii) by the hypothesis (iii′) $X(\mathbb{A}_k) \neq \varnothing$, then we get (a slightly simplified statement of) [Cl, Thm. 1]. Suppose now that $X(\mathbb{A}_k) = \varnothing$ but for some nontrivial quadratic extension l_0/k we have $\mathcal{T}(X, \iota, l_0/k)(\mathbb{A}_k) \neq \varnothing$. Put $Y = \mathcal{T}(X, \iota, l_0/k)$. The canonical bijection $X(\overline{k}) \to Y(\overline{k})$ induces bijections on the sets of ι-fixed points and of k-rational ι-fixed points, so conditions (i) and (ii) hold for Y. By our assumption, hypothesis (iii′) holds for Y. And as above we have a canonical isomorphism $(X/\iota) \to (Y/\iota)$. So we may apply [Cl, Thm. 1] to Y in place of X, getting the conclusion that infinitely many quadratic twists of (Y, ι) – equivalently, of (X, ι) – violate the Hasse Principle over k.

c) Similarly, if $k = \mathbb{Q}$ and we replace (iii) by (iii′) $X(\mathbb{A}_{\mathbb{Q}}) \neq \varnothing$, then we may apply [Cl08, Thm. 2] to get that the set of prime numbers p such that $\mathcal{T}(X, \iota, \mathbb{Q}(\sqrt{p})/\mathbb{Q})$ violates the Hasse Principle has positive density, and thus that the number of quadratic twists by squarefree d with $|d| \leq B$ is $\gg \frac{B}{\log B}$, the number of primes up to B. This conclusion applies to some quadratic twist $Y = \mathcal{T}(X, \iota, \mathbb{Q}(\sqrt{d_0})/\mathbb{Q})$ and thus it also applies (with a different value of the suppressed constant) to X. ☐

2.2. **Results on Shimura curves.**

Theorem 4. *If $X^D_{/\mathbb{Q}}$ is the Shimura curve associated to an indefinite rational quaternion algebra of reduced discriminant D and w_D is the main Atkin-Lehner involution, then the following hold:*

a) (Shimura [Sh75]) *We have $X^D(\mathbb{R}) = \varnothing$.*

b) (Clark [Cl03, Main Theorem 2]) *We have $(X^D/w_D)(\mathbb{A}_{\mathbb{Q}}) \neq \varnothing$.*

Corollary 5. *Let $d \in \mathbb{Q}^\times \setminus \mathbb{Q}^{\times 2}$, and let $Y_d = \mathcal{T}(X^D, w_D, \mathbb{Q}(\sqrt{d})/\mathbb{Q})$. Then $Y_d(\mathbb{R}) = \varnothing \iff d > 0$.*

Proof. If $d > 0$, then $Y_d \cong_{\mathbb{R}} X^D$, so $Y_d(\mathbb{R}) = \varnothing$ by Theorem 4a). By Theorem 4b), there is $P \in (X^D/w_D)(\mathbb{R})$. Let $q : X^D \to X^D/w_D$ be the quotient map, defined over $\mathbb{R}$. Because $X^D(\mathbb{R}) = \varnothing$, the fiber of q consists of a pair of $\mathbb{C}$-conjugate $\mathbb{C}$-valued points, say Q and $\overline{Q} = \iota(Q)$. Thus $Q = \iota(\overline{Q})$, so $Q \in Y_d(\mathbb{R})$. ☐

Recall that for $d \in \mathbb{Z}^{<0}$, there is an order $\mathcal{O}$ of discriminant d in a quadratic field iff $d \equiv 0, 1 \pmod 4$. If $d \equiv 2, 3 \pmod 4$ we put $h'(d) = 0$. If $d \equiv 0, 1 \pmod 4$ we define $h'(d) = \# \operatorname{Pic} \mathcal{O}$, the class number of the quadratic order $\mathcal{O}$ of discriminant d.

Lemma 6. *Let $D > 1$ be the discriminant of an indefinite quaternion algebra $B_{/\mathbb{Q}}$.*

a) *The set $\{P \in X^D(\mathbb{Q}) \mid w_D(P) = P\}$ is empty.*

b) *We have $\#\{P \in X^D(\overline{\mathbb{Q}}) \mid w_D(P) = P\} = h'(-D) + h'(-4D) > 0$.*

c) *The genus of X^D is*

$$g_D := 1 + \frac{\varphi(D)}{12} - \frac{\prod_{p|D}\left(1-\left(\frac{-4}{p}\right)\right)}{4} - \frac{\prod_{p|D}\left(1-\left(\frac{-3}{p}\right)\right)}{3}. \tag{2}$$

d) *The genus of X^D/w_D is*

$$1 + \frac{\varphi(D)}{24} - \frac{\prod_{p|D}\left(1-\left(\frac{-4}{p}\right)\right)}{8} - \frac{\prod_{p|D}\left(1-\left(\frac{-3}{p}\right)\right)}{6} - \frac{h'(-D)+h'(-4D)}{4}. \tag{3}$$

Proof. a) This is immediate from Theorem 4a). b), c), d). See, e.g., [Cl03, §0.3.1]. □

Lemma 7.

a) *If $D \in \{6, 10, 14, 15, 21, 22, 26, 33, 34, 35, 38, 39, 46, 51, 55, 62, 69, 74, 86, 87, 94, 95, 111, 119, 134, 146, 159, 194, 206\}$, then $X^D/w_D \cong_{\mathbb{Q}} \mathbb{P}^1$.*

b) *If $D \in \{57, 58, 65, 77, 82, 106, 118, 122, 129, 143, 166, 210, 215, 314, 330, 390, 510, 546\}$, then X^D/w_D is an elliptic curve of positive rank.*

c) *For all other D – in particular, for all $D > 546$ – the set $(X^D/w_D)(\mathbb{Q})$ is finite.*

Proof. Using (3), one sees that X^D/w_D has genus 0 iff D is one of the discriminants listed in part a) and that X^D/w_D has genus 1 iff D is one of the discriminants listed in part b). Thus for all other D, X^D/w_D has genus at least 2 and $(X^D/w_D)(\mathbb{Q})$ is finite by Faltings' Theorem, establishing part c). By Theorem 4b) the curve $(X^D/w_D)_{/\mathbb{Q}}$ has points everywhere locally, so when it has genus 0 it is isomorphic to $\mathbb{P}^1$, establishing part a). The case in which X^D/w_D has genus 1 is handled by work of Rotger. In every case he shows that there is a class number one imaginary quadratic field K and a point corresponding to an abelian surface isogenous to the product of two elliptic curves with CM by $\mathbb{Z}_K$ (a $\mathbb{Z}_K$-CM point) on X^D. This induces a $\mathbb{Q}$-rational point on X^D/w_D, so X^D/w_D is an elliptic curve. Moreover, Rotger identifies X^D/w_D with a modular elliptic curve in Cremona's tables – see [Ro02, Table III]. All of these elliptic curves have rank one.[6] This establishes part a). □

3. Proof of the Main Theorem, part a)

Let D be the discriminant of a nonsplit indefinite rational quaternion algebra. Assume moreover that D does not appear in Lemma 7 a) or b); in particular, this holds for all $D > 546$. We will prove the Main Theorem by verifying that the pair (X^D, w_D) satisfies the hypotheses of Theorem 3: then by Theorem 3c), the number of quadratic twists by d with $d \le |X|$ which violate the Hasse Principle is $\gg \frac{X}{\log X}$.

Parts a) and b) of Lemma 6 show that conditions (i) and (ii) hold, and part c) of Lemma 7 shows that condition (iv) holds. For $d \in \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$, put

$$Y_d = \mathcal{T}(X^D, w_D, \mathbb{Q}(\sqrt{d})/\mathbb{Q}).$$

By Corollary 5, $Y_d(\mathbb{R}) \neq \varnothing \iff d < 0$. Henceforth we assume that $d < 0$.

Recall that g_D is the genus of X^D, and put $\overline{D} = \begin{cases} D & 2 \nmid D, \\ \frac{D}{2} & 2 \mid D. \end{cases}$

[6] This is not a coincidence. By Atkin-Lehner theory and the Jacquet-Langlands correspondence, every elliptic curve which is a $\mathbb{Q}$-isogeny factor of the Jacobian of X^D has odd analytic rank.

For $n \in \mathbb{Z}^+$, let $\omega(n)$ be the number of distinct prime divisors of n. Put

$$e_D = \omega(\overline{D}) + 2. \tag{4}$$

Let $\mathcal{S}_D$ be the set of prime numbers ℓ satisfying:

(a) $\ell \equiv 3 \pmod 8$ and

(b) for all primes $q \mid \overline{D}$ we have $\left(\frac{-\ell}{q}\right) = -1$.

Let η_D be the set of all negative integers d such that:

(c) $-d = \prod_{i=1}^{2r-1} \ell_i$ is the product of an odd number of distinct primes $\ell_i \in \mathcal{S}_D$ and

(d) for all primes $p \in (2, 4g^2]$ such that $p \nmid D$, we have $\left(\frac{d}{p}\right) = -1$.

The set η_D is infinite: indeed, by the Chinese Remainder Theorem and Dirichlet's Theorem it contains infinitely many elements $d = -\ell$ with $\ell \in \mathcal{S}_D$. Moreover:

(e) For $d \in \eta_D$ we have $d \equiv 5 \pmod 8$ and thus 2 is inert in $\mathbb{Q}(\sqrt{d})$.

(f) For $\ell \in \mathcal{S}_D$ we have $\left(\frac{-D}{\ell}\right) = -1$. To see this, first suppose $2 \nmid D$, so $D = \prod_{i=1}^{2a} q_i$ with $q_1, \ldots, q_{2a}$ distinct odd primes. Then

$$\left(\frac{-D}{\ell}\right) = \left(\frac{-1}{\ell}\right) \prod_{i=1}^{2a} \left(\frac{q_i}{\ell}\right) = -\prod_{i=1}^{2a} \left(\frac{-\ell}{q_i}\right) = -\prod_{i=1}^{2a} -1 = -1. \tag{5}$$

Now suppose $2 \mid D$, so $D = 2\prod_{i=1}^{2a-1} q_i$ with $q_1, \ldots, q_{2a-1}$ distinct odd primes. Then

$$\left(\frac{-D}{\ell}\right) = \left(\frac{-1}{\ell}\right)\left(\frac{2}{\ell}\right) \prod_{i=1}^{2a-1} \left(\frac{q_i}{\ell}\right) = (-1)(-1) \prod_{i=1}^{2a-1} \left(\frac{-\ell}{q_i}\right) = (-1)^{2a-1} = -1.$$

We claim that for all $d \in \eta_D$ we have $Y_d(\mathbb{A}_{\mathbb{Q}}) \neq \varnothing$. Indeed:

- As above, since $d < 0$ we have $Y_d(\mathbb{R}) \neq \varnothing$.
- By [St14, Thm. 4.1.3] and [St14, Thm. 4.1.5], we have $Y_d(\mathbb{Q}_p) \neq \varnothing$ for all $p \mid d$.
- Since 2 is inert in $\mathbb{Q}(\sqrt{d})$, we have $Y_d(\mathbb{Q}_2) \neq \varnothing$ by either [St14, Thm. 5.1] in the case where $2 \mid D$ or [St14, Cor 3.17] when $2 \nmid D$.
- If $p \mid \overline{D}$, then p is inert in $\mathbb{Q}(\sqrt{d})$, so by [St14, Cor. 5.2] we have $Y_d(\mathbb{Q}_p) \neq \varnothing$.
- If $p \nmid Dd$ and $p > 4g^2$, then by [St14, Thm. 3.1] we have $Y_d(\mathbb{Q}_p) \neq \varnothing$.
- If $p \nmid D$ and $p \in (2, 4g^2]$, then p is inert in $\mathbb{Q}(\sqrt{d})$, so by [St14, Cor. 3.17] we have $Y_d(\mathbb{Q}_p) \neq \varnothing$. Note that by (d), if $p < 4g^2$, then $p \nmid d$.

4. Proof of the Main Theorem, parts b) and c)

4.1. Restating the theorem. Let the set of "adelic discriminants" be

$$\mathfrak{A}_D = \{\text{squarefree } d \in \mathbb{Z} \mid \mathcal{T}(X^D, w_D, \mathbb{Q}(\sqrt{d})/\mathbb{Q})(\mathbb{A}_{\mathbb{Q}}) \neq \varnothing\}.$$

For $X \geq 1$, put

$$\mathcal{S}_D(X) = \mathcal{S}_D \cap [1, X],$$

$$\eta_D(X) = \#\eta_D \cap [-X, -1],$$

$$\mathfrak{A}_D(X) = \#\mathfrak{A}_D \cap [-X, X] = \#\mathfrak{A}_D \cap [-X, -1].$$

For an imaginary quadratic discriminant Δ, let $H_\Delta(X)$ be the Hilbert class polynomial [Co89, p. 285].

Theorem 8. *Fix $D > 1$ an indefinite quaternionic discriminant. Then*

a) *Suppose $D \notin \mathcal{E} := \{6, 10, 14, 15, 21, 22, 33, 34, 38, 46, 58, 82, 94\}$. Then there is a positive constant c_D such that as $X \to \infty$ we have*

$$\eta_D(X) = c_D \frac{X}{\log^{(1-2^{-e_D})} X} + O\left(\frac{X}{\log^{(2-2^{-e_D})} X}\right). \tag{6}$$

b) *Let h_D be the class number of the field $\mathbb{Q}(\sqrt{-D})$. As $X \to \infty$ we have*

$$\mathfrak{A}_D(X) = O\left(\frac{X}{\log^{(1-2^{-e_D}-(2h_D)^{-1})} X}\right) = O\left(\frac{X}{\log^{5/8}(X)}\right). \tag{7}$$

Suppose X^D/w_D has genus at least 2. Then $D \notin \mathcal{E}$ by Lemma 7. Moreover, only finitely many quadratic twists of (X^D, w_D) have $\mathbb{Q}$-points, so

$$\mathfrak{A}_D(X) = \mathfrak{H}_D(X) + O(1).$$

Since $\eta_D \subset \mathfrak{A}_D$, this gives

$$\mathfrak{H}_D(X) \geq \eta_D(X) + O(1),$$

and thus (6) implies part b) of the Main Theorem. Since $\mathfrak{H}_D \subset \mathfrak{A}_D$, (7) implies part c) of the Main Theorem.

Thus Theorem 8 is a moderately sharpened form of parts b) and c) of the Main Theorem. The remainder of the section is devoted to its proof.

4.2. **A preliminary lemma.**

Lemma 9. *If $D \notin \mathcal{E}$, then for all prime numbers $p \mid D$, we have $p < 4g_D^2$.*

Proof. It is an easy consequence of Lemma 6c) that

$$4g_D^2 \geq \frac{1}{36}(12 + \varphi(D) - 7 \cdot 2^{\omega(D)})^2.$$

So to get $p < 4g_D^2$ it will suffice to show that

$$\forall p \mid D,\ 6\sqrt{p} < 12 + \varphi(D) - 7 \cdot 2^{\omega(D)}. \tag{8}$$

Case 1. Suppose $\omega(D) = 2$, so $D = pq$ for prime numbers $p < q$. For all $q > 66$,

$$12 + \varphi(D) - 7 \cdot 2^{\omega(D)} = (p-1)(q-1) - 16 \geq q - 17 > 6\sqrt{q} > 6\sqrt{p},$$

so (8) holds. Using the genus formula we test all discriminants $D = pq$ with $q \leq 66$; the ones for which $q \geq 4g_D^2$ are precisely the set $\mathcal{E}$.

Case 2. Suppose $\omega(D) \geq 4$. Put $\beta(D) = \prod_{p \mid D} \frac{p-1}{2}$.

Let us first suppose that $\beta(D) > 49$. Then, since $\frac{y-1}{2} \geq \frac{y}{4}$ for all $y \geq 2$, we have

$$\beta(D) - 7 \geq 6\sqrt{\beta(D)} \geq 6\sqrt{\prod_{p \mid D} \frac{p}{4}}$$

and thus for all $p \mid D$ we have

$$12 + \varphi(D) - 7 \cdot 2^{\omega(D)} > \varphi(D) - 7 \cdot 2^{\omega(D)} \geq 6\sqrt{D} \geq 6\sqrt{p}.$$

If $\omega(D) \geq 4$ and $\beta(D) \leq 49$, then $D = p_1 p_2 p_3 p_4$ for primes $p_1 < p_2 < p_3 < p_4 < 101$. Using the genus formula we find that $p_4 < 4g_D^2$ in all cases. $\square$

Thus for $D \notin \mathcal{E}$, η_D is the set of squarefree negative $d \in \mathbb{Z}$ with an odd number of prime divisors all in $\mathcal{S}_D$ such that all primes less than $4g_D^2$ are inert in $\mathbb{Q}(\sqrt{d})$.

4.3. **Proof of Theorem 8a).** Suppose $D \notin \mathcal{E}$. Let I be the product of the primes less than $4g_D^2$ that do not divide $2D$. Let $\{u_i\} \subset \{1, \ldots, I\}$ be the subset of integers such that $\left(\frac{u_i}{\ell}\right) = -1$ for all $\ell \mid I$. Let $\{v_j\} \subset \{1, \ldots, 8\overline{D}\}$ be the subset of integers such that $v_j \equiv 3 \bmod 8$ and for all $q \mid D$ odd, $\left(\frac{v_j}{q}\right) = -1$. The size of $\{v_j\}$ is $\frac{\varphi(8\overline{D})}{2^{e_D}}$. Let μ be the Möbius function, let b_n be the multiplicative function such that for any power of a prime p,

$$b_{p^m} = \begin{cases} 1, & p \equiv v_j \bmod 8\overline{D} \text{ for some } j, \\ 0 & \textit{else}. \end{cases}$$

We let

$$a_n = b_n \left(\frac{1}{\varphi(I)} \sum_{\chi} \sum_{i} \overline{\chi(-u_i)} \chi(n) \right) \left(\frac{\mu^2(n) - \mu(n)}{2} \right),$$

where χ runs over the mod I Dirichlet characters.

Here $a_n = 1$ if and only if $-n \equiv u_i \bmod I$ for some i, n (and thus $-n$) is square-free with an odd number of prime factors, and each prime dividing n is congruent to some v_j. That is, a_n is the indicator function for $-\eta_D$.

Consider the function $f(s) = \sum_n a_n n^{-s}$, holomorphic on $\Re(s) > 1$. Note that the a_n are not necessarily multiplicative. We however reduce to this case as we write the Dirichlet series $f_{k,\chi}(s) = \sum_{n \geq 0} b_n \mu^k(n) \chi(n) n^{-s}$, again converging in the half-plane $\Re(s) > 1$. We therefore have

$$f(s) = \frac{1}{2\varphi(I)} \sum_{\chi} \left(\left(\sum_i \overline{\chi(-u_i)} \right) (f_{2,\chi}(s) - f_{1,\chi}(s)) \right).$$

We begin by showing that with the exception of $(k, \chi) = (2, \mathbf{1})$, these are in fact holomorphic in the region $\Re(s) \geq 1$.

Consider

$$\begin{aligned} \log(f_{k,\chi}(s)) &= \sum_p \log(\sum_{m \geq 0} b_{p^m} \mu^k(p^m) \chi(p^m) p^{-ms}) \\ &= \sum_p \log(1 + b_p (-1)^k \chi(p) p^{-s}) \\ &= (-1)^k \sum_p \frac{b_p \chi(p)}{p^s} + \beta_{k,\chi}(s), \end{aligned}$$

where $\beta_{k,\chi}(s)$ is holomorphic on $\Re(s) > 1/2$.

Now use the fact that

$$b_p = \frac{1}{\varphi(8\overline{D})} \sum_{\psi \bmod 8\overline{D}} \sum_j \overline{\psi(v_j)} \psi(p).$$

Therefore

$$\log(f_{k,\chi}(s)) = (-1)^k \frac{1}{\varphi(8\overline{D})} \sum_{\psi} \sum_j \overline{\psi(v_j)} \log(L(s, \chi\psi)) + \rho_{k,\chi}(s),$$

where $\rho_{k,\chi}$ is holomorphic for $\Re(s) > 1/2$.

It follows that zero-free regions for L-functions of Dirichlet characters give zero-free regions for the $f_{j,\chi}$ and thus holomorphic regions for f. In particular, if ϵ is a Dirichlet character and $\delta_\epsilon = 1$ for $\epsilon = \mathbf{1}$ (the trivial character) and zero otherwise, then there are positive numbers A_ϵ, B_ϵ such that $\log(L(s,\epsilon)) - \delta_\epsilon \log(1/(s-1))$ is holomorphic on $\Re(s) \geq 1 - B_\epsilon / \log^{A_\epsilon}(2 + |\Im(s)|)$ [Se76, Prop. 1.7].

Now we note that since $(I, 8\overline{D}) = 1$, $\chi\psi = \mathbf{1}$ if and only if $\chi = \mathbf{1}$ and $\psi = \mathbf{1}$. Therefore by exponentiating, we find a holomorphic, nonzero function $g_{k,\chi}$ on the same region in $\mathbb{C}$ such that

$$f_{k,\chi}(s) = \left(\frac{1}{s-1}\right)^{\delta_\chi (-1)^k / 2^{e_D}} g_{k,\chi}(s).$$

Thus there is a function g holomorphic on the intersection of the A_ϵ, B_ϵ regions such that $f(s) = \left(\frac{1}{s-1}\right)^{2^{-e_D}} g(s)$. Finally, we may apply the method of Serre and Watson [Se76, Thm. 2.8] to get our asymptotic for $\sum_{n \leq X} a_n = \#\eta_D(X)$.

4.4. **Proof of Theorem 8b).** We define a set $\mathcal{S}'_D$ of primes, as follows:

- If $D \not\equiv 3 \pmod 4$, then $\ell \in \mathcal{S}'_D$ iff $\left(\frac{-D}{\ell}\right) = 1$ and $H_{-4D}(X)$ has a root modulo ℓ.
- If $D \equiv 3 \pmod 4$, then $\ell \in \mathcal{S}'_D$ iff $\left(\frac{-D}{\ell}\right) = 1$ and at least one of $H_{-D}(X)$ and $H_{-4D}(X)$ has a root modulo ℓ.

By (5) the sets $\mathcal{S}_D$ and $\mathcal{S}'_D$ are disjoint. Moreover, by [St14, Thm. 4.1], if $d \in \mathfrak{A}_D$, then for all primes $p \mid d$ we have

$$p \in \mathcal{C}_D := \mathcal{S}_D \cup \mathcal{S}'_D \cup \{\text{prime divisors of } 2D\}.$$

Step 1. We show that $\mathcal{S}'_D$ is a Chebotarev set of density $\frac{1}{2h_D}$.

For any imaginary quadratic discriminant $\Delta < 0$, the field

$$K_\Delta = \mathbb{Q}(\sqrt{-\Delta})[X]/(H_\Delta(X))$$

is the ring class field of discriminant Δ. This field is Galois over $\mathbb{Q}$ of degree twice the class number of the imaginary quadratic order of discriminant Δ.

- Suppose $D \not\equiv 3 \pmod 4$. Then up to a finite set, $\mathcal{S}'_D$ is the set of primes which split completely in K_{-4D}, which in this case is the Hilbert class field of $\mathbb{Q}(\sqrt{-D})$. Thus $\mathcal{S}'_D$ is Chebotarev of density $\frac{1}{2h_D}$.
- Suppose $D \equiv 3 \pmod 4$. Then K_{-D} is the Hilbert class field of $\mathbb{Q}(\sqrt{-D})$. The ring class field K_{-4D} contains K_{-D}, so the set of primes splitting completely in K_{-D} or in K_{-4D} is the same as the set of primes splitting completely in K_{-D}. Thus again $\mathcal{S}'_D$ is Chebotarev of density $\frac{1}{2h_D}$.

Step 2. Since $\mathcal{S}_D$ is a Chebotarev set of density $\frac{1}{2^{e_D}}$ defining a set of conjugacy classes for $K_\Delta/\mathbb{Q}$, and since $\mathcal{S}'_D$ is a disjoint Chebotarev set of primes defined in terms of K_Δ, and only finitely many primes divide $2D$, it follows that $\mathcal{C}_D$ is a Chebotarev set[7] of density

$$\delta_D = \frac{1}{2^{e_D}} + \frac{1}{2h_D}.$$

[7]If there were different implied Galois extensions for $\mathcal{S}_D$ and $\mathcal{S}'_D$, $\mathcal{C}_D$ would have the same density. On the other hand it might only be *regular in the sense of Delange* because the Frobenii of the primes might not be closed under conjugation in the Galois group of the compositum. In any case our results here would be unchanged [Se76, Thm 2.4].

Let $\mathfrak{A}'_D$ be the set of $n \in \mathbb{Z}^+$ with all prime divisors lying in $\mathcal{C}_D$, and for $X \geq 1$, put $\mathfrak{A}'_D(X) = \mathfrak{A}'_D \cap [1, X]$. By [Se76, Thm. 2.8] we have that as $X \to \infty$,

$$\mathfrak{A}'_D(X) = O\left(\frac{X}{\log^{1-\delta_D} X}\right).$$

Since $-\mathfrak{A}_D \subset \mathfrak{A}'_D$, we get

$$\mathfrak{A}_D(X) = O\left(\frac{X}{\log^{1-\delta_D} X}\right).$$

Step 3. By the genus theory of binary quadratic forms [Co89, Prop. 3.11], h_D is even. Since $e_D = \omega(\overline{D}) + 2 \geq 3$, we get $\delta_D \leq \frac{3}{8}$, so $\mathfrak{A}_D(X) = O\left(\frac{X}{\log^{5/8} X}\right)$.

5. Final remarks

Suppose we want to find Hasse Principle violations for X^D over a number field k. Since $X^D(\mathbb{R}) = \varnothing$, such a k cannot have a real place, and thus the case of an imaginary quadratic field is in a certain sense minimal. Here is such a result.

Theorem 10 (Clark [Cl09, Thm. 1])**.** *If $D > 546$, then there are infinitely many quadratic fields $l/\mathbb{Q}$ such that $X^D_{/l}$ violates the Hasse Principle.*

The global input of Theorem 10 is a result of Harris-Silverman [HS91, Cor. 3]: if $X_{/k}$ is a curve, then X has infinitely many quadratic points iff X admits a degree 2 k-morphism to $\mathbb{P}^1$ or to an elliptic curve of positive rank.[8] In [Ro02], Rotger shows that the Shimura curves X^D with infinitely many quadratic points are precisely those in which X^D/w_D is $\mathbb{P}^1$ or an elliptic curve, i.e., the values of D recorded in parts a) and b) of Lemma 7. Thus under the hypotheses of the Main Theorem, for all but finitely many quadratic fields l such that $\mathcal{T}(X^D, w_D, l/\mathbb{Q})$ violate the Hasse Principle over $\mathbb{Q}$, also $X^D_{/l}$ violates the Hasse Principle: we recover Theorem 10.

Assuming the hypotheses both of Harris-Silverman and of Theorem 3, $X(k) = \varnothing$ implies that there are infinitely many quadratic extensions l/k such that $X_{/l}$ violates the Hasse Principle. However, this was already known. Consider the following.

Theorem 11 (Clark [Cl09, Thm. 7])**.** . *Let $X_{/k}$ be a curve over a number field. Assume*

(i) *We have $X(k) = \varnothing$.*
(ii) *There is no degree 2 morphism from X to $\mathbb{P}^1$ or to an elliptic curve $E_{/k}$ with positive rank.*
(iii) *For every place v, the curve $X_{/K_v}$ over the completion K_v of K at v has a closed point of degree at most 2.*

Then there are infinitely many quadratic extensions l/k such that $X_{/l}$ violates the Hasse Principle.

As in [Cl09], for a curve X over a field k, we denote by $m(X)$ the least degree of a closed point on X. For a curve X defined over a number field k, we put

$$m_{\mathrm{loc}}(X) = \operatorname{lcm} m(X_{/K_v}),$$

[8]Harris-Silverman state their result under the hypothesis that X admits no degree 2 morphism to $\mathbb{P}^1$ or to a curve of genus 1. Their argument immediately gives the stronger result, as has been noted by several authors, e.g., [Ro02, Thm. 8]. Note that the result relies on an extraordinarily deep theorem of Faltings classifying k-rational points on subvarieties of abelian varieties [Fa94].

the lcm ranging over all places of K ($m(X_{/K_v}) = 1$ for all but finitely many v). Condition (iii) in Theorem 11 is $m_{\mathrm{loc}}(X) \leq 2$. For a curve equipped with an involution defined over a number field $(X, \iota)_{/k}$, consider the following local hypotheses:

(L1) $\mathcal{T}(X, \iota, l/k)(\mathbb{A}_k) \neq \varnothing$ for some quadratic extension l/k (we allow $l = k$).

(L2) $(X/\iota)(\mathbb{A}_k) \neq \varnothing$.

(L3) $m_{\mathrm{loc}}(X) \leq 2$.

Clearly (L1) $\implies$ (L2) $\implies$ (L3). Whereas Theorem 11 uses condition (L3) to get Hasse Principle violations over quadratic extensions l/k, the Main Theorem uses condition (L1) to get Hasse Principle violations over k.

To prove Theorem 10 it suffices to establish that $m_{\mathrm{loc}}(X^D) \leq 2$ (and thus clearly $m_{\mathrm{loc}}(X^D) = 2$ since $X^D(\mathbb{R}) = \varnothing$). This is [Cl09, Thm. 8a]. We want to emphasize that the local analysis from [St14] needed to show that X^D satisfies (L1) lies considerably deeper. Thus it is our perspective that the Main Theorem, which gives Hasse Principle violations over $\mathbb{Q}$, is a deeper result than Theorem 10, which gives Hasse Principle violations over quadratic fields.

It is natural to ask whether (L1), (L2), and (L3) may in fact be equivalent.

Example 12. Lemma 7a) gives $X^{55}/w_{55} \cong \mathbb{P}^1$. Now consider the Atkin-Lehner involution w_5 on X^{55}. A result of Ogg implies that since $\left(\frac{11}{5}\right) = 1$, we have $(X^{55}/w_5)(\mathbb{R}) = \varnothing$. (See [Cl03, Thm. 57 and Cor. 42].) Since $(X^{55}, w_{55})_{/\mathbb{Q}}$ satisfies (L2), the curve $X^{55}/\mathbb{Q}$ satisfies (L3). So $(X^{55}, w_5)_{/\mathbb{Q}}$ satisfies (L3) but not (L2).

Whether (L2) implies (L1) we leave as an open question. But we observe: if $(X/\iota)(k) \neq \varnothing$, then (as in the proof of Theorem 3) some quadratic twist of (X, ι) has a k-rational point, so condition (L1) holds. So if (X, ι) satisfies (L2) but not (L1), then X/ι violates the Hasse Principle over k.

There are no known values of D in which X^D/w_D violates the Hasse Principle over $\mathbb{Q}$. When there is a class number one imaginary quadratic field K such that every prime p dividing D is inert in K, then there is a CM point on $(X^D/w_D)(\mathbb{Q})$, and these conditions apply for many small values of D. However, for a set of D's of relative density one there are no CM points on $(X^D/w_D)(\mathbb{Q})$, and the folk wisdom is that we ought to have $(X^D/w_D)(\mathbb{Q}) = \varnothing$ for "most" such D (perhaps all but finitely many), and thus it seems likely that X^D/w_D violates the Hasse Principle over $\mathbb{Q}$ for a density 1 set of D's. This is a close analogue of the problem of determining all $\mathbb{Q}$-points on $X_0(N)/w_N$. Both problems are wide open.

References

[BGW] M. Bhargava, B. Gross, and X. Wang, *Pencils of quadrics and the arithmetic of hyperelliptic curves*, to appear, J. Amer. Math. Soc.

[Cl] P. L. Clark, *Curves over global fields violating the Hasse Principle*, `alpha.math.uga.edu/~pete/HasseBjornODD.pdf`

[Cl03] Pete L. Clark, *Rational points on Atkin-Lehner quotients of Shimura curves*, ProQuest LLC, Ann Arbor, MI, 2003. Thesis (Ph.D.)–Harvard University. MR2704676

[Cl08] Pete L. Clark, *An "anti-Hasse principle" for prime twists*, Int. J. Number Theory **4** (2008), no. 4, 627–637. MR2441796

[Cl09] Pete L. Clark, *On the Hasse principle for Shimura curves*, Israel J. Math. **171** (2009), 349–365. MR2520114

[Co89] David A. Cox, *Primes of the form x^2+ny^2*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1989. Fermat, class field theory and complex multiplication. MR1028322

[El04] Jordan S. Ellenberg, *$\mathbb{Q}$-curves and Galois representations*, Modular curves and abelian varieties, Progr. Math., vol. 224, Birkhäuser, Basel, 2004, pp. 93–103. MR2058645

[Fa94] Gerd Faltings, *The general case of S. Lang's conjecture*, Barsotti Symposium in Algebraic Geometry (Abano Terme, 1991), Perspect. Math., vol. 15, Academic Press, San Diego, CA, 1994, pp. 175–182. MR1307396

[HS91] Joe Harris and Joe Silverman, *Bielliptic curves and symmetric products*, Proc. Amer. Math. Soc. **112** (1991), no. 2, 347–356. MR1055774

[Oz12] Ekin Ozman, *Points on quadratic twists of* $X_0(N)$, Acta Arith. **152** (2012), no. 4, 323–348. MR2890545

[Po06] Bjorn Poonen, *Heuristics for the Brauer-Manin obstruction for curves*, Experiment. Math. **15** (2006), no. 4, 415–420. MR2293593

[Po10] B. Poonen, *Curves over every global field violating the local-global principle* (English, with English and Russian summaries), Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **377** (2010), no. Issledovaniya po Teorii Chisel. 10, 141–147, 243–244; English transl., J. Math. Sci. (N.Y.) **171** (2010), no. 6, 782–785. MR2753654

[Ro02] Victor Rotger, *On the group of automorphisms of Shimura curves and applications*, Compositio Math. **132** (2002), no. 2, 229–241. MR1915176

[Se76] Jean-Pierre Serre, *Divisibilité de certaines fonctions arithmétiques*, Enseignement Math. (2) **22** (1976), no. 3-4, 227–260. MR0434996

[Sh75] Goro Shimura, *On the real points of an arithmetic quotient of a bounded symmetric domain*, Math. Ann. **215** (1975), 135–164. MR0572971

[St14] James Stankewicz, *Twists of Shimura curves*, Canad. J. Math. **66** (2014), no. 4, 924–960. MR3224271

Department of Mathematics, University of Georgia, Boyd Graduate Studies Research Center, Athens, Georgia 30602-7415

Email address: plclark@gmail.com

University of Bristol Department of Mathematics/Heilbronn Institute for Mathematical Research, Howard House, Queens Avenue, Bristol, BS8 1SN, United Kingdom

Current address: IDA Center for Computing Sciences, 17100 Science Drive, Bowie, Maryland 20715

Email address: stankewicz@gmail.com