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A CHARACTERIZATION OF TEMPORAL HOMOGENEITY FOR ADDITIVE PROCESSES

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ABSTRACT. In this paper, temporal homogeneity of an $\mathbb{R}^d$-valued additive process is studied. When an additive process has the stationary independent increments property, the process is called a temporally homogeneous additive process or Lévy process. Moreover, an additive process is said to have the independent increments property in a strong sense if the process has the independent increments property at every finite stopping time (that is, its increments starting at any finite stopping time and events before the stopping time are independent). This paper shows that if an additive process has the independent increments property in a strong sense, then the process is temporally homogeneous, provided the process is immediately random. In particular, in the case of additive processes with Poisson distributed independent increments, it follows that, under some non-degeneracy conditions, the temporal homogeneity is equivalent to the independent increments property at the first jumping time and that, in the degenerate cases, whether each process has the independent increments property at the first jumping time is determined.

1. PRELIMINARIES

Let $\{X(t)\}_{t\geq 0}$ be a stochastically continuous $\mathbb{R}^d$-valued càdlàg process defined on a probability space $(\Omega, \mathcal{F}, P)$ with $X(0) = 0$: a càdlàg process means that its almost every sample function is càdlàg, i.e., right-continuous on $[0, \infty)$ with left hand limits on $(0, \infty)$. For simplicity, we assume that every sample function of $\{X(t)\}_{t\geq 0}$ is càdlàg. If $\{X(t)\}_{t\geq 0}$ has the independent increments property, that is, if $X(t_0)$, $X(t_1)-X(t_0)$, $X(t_2)-X(t_1), \ldots, X(t_n)-X(t_{n-1})$ are independent for any $n \geq 1$ and $0 \leq t_0 < t_1 < \cdots < t_n$, it is called an additive process. If an additive process has temporal homogeneity, that is, if the distribution of $X(t+s)-X(s)$ does not depend on $s \geq 0$ for each $t > 0$, it is called a temporally homogeneous additive process or Lévy process. We use two filtrations $\{\mathcal{B}_t\}_{t\geq 0}$ and $\{\mathcal{F}_t\}_{t\geq 0}$: one is given by $\mathcal{B}_t = \sigma(X(s); 0 \leq s \leq t)$, the σ-field generated by $\{X(s); 0 \leq s \leq t\}$, and the other is the minimal augmentation of $\{\mathcal{B}_t\}_{t\geq 0}$ which satisfies the usual condition (see [2], pp. 2–3 and [3], p. 1 for the augmentation and the usual condition of a filtration), that is,

$$\mathcal{F}_t = \mathcal{B}_{t+} \vee \mathcal{N} = \bigcap_{\varepsilon>0} (\mathcal{B}_{t+\varepsilon} \vee \mathcal{N}),$$

where $\mathcal{N}$ denotes the family of null sets of the P-completion of the σ-field $\mathcal{F}$ and $\mathcal{B}_{t+} \vee \mathcal{N}$ is the σ-field generated by $\mathcal{B}_{t+} \cup \mathcal{N}$.

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Clearly the independent increments property is equivalent to the property that

$$X(t+s)-X(s) \text{ and } \mathcal{B}_s \text{ are independent for each } s,\, t \geq 0.$$

It is known that this is equivalent also to the property that

$$X(t+s)-X(s) \text{ and } \mathcal{F}_s \text{ are independent for each } s,\, t \geq 0.$$

Let us consider an analogous property at a stopping time. A mapping σ from Ω into $[0,\infty]$ is called a $\{\mathcal{B}_t\}$-stopping time if $\{\sigma \leq t\} \in \mathcal{B}_t$ for every $t \geq 0$. For any $\{\mathcal{B}_t\}$-stopping time σ, the σ-field $\mathcal{B}_\sigma$ is defined as the class of all sets $B \in \mathcal{F}$ satisfying $B \cap \{\sigma \leq t\} \in \mathcal{B}_t$ for $t \geq 0$. For a finite $\{\mathcal{B}_t\}$-stopping time σ, we say that $\{X(t)\}_{t\geq 0}$ has independent increments property at σ if

$$X(t+\sigma)-X(\sigma) \text{ and } \mathcal{B}_\sigma \text{ are independent for each } t \geq 0.$$

We say that $\{X(t)\}_{t\geq 0}$ has the *independent increments property in a strong sense* with respect to $\{\mathcal{B}_t\}$ if it has the independent increments property at every finite $\{\mathcal{B}_t\}$-stopping time σ. Similarly we define the notion of an $\{\mathcal{F}_t\}$-stopping time σ, the σ-field $\mathcal{F}_\sigma = \{B \in \mathcal{F} \vee \mathcal{N}\colon B \cap \{\sigma \leq t\} \in \mathcal{F}_t \text{ for } t \geq 0\}$ for an $\{\mathcal{F}_t\}$-stopping time σ, the independent increments property at an $\{\mathcal{F}_t\}$-stopping time σ, and the independent increments property in a strong sense with respect to $\{\mathcal{F}_t\}$. Since any $\{\mathcal{B}_t\}$-stopping time is an $\{\mathcal{F}_t\}$-stopping time, the independent increments property in a strong sense with respect to $\{\mathcal{F}_t\}$ yields the same property with respect to $\{\mathcal{B}_t\}$. The converse also holds as described in Remark 2.4 (i) below.

It is known that any Lévy process has the independent increments property in a strong sense with respect to $\{\mathcal{F}_t\}$, hence also with respect to $\{\mathcal{B}_t\}$. However, the problem of what additive processes have the independent increments property in a strong sense has not yet been studied to the author's knowledge. Some people might think a large class of temporally inhomogeneous additive processes have this property. A main result of this paper is that, on the contrary, under a weak condition called immediate randomness, an additive process has the independent increments property in a strong sense (with respect to either $\{\mathcal{F}_t\}$ or $\{\mathcal{B}_t\}$) if and only if it is temporally homogeneous.

Given an $\mathbb{R}^d$-valued additive process $\{X(t)\}_{t\geq 0}$, let us introduce its *randomness time* t_* as

$$t_* = \inf\{t \geq 0\colon 0 < P(B) < 1 \text{ for some } B \in \mathcal{B}_t\}.$$

We say that $\{X(t)\}_{t\geq 0}$ is *immediately random* if $t_* = 0$. Let us describe t_* by the distribution of $X(t)$ for $t \geq 0$. For any $\mathbb{R}^d$-valued additive process $\{X(t)\}_{t\geq 0}$, the distribution of $X(t)$ is infinitely divisible and hence its characteristic function has the Lévy–Khintchine representation

$$E[e^{i\langle X(t),z\rangle}] = \exp\left[-\frac{1}{2}\langle V(t)z,z\rangle + i\langle m(t),z\rangle + \int_{\mathbb{R}^d}(e^{i\langle z,u\rangle} - 1 - i\langle z,u\rangle 1_{\{|u|\leq 1\}}(u))n_t(du)\right]$$

for $z \in \mathbb{R}^d$ in a unique way, where $V(t)$ is a symmetric nonnegative-definite $d \times d$ matrix, $m(t) \in \mathbb{R}^d$, and n_t is a measure on $\mathbb{R}^d$ with $n_t(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|u|^2 \wedge 1)n_t(du) < \infty$. We call $\{(V(t), n_t, m(t))\}_{t\geq 0}$ the system of generating triplets of $\{X(t)\}_{t\geq 0}$. This system satisfies the conditions that $(V(t), n_t, m(t)) = (0,0,0)$ for $t = 0$, $V(t) - V(s)$ is nonnegative-definite and $n_t - n_s \geq 0$ for $t > s$, and $V(t)$, $m(t)$, and $n_t(B)$ are continuous in $t \geq 0$ for any Borel set B strictly away from 0. Moreover, every increment $X(t) - X(s)$ $(t \geq s)$ of the additive process has an infinitely divisible distribution with generating triplet $(V(t) - V(s), n_t - n_s, m(t) - m(s))$. (See [5], Theorems 8.1 and 9.8.) The randomness time t_* is characterized as

$$t_* = \inf\{t \geq 0\colon V(t) \neq 0 \text{ or } n_t \neq 0\},$$

since $X(s) = m(s)$ for $s \in [0, t_*]$ a.s.

2. Results on temporal homogeneity

Let us prove the following.

Theorem 2.1. *Let $\{X(t)\}_{t\geq 0}$ be an immediately random $\mathbb{R}^d$-valued additive process. Then it has the independent increments property in a strong sense with respect to $\{\mathcal{B}_t\}$ if and only if it is a Lévy process.*

It is enough to show the "only if" part. We prepare two lemmas.

Lemma 2.2. *Let $\{X(t)\}_{t\geq 0}$ be an $\mathbb{R}^d$-valued additive process with the system of generating triplets $\{(V(t), n_t, m(t))\}_{t\geq 0}$ and let σ be a finite $\{\mathcal{F}_t\}$-stopping time with distribution μ_σ. Then $\{X(t)\}_{t\geq 0}$ has the independent increments property at σ if and only if, for each $t \geq 0$, the distribution of $X(t+r) - X(r)$ does not depend on r as long as $r \in \operatorname{supp}(\mu_\sigma)$, that is, if and only if $V(t+r) - V(r)$, $n_{t+r} - n_r$ and $m(t+r) - m(r)$ do not depend on r of $\operatorname{supp}(\mu_\sigma)$.*

Proof. Given $t \geq 0$ and a bounded continuous function $f(x)$ on $\mathbb{R}^d$, let

$$\begin{aligned} g_{t;f}(r) &= E[f(X(t+r) - X(r))] \quad \text{for } r \geq 0, \\ c_{t;f} &= E[f(X(t+\sigma) - X(\sigma))]. \end{aligned}$$

First we show that the equality

$$E[f(X(t+\sigma) - X(\sigma))1_A] = E[f(X(t+\sigma) - X(\sigma))]P(A) \quad \text{for every } A \in \mathcal{F}_\sigma \tag{2.1}$$

implies the equality

$$g_{t;f}(r) = c_{t;f} \quad \text{for every } r \in \operatorname{supp}(\mu_\sigma) \tag{2.2}$$

and vice versa. Notice that $g_{t;f}(r)$ is continuous in (t, r), because $\{X(t)\}$ is stochastically continuous. Let $\sigma_n = \sum_{k=1}^\infty k2^{-n}1_{\{(k-1)2^{-n}\leq\sigma<k2^{-n}\}}$. Then $\sigma_n \downarrow \sigma$ as $n \to \infty$. For $A \in \mathcal{F}_\sigma$ we have

$$\begin{aligned} E[f(X(t+\sigma) - X(\sigma))1_A] &= \lim_{n\to\infty} E[f(X(t+\sigma_n) - X(\sigma_n))1_A] \\ &= \lim_{n\to\infty} \sum_{k=1}^\infty E[f(X(t+k2^{-n}) - X(k2^{-n}))1_{A\cap\{(k-1)2^{-n}\leq\sigma<k2^{-n}\}}] \\ &= \lim_{n\to\infty} \sum_{k=1}^\infty E[f(X(t+k2^{-n}) - X(k2^{-n}))]P(A \cap \{(k-1)2^{-n} \leq \sigma < k2^{-n}\}) \\ &= \lim_{n\to\infty} \sum_{k=1}^\infty g_{t;f}(k2^{-n})P_A((k-1)2^{-n} \leq \sigma < k2^{-n}) \\ &= \int_{[0,\infty)} g_{t;f}(r)P_A(\sigma \in dr), \end{aligned}$$

where we define $P_A(B) = P(A \cap B)$ for $B \in \mathcal{F} \vee \mathcal{N}$. In the third equality above we made use of independent increments property and $A \cap \{(k-1)2^{-n} \le \sigma < k2^{-n}\} \in \mathcal{F}_{k2^{-n}}$. In particular we have

$$c_{t;f} = \int_{[0,\infty)} g_{t;f}(r)\mu_\sigma(dr), \tag{2.3}$$

letting $A = \Omega$. If the equality (2.1) holds, then we get

$$\int_{[0,\infty)} g_{t;f}(r)P_A(\sigma \in dr) = c_{t;f}P(A).$$

Letting $A = \{\sigma \ge s\}$, we see that

$$\int_{[s,\infty)} g_{t;f}(r)\mu_\sigma(dr) = c_{t;f}\int_{[s,\infty)} \mu_\sigma(dr).$$

It follows that

$$g_{t;f}(r) = c_{t;f} \quad \text{for } \mu_\sigma\text{-a.e. } r \in [0,\infty).$$

Since $g_{t;f}(r)$ is continuous, we have, for any $r \in \operatorname{supp}(\mu_\sigma)$,

$$|g_{t;f}(r) - c_{t;f}| = \lim_{\delta\to 0} \frac{1}{\mu_\sigma([r-\delta, r+\delta])}\int_{[r-\delta,r+\delta]} |g_{t;f}(s) - c_{t;f}|\mu_\sigma(ds) = 0.$$

Thus we get the equality (2.2). Conversely, assume that the equality (2.2) holds. For $A \in \mathcal{F}_\sigma$, let $\mu_{A;\sigma}(dr) = P_A(\sigma \in dr)$. Then $\mu_{A;\sigma} \le \mu_\sigma$ and hence $\operatorname{supp}(\mu_{A;\sigma}) \subset \operatorname{supp}(\mu_\sigma)$. Therefore

$$g_{t;f}(r) = c_{t;f} \quad \text{for all } r \in \operatorname{supp}(\mu_{A;\sigma})$$

and it implies

$$E[f(X(t+\sigma) - X(\sigma))1_A] = \int_{[0,\infty)} g_{t;f}(r)P_A(\sigma \in dr) = c_{t;f}P(A);$$

that is, the equality (2.1) is verified. Thus, by the arbitrariness of f, we see that the independent increments property at σ yields that the distribution of $X(t+r) - X(r)$ is common to all $r \in \operatorname{supp}(\mu_\sigma)$ and the converse also holds. Finally, since the distribution of $X(t+r) - X(r)$ has the generating triplet $(V(t+r) - V(r), n_{t+r} - n_r, m(t+r) - m(r))$, the distribution and its generating triplet do not depend on $r \in \operatorname{supp}(\mu_\sigma)$ simultaneously. □

Lemma 2.3. *Let $\{X(t)\}_{t\ge 0}$ be an $\mathbb{R}^d$-valued additive process with randomness time t_*. Then, for any $s > t_*$, there is a finite $\{\mathcal{B}_t\}$-stopping time σ satisfying* $\operatorname{supp}(\mu_\sigma) = [s,\infty)$.

Proof. Let $s > t_*$. Then we have either $V(s) \ne 0$ or $n_s \ne 0$.

Case 1. Suppose that $n_s \ne 0$. Choose $\varepsilon > 0$ such that $n_s(\{|x| > \varepsilon\}) = c > 0$. By the well-known Lévy–Itô decomposition of sample functions of an additive process, the number of jumps up to time s with magnitude $> \varepsilon$ has mean c; more precisely, the number J of $t \in (0, s]$ satisfying $|X(t) - X(t-)| > \varepsilon$ has Poisson distribution with mean c. Let $A_j = \{J = j\}$ for $j = 0, 1, 2, \ldots$. Then $P(A_j) > 0$ and $A_j \in \mathcal{B}_s$ for all j and $\{A_j\}_{j=0}^\infty$ gives a countable decomposition of Ω. For any sequence $\{t_j\}_{j=0}^\infty$ in $[s,\infty)$ with $t_j \ne t_k$ for $j \ne k$, let $\sigma = \sum_{j=0}^\infty t_j 1_{A_j}$. Then σ is a finite $\{\mathcal{B}_t\}$-stopping time (suggested by [4], p. 71, Proof of Proposition (3.5)). Choosing $\{t_j\}_{j=0}^\infty$ dense in $[s,\infty)$, we obtain $\operatorname{supp}(\mu_\sigma) = [s,\infty)$.

Case 2. Suppose that $n_s = 0$. Then $X(s)$ is Gaussian-distributed with mean $m(s)$ and covariance $V(s) \neq 0$. The distribution is supported on a hyperplane in $\mathbb{R}^d$ with dimension $= \operatorname{rank} V(s) \geq 1$. Therefore there exists a countable decomposition of Ω with the same property as in Case 1, and the rest of the proof is similar. $\square$

Proof of Theorem 2.1. Since $t_* = 0$, we can apply Lemma 2.3 to any $s > 0$ and find a $\{\mathcal{B}_t\}$-stopping time σ with $\operatorname{supp}(\mu_\sigma) = [s, \infty)$. It follows from the independent increments property in a strong sense with respect to $\{\mathcal{B}_t\}$ and from Lemma 2.2 that the distribution of $X(t + r) - X(r)$ is common to all $r \in [s, \infty)$. Since s is arbitrary, this means temporal homogeneity. $\square$

Remark 2.4. (i) Theorem 2.1 remains true if "with respect to $\{\mathcal{B}_t\}$" is replaced by "with respect to $\{\mathcal{F}_t\}$". Indeed, on the one hand, the independent increments property in a strong sense with respect to $\{\mathcal{F}_t\}$ implies the property in a strong sense with respect to $\{\mathcal{B}_t\}$ by definition, and on the other hand it is known that any Lévy process has the property in a strong sense with respect to $\{\mathcal{F}_t\}$. Recall that hitting times of a set in a large class are shown to be $\{\mathcal{F}_t\}$-stopping times, not $\{\mathcal{B}_t\}$-stopping times.

(ii) If every sample function of the process $\{X(t)\}_{t\geq 0}$ is not always càdlàg, it is hard to check the conclusion of Lemma 2.3 (but the proof remains valid if $\{\mathcal{B}_t\}$ is replaced by $\{\mathcal{F}_t\}$). However, Theorem 2.1 remains true. Indeed, for any $t_1, t_2 > t_*$ with $t_1 \neq t_2$, there is a $\{\mathcal{B}_t\}$-stopping time σ with $\operatorname{supp}(\mu_\sigma) = \{t_1, t_2\}$. Hence it follows that the union of $\operatorname{supp}(\mu_\sigma)$ for such stopping times σ is dense in $[t_*, \infty)$.

(iii) For any $\{\mathcal{F}_t\}$-stopping time σ with $P(\sigma < \infty) > 0$, the notion of the independent increments property at σ is generalized in a natural way. Then Lemma 2.2 is true for such a stopping time σ. The proof remains valid and (2.2) holds with $\operatorname{supp}(\mu_\sigma)$ replaced by $\operatorname{supp}(\mu_\sigma) \cap [0, \infty)$.

(iv) Lemma 2.2 (or its generalization mentioned just above) says the following as a special case. If an $\mathbb{R}^d$-valued additive process $\{X(t)\}_{t\geq 0}$ has the independent increments property at an $\{\mathcal{F}_t\}$-stopping time σ with full support (that is, $\operatorname{supp}(\mu_\sigma) \cap [0, \infty) = [0, \infty)$), then $\{X(t)\}_{t\geq 0}$ is a Lévy process.

An additive process $\{X(t)\}_{t\geq 0}$ with $0 < t_* < \infty$ is not temporally homogeneous, since, for sufficiently small $s > 0$, $X(s) - X(0)$ is nonrandom but $X(s + t_*) - X(t_*)$ is truly random. However, Theorem 2.1 is extended in the following way (due to Sato [7]).

Theorem 2.5. *Let $\{X(t)\}_{t\geq 0}$ be an $\mathbb{R}^d$-valued additive process with $t_* < \infty$ and let $Y(t) = X(t+t_*) - X(t_*)$. Then $\{X(t)\}_{t\geq 0}$ has the independent increments property in a strong sense with respect to $\{\mathcal{B}_t\}$ if and only if $\{Y(t)\}_{t\geq 0}$ is a Lévy process.*

Proof. We have $X(t_*) = m(t_*)$ a.s., as is mentioned at the end of Section 1. In general the process $\{Y(t)\}_{t\geq 0}$ is an additive process. Let $\mathcal{B}^Y_t = \sigma(Y(s); 0 \leq s \leq t)$. Then we have $\mathcal{B}^Y_t = \mathcal{B}_{t+t_*}$ for $t \geq 0$. If τ is a $\{\mathcal{B}^Y_t\}$-stopping time, then $\sigma := \tau + t_*$ is a $\{\mathcal{B}_t\}$-stopping time and $\mathcal{B}_\sigma = \mathcal{B}^Y_\tau$. Conversely, for any $\{\mathcal{B}_t\}$-stopping time σ, $P(\sigma < t_*) = 0$ or 1 and in the case $P(\sigma < t_*) = 0$, $\tau := \sigma - t_*$ is a $\{\mathcal{B}^Y_t\}$-stopping time; in the case $P(\sigma < t_*) = 1$, σ is a constant time less than t_*. Hence, $\{X(t)\}_{t\geq 0}$ has the independent increments property in a strong sense with respect to $\{\mathcal{B}_t\}$ if and only if $\{Y(t)\}_{t\geq 0}$ has the independent increments property in a strong sense with respect to $\{\mathcal{B}^Y_t\}$. Thus application of Theorem 2.1 to $\{Y(t)\}_{t\geq 0}$ finishes the proof of the theorem. $\square$

3. Examples

We study some examples of additive processes and stopping times σ. Applying Lemma 2.2 or its generalization mentioned in Remark 2.4 (iii), we examine whether the process has the independent increments property at σ. First consider Poisson distributed additive processes and their first jumping times (see [1], 1.4, Theorems 1, 2 and 3 for the basic properties of such processes; there such processes are called Lévy processes of Poisson type unlike the present usage of the word Lévy process).

Proposition 3.1. *Let $\{X(t)\}_{t\geq 0}$ be a one-dimensinal additive process such that $X(t)$ has Poisson distribution with mean $\lambda(t) \not\equiv 0$ and let σ be the first jumping time for the process. Then the following hold:*

(i) Assume that $\lambda(t)$ is strictly increasing and $\lim_{t\to\infty}\lambda(t) = \infty$. Then $\{X(t)\}_{t\geq 0}$ is temporally homogeneous, i.e., $\lambda(t) = \lambda(1)t$ for $t \geq 0$, if and only if it has the independent increments property at σ.

(ii) Assume that $\lambda(t)$ is either not strictly increasing or bounded. Then $\lambda(t) = \lambda(1 + t_0)((t - t_0) \vee 0)$ for $t \geq 0$ with some $t_0 > 0$ if and only if $\{X(t)\}_{t\geq 0}$ has the independent increments property at σ.

Proof. In general $\lambda(t)$ is continuous, increasing, and $\lambda(0) = 0$. The process $\{X(t)\}_{t\geq 0}$ is called a Poisson distributed additive process with mean function $\lambda(t)$: It is realized as $X(t) = Z(\lambda(t))$ by time change of Poisson process $\{Z(t)\}_{t\geq 0}$ with parameter 1, and its system of generating triplets is given by $\{(0, \lambda(t)\delta_1, \lambda(t))\}_{t\geq 0}$, where δ_1 denotes the probability measure concentrated at 1. Moreover, the first jumping time σ is a $\{\mathcal{B}_t\}$-stopping time and $\mu_\sigma(dr) = -de^{-\lambda(r)}$.

(i) It is enough to check the "if" part. From the assumption it follows that μ_σ has full support. Hence we see the "if" part from Remark 2.4 (iv).

(ii) The "only if" part is immediate from Theorem 2.5. It is enough to check the "if" part. Take the left-continuous inverse $\lambda^-(u)$ of $\lambda(t)$:

$$\lambda^-(u) = \inf\{t \geq 0 : \lambda(t) \geq u\} \quad (u \geq 0).$$

Then $\lambda^-(u)$ is an extended-real-valued left-continuous increasing function of $u \in [0, \infty)$ and it is strictly increasing on $[0, \lambda(\infty))$, where $\lambda(\infty) = \lim_{t\to\infty}\lambda(t)$. Moreover, it fulfills the following conditions:

$$\lambda^-(\lambda(t)) \leq t \quad (t \in [0, \infty));\ \lambda(\lambda^-(u)) = u \quad (u \in [0, \lambda(\infty))).$$

Let

$$D(\lambda^-) = \{u \in [0, \infty) : \lambda^-(u+) - \lambda^-(u) > 0\}.$$

Then we see that

$$\lambda^-([0, \infty)) \cap (0, \infty) \subset \operatorname{supp}(\mu_\sigma) \cap (0, \infty) = (0, \infty) \setminus \bigcup_{u\in D(\lambda^-)} (\lambda^-(u), \lambda^-(u+)).$$

First consider the case $D(\lambda^-) \cap (0, \infty) \neq \emptyset$. Take a $u_0 \in D(\lambda^-) \cap (0, \infty)$. Then for $t \in (0, (\lambda^-(u_0+) - \lambda^-(u_0))/2)$, choose a $\delta > 0$ so small that $0 < \lambda^-(u_0) - \lambda^-(u) < t$ for any $u \in (u_0 - \delta, u_0)$. For such u, it holds

$$\lambda^-(u) < \lambda^-(u_0) < t + \lambda^-(u) < \lambda^-(u_0+).$$

Since $\lambda(v) = u_0$ for $v \in [\lambda^-(u_0), \lambda^-(u_0+)]$, we have

$$\lambda(t + \lambda^-(u)) - \lambda(\lambda^-(u)) = u_0 - u \text{ for all } u \in (u_0 - \delta, u_0),$$

which depends on $\lambda^-(u) \in \operatorname{supp}(\mu_\sigma) \cap (0,\infty)$. Hence we see that the process does not have the independent increments property at σ. Second consider the case $D(\lambda^-) = \{0\}$. Then $t_0 = \lambda^-(0+) > 0$ and $\lambda(t)$ is strictly increasing on $[t_0,\infty)$ with $\lambda(t_0) = 0$; hence $\operatorname{supp}(\mu_\sigma) \cap [0,\infty) = [t_0,\infty)$. Put $Y(t) = X(t+t_0) - X(t_0)$. Then $\{Y(t)\}_{t\geq 0}$ is a Poisson distributed additive process with mean function $\lambda(t+t_0)$. Therefore, by using Theorem 2.5, Remark 2.4 (iv) and the first part (i) of the proposition, we can see that if $\lambda(t) \not\equiv \lambda(1+t_0)((t-t_0)\vee 0)$ $(t \geq 0)$, then $\{X(t)\}_{t\geq 0}$ does not have the independent increments property at σ. Finally consider the last case $D(\lambda^-) = \emptyset$. In this case, $\lambda(t)$ is strictly increasing and bounded on $[0,\infty)$ and hence $\operatorname{supp}(\mu_\sigma) \cap [0,\infty) = [0,\infty)$. Using Remark 2.4 (iv) and the first part (i) of the proposition, we see that $\{X(t)\}_{t\geq 0}$ does not have the independent increments property at σ. Consequently, the "if" part is verified. □

Continue to consider a Poisson distributed additive process with mean function $\lambda(t)$ briefly. In the case of $\lambda(t) \not\equiv \lambda(1)t$ and $\lambda^-(0+) = 0$, the process is immediately random and hence the independent increments property is violated at some finite stopping time. But, whether it has the independent increments property at a specific finite stopping time is another problem.

The first part (i) of the proposition above is somewhat generalized as follows: Let $\{X(t)\}_{t\geq 0}$ be a pure jump type $\mathbb{R}^d$-valued additive process with $n_t(\mathbb{R}^d) < \infty$ for $t \geq 0$ such that

$$E[e^{i\langle X(t),z\rangle}] = \exp\left[\int_{\mathbb{R}^d} (e^{i\langle z,u\rangle} - 1) n_t(du)\right].$$

Assume that $n_t(\mathbb{R}^d)$ is strictly increasing in t and tends to infinity as $t \to \infty$. Then the temporal homogeneity of the process is equivalent to the independent increments property at the first jumping time.

By using the idea of time change, we can obtain the same result as Proposition 3.1 (i) for a one-dimensional continuous additive process with mean 0 (that is, an additive process with the system of generating triplets $\{(V(t),0,0)\}_{t\geq 0}$). It is enough to replace σ, $\lambda(t)$, and $\{Z(t)\}_{t\geq 0}$ by the first leaving time from the interval $(-1,1)$, $V(t)$, and a Wiener process $\{B(t)\}_{t\geq 0}$, respectively.

We should notice that the idea of time change has already been used in [6] to analyze some singular additive processes.

Finally we consider an additive process defined as the sum of a Poisson process $\{Z(t)\}_{t\geq 0}$ and a temporally inhomogeneous increasing function (due to Sato [7]).

Example 3.2. Let $X(t) = Z(t) + (t \wedge 1)$ and $\sigma = \inf\{t > 0\colon X(t) \geq 1\}$. Then the additive process $\{X(t)\}_{t\geq 0}$ does not have the independent increments property at σ, $\operatorname{supp}(\mu_\sigma) \neq [0,\infty)$, and it is not temporally homogeneous.

Indeed, since the process involves a nonlinear motion $t \wedge 1$, it is not temporally homogeneous. We have $\sigma = \tau \wedge 1$, where τ is the first jumping time for $\{Z(t)\}_{t\geq 0}$, and $\{X(t)\}_{t\geq 0}$ has the system of generating triplets $\{(0, t\delta_1, m(t))\}_{t\geq 0}$ with $m(t) = t + (t \wedge 1)$. For $t \geq 1$, $m(t+r) - m(r)$ depends on $r \in \operatorname{supp}(\mu_\sigma) = [0,1]$, which shows that $X(t+\sigma) - X(\sigma)$ and $\mathcal{F}_\sigma$ are not independent.

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