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ROBUSTNESS OF EXPONENTIAL ATTRACTORS FOR DAMPED KORTEWEG-DE VRIES EQUATIONS

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ABSTRACT. In this paper, we study the long-time behaviour of the solutions of the Korteweg-de Vries equations with localized dampings in a bounded domain. It is shown that, under appropriate assumptions on the dampings, these equations possess robust families of exponential attractors in the corresponding phase space.

1. INTRODUCTION

In this paper, we consider the Korteweg-de Vries (KdV) equations posed on a finite interval with localized dampings:

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x + a^\varepsilon(x)u = 0 & \text{in } I \times (0, +\infty), \\ u(0,t) = u(L,t) = u_x(L,t) = 0 & \text{in } (0, +\infty), \\ u(x,0) = u_0(x) & \text{in } I, \end{cases} \tag{1.1}$$

where $I = (0, L)(L > 0)$ and $\{a^\varepsilon\}_{0 \leq \varepsilon \leq 1}$ is a family of given nonnegative functions. Our aim is to study the existence and the robustness of exponential attractors for (1.1).

The study of the long-time behavior of equations arising from mechanics and physics is very important, as it is essential, for practical purposes, to understand and predict the asymptotic behavior of the system. The natural object which is used to describe the behavior of solutions of nonlinear evolutionary partial differential equations is the global attractor of these equations. The global attractor is the unique invariant compact set which (uniformly) attracts bounded sets of initial data. The existence of global attractors and estimates of their fractal and Hausdorff dimension were obtained for many equations of mathematical physics. Nevertheless, the global attractor may present two major defaults for practical purposes. Indeed, the rate of attraction of the trajectories may be small and it may be sensible to perturbations. In order to overcome these difficulties, Foias, Sell, and Temam proposed in [6] the notion of inertial manifold, which is a smooth finite-dimensional hyperbolic (and thus robust) positively invariant manifold which contains the global attractor and exponentially attracts the trajectories. But the conditions under which it is possible to prove the existence of an inertial manifold (the gap condition

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on the spectrum of the principal linear part) is very restrictive. In many cases inertial manifolds do not exist. Thus, as an intermediate object between the global attractor and the inertial manifold, Eden, Foias, Nicolaenko, and Temam proposed in [4] the notion of exponential attractor, which is a compact positively invariant set which contains the global attractor, has finite fractal dimension, and exponentially attracts the trajectories.

The KdV equation

$$u_t + u_{xxx} + uu_x = 0$$

was first derived by Korteweg and de Vries [11] in 1895 (or by Boussinesq [3] in 1877) as a model for propagation of some surface water waves along a channel. x is often proportional to distance in the direction of propagation, and t is proportional to the elapsed time. The equation is now commonly accepted as a mathematical model for the unidirectional propagation of small amplitude long waves in nonlinear dispersive systems. In many real situations, however, one cannot neglect energy dissipation mechanisms and external excitation, thus many authors considered the KdV equation in the form

$$u_t + u_{xxx} + uu_x + \mathcal{L}(u) = f, \tag{1.2}$$

where f represents the external excitation and $\mathcal{L}(u)$ is the damping term. Depending on the physical situation, $\mathcal{L}(u)$ can be a differential operator or even a pseudo-differential operator. In many articles, the authors consider (1.2) with $\mathcal{L}(u) = \gamma u$, where γ is a positive constant. The global attractors for the KdV type equations with damping term $\mathcal{L}(u) = \gamma u$ were considered in [10, 12, 14, 16, 20, 21] for $x \in \mathbb{R}$ and in [7–9, 13, 22] for $x \in \mathbb{T}$.

In applications, we may observe the propagation of water waves in a bounded channel; however, there are to date few results concerning the attractors for the KdV type equations in a bounded domain. Notice that, as it was suggested in [2], the extra term u_x should be incorporated in the equation in order to obtain an appropriate model for water waves in a uniform channel. In this work, we study the long-time behavior for (1.1). The damping term $a^\varepsilon u$ in (1.1) is weaker than γu if we require that the support set of a^ε is contained in an open subset of I; consequently, we cannot obtain the decay of the solution by Gronwall inequality directly. Assuming that $a^\varepsilon = a$ satisfies

$$a \in H^1(I) \text{ and } a(x) \geq a_0 > 0 \text{ a.e. in } \omega,$$

where ω is any nonempty open subset of I, [15] and [18] investigated the exponential decay for the solution of (1.1).

The rest of this paper is organized as follows. Section 2 is devoted to some abstract theories which will be used in this paper. In Section 3, we prove that (1.1) possesses robust families of exponential attractors in the corresponding phase space.

2. Abstract theory

Let X be a Banach space; we usually denote the norm in X by $\|\cdot\|_X$. We indicate by

$$\operatorname{dist}_X(B_1, B_2) := \sup_{U\in B_1} \inf_{V\in B_2} \|U - V\|_X$$

the Hausdorff semidistance in X from a set B_1 to a set B_2. Let $\mathrm{dist}_{\mathrm{sym}}(B_1, B_2)$ denote the symmetric Hausdorff distance between B_1 and B_2 defined by

$$\mathrm{dist}_{\mathrm{sym}}(B_1, B_2) := \max\{\mathrm{dist}_X(B_1, B_2), \mathrm{dist}_X(B_2, B_1)\}.$$

Let $\mathcal{X}$ be a subset of X and let $S(t)$, $0 \le t < \infty$, be a family of continuous mappings from $\mathcal{X}$ into itself with the properties: (i) $S(0) = I$ (the identity mapping) and (ii) $S(t)S(s) = S(t+s)$, $0 \le t, s < \infty$ (the semigroup property). Such a family is called a (nonlinear) semigroup acting on $\mathcal{X}$. For each $U_0 \in \mathcal{X}$, $S(t)U_0$ defines a function for $t \in [0, \infty)$ with values in $\mathcal{X}$; this function is called a trajectory starting from U_0. The space of all trajectories is called a dynamical system with phase space $\mathcal{X}$ in the universal space X and is denoted by $(S(t), \mathcal{X}, X)$.

Now, we present the definition of the exponential attractor which can be found in [1, 4].

Definition 2.1. A nonempty subset $\mathcal{M} \subset \mathcal{X}$ is called an exponential attractor for $(S(t), \mathcal{X}, X)$ if

- $\mathcal{M}$ is a compact subset of X with finite fractal dimension;
- $\mathcal{M}$ is a positively invariant set of $S(t)$, namely, $S(t)\mathcal{M} \subset \mathcal{M}$ for every $t \ge 0$;
- $\mathcal{M}$ attracts $\mathcal{X}$ exponentially in the following sense:

$$\mathrm{dist}_X(S(t)\mathcal{X}, \mathcal{M}) \le Ce^{-\delta t},\ t \ge 0,$$

with some exponent $\delta > 0$ and a constant $C > 0$.

In the next section, we will use the following abstract result which was proved in [5].

Theorem 2.1. *Let X and Z be Banach spaces such that the embedding $Z \hookrightarrow X$ is compact. Let $(S_\varepsilon(t), \mathcal{X}_\varepsilon, X)$ be a family of dynamical systems which are defined for $0 \le \varepsilon \le 1$ with compact phase spaces $\mathcal{X}_\varepsilon$ of X for all $0 \le \varepsilon \le 1$. Assume that*

- *there exists a uniform absorbing set $B \in \bigcap_{0\le\varepsilon\le 1} \mathcal{X}_\varepsilon$ and $T^* > 0$ such that*

$$S_\varepsilon(t)\mathcal{X}_\varepsilon \subset B \text{ for every } t \ge T^* \tag{2.1}$$

for all $0 \le \varepsilon \le 1$;
- *$S_\varepsilon(T^*)$, $0 \le \varepsilon \le 1$, satisfy a compact Lipschitz condition*

$$\|S_\varepsilon(T^*)U - S_\varepsilon(T^*)V\|_Z \le L\|U - V\|_X,\ U, V \in \mathcal{X}_\varepsilon \tag{2.2}$$

with some uniform constant $L > 0$;
- *$S_\varepsilon(t)$, $0 \le \varepsilon \le 1$, satisfy a Lipschitz condition*

$$\|S_\varepsilon(t_1)U - S_\varepsilon(t_2)V\|_X \le D(|t-s| + \|U-V\|_X),\ 0 \le t_1, t_2 \le T^*,\ U, V \in \mathcal{X}_\varepsilon \tag{2.3}$$

on the interval $[0, T^*]$ with some uniform constant $D > 0$;
- *there exists some constant $K > 0$ such that*

$$\sup_{U\in B}\ \sup_{0\le t\le T^*} \|S_\varepsilon(t)U - S_0(t)U\|_X \le K\varepsilon \tag{2.4}$$

for all $0 \le \varepsilon \le 1$.

Then, there exist exponential attractors $\mathcal{M}_\varepsilon$ for $(S_\varepsilon(t), \mathcal{X}_\varepsilon, X)$, $0 \le \varepsilon \le 1$, respectively, for which the estimate

$$dist_{sym}(\mathcal{M}_\varepsilon, \mathcal{M}_0) \le C\varepsilon^\kappa,\ 0 < \varepsilon \le 1,$$

holds with some exponent $0 < \kappa < 1$ and constant $C > 0$.

3. Existence and robustness of exponential attractors

In this section, we proceed to show the existence and robustness of the exponential attractors for (2.1) applying Theorem 2.1.

We begin with some assumptions on $\{a^\varepsilon\}_{0\le\varepsilon\le1}$. Let $\{\omega^\varepsilon\}_{0\le\varepsilon\le1}$ be a family of open, nonempty subsets of I. Assume that $\{a^\varepsilon\}_{0\le\varepsilon\le1}$ is a family of nonnegative $H^1(I)$-valued functions satisfying

$$\omega^\varepsilon \subset \operatorname{supp} a^\varepsilon,\ \|a^\varepsilon\|_{H^1(I)} \le C \text{ and } \|a^\varepsilon - a^0\|_{L^2(I)} \le C\varepsilon \text{ for any } \varepsilon \in [0,1]; \tag{3.1}$$

here $C > 0$ is a constant independent of ε.

In what follows, unless otherwise specified, C denotes a generic positive constant whose value can change from line to line. If it is essential, the dependence of a constant C on some parameters, say " $\cdot$ " , will be written by $C(\cdot)$.

For any $0 \le s \le 3$, Let X_s be the collection of all functions u_0 in the space $H^s(I)$ satisfying the compatibility conditions

$$\begin{cases} u_0(0) = u_0(L) = 0, & 1/2 < s \le 3/2, \\ u_0(0) = u_0(L) = u_0'(L) = 0, & 3/2 < s \le 3 \end{cases}$$

with its usual topology, and let

$$Y_{s,[a,b]} = C([a,b]; X_s) \cap L^2(a,b; H^{s+1}(I))$$

with its usual topology. For simplicity, we denote $Y_{s,[0,T]}$ by $Y_{s,T}$ if $[a,b] = [0,T]$.

The well-posedness of (2.1) can be found in [18].

Proposition 3.1 ([18]). *Let $T > 0$ and $0 \le s \le 3$ be given. Suppose $\{a^\varepsilon\}_{0\le\varepsilon\le1}$ satisfy* (3.1). *Then, for any $\varepsilon \in [0,1]$ and any $u_0 \in X_s$,* (1.1) *admits a unique solution $u^\varepsilon \in Y_{s,T}$ which also satisfies*

$$\|u^\varepsilon\|_{Y_{s,T}} \le \alpha_s(\|u_0\|_{L^2(I)})\|u_0\|_{H^s(I)},$$
$$\|u_t^\varepsilon\|_{Y_{0,T}} \le \alpha_3(\|u_0\|_{L^2(I)})\|u_0\|_{H^3(I)},$$

where $\alpha_s : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing continuous function. Moreover, system (1.1) *is globally uniformly exponentially stable in the space $H^3(I)$, i.e., there exists a uniform constant $\nu > 0$ and a continuous nonnegative function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ such that for a given $u_0 \in X_3$, the corresponding solution u^ε satisfies*

$$\|u^\varepsilon(\cdot,t)\|_{H^3(I)} \le \beta(\|u_0\|_{L^2(I)})\|u_0\|_{H^3(I)}e^{-\nu t} \ \forall\ t \ge 0.$$

Remark 3.1. Following the proofs of Theorem 1.2 and 1.5 in [18] and assumption (3.1), we can guarantee that functions α_s, β and constant ν are all independent of ε.

In this section, we consider the semigroup $S_\varepsilon(t)$ associated with (1.1). Set the universal space X by $L^2(I)$ and Z by $H^3(I)$. Let $R > 0$ be a given constant and consider the set

$$B = \{v \in X_3 :\ \|v\|_{H^3(I)} \le R\}.$$

According to Proposition 3.1, there exists a time $T_R > 0$ independent of ε such that

$$S_\varepsilon(t)B \subset B \ \forall\ t \ge T_R.$$

We then set the phase space $\mathcal{X}_\varepsilon$ by

$$\mathcal{X}_\varepsilon = \bigcup_{0\le t\le\infty} S_\varepsilon(t)B = \bigcup_{0\le t\le T_R} S_\varepsilon(t)B \ \forall\ \varepsilon \in [0,1].$$

It is not difficult to verify that for any $\varepsilon \in [0,1]$, $B \subset \mathcal{X}_\varepsilon \subset X_3$, $\mathcal{X}_\varepsilon$ is a positively invariant set, i.e., $S_\varepsilon(t)\mathcal{X}_\varepsilon \subset \mathcal{X}_\varepsilon$ for every $t \geq 0$, and is a compact set of X. Thus $(S_\varepsilon(t), \mathcal{X}_\varepsilon, X)$ $(0 \leq \varepsilon \leq 1)$ define the dynamical systems we need. Applying Proposition 3.1 again, there exists a constant $M > 0$ independent of ε such that any element U in $\mathcal{X}_\varepsilon$ satisfies

$$\|U\|_{H^3(I)} \leq M. \tag{3.2}$$

We are now ready to state the main result in this paper.

Theorem 3.1. *Suppose* $\{a^\varepsilon\}_{0\leq\varepsilon\leq 1}$ *satisfy* (3.1). *Then there exist exponential attractors* $\mathcal{M}_\varepsilon$ *for dynamical systems* $(S_\varepsilon(t), \mathcal{X}_\varepsilon, L^2(I))$ *associated with* (1.1), $0 \leq \varepsilon \leq 1$, *respectively, for which the estimate*

$$dist_{sym}(\mathcal{M}_\varepsilon, \mathcal{M}_0) \leq C\varepsilon^\kappa, \ 0 < \varepsilon \leq 1$$

holds with some exponent $0 < \kappa < 1$ *and constant* $C > 0$.

Proof. In order to apply Theorem 2.1, it is sufficient to verify (2.1)–(2.4) for $(S_\varepsilon(t), \mathcal{X}_\varepsilon, L^2(I))$. Therefore, we divide the proof into four steps.

Step 1. Let $T^* = \ln(\nu\beta(M)M/R)$. According to Proposition 3.1 and (3.2), it is easy to show that for any $\varepsilon \in [0,1]$ and any $u_0 \in \mathcal{X}_\varepsilon$, we have

$$\|u^\varepsilon(\cdot,t)\|_{H^3(I)} \leq R \text{ for every } t \geq T^*.$$

This implies (2.1).

Step 2. For $\varepsilon \in [0,1]$, let $u_0^\varepsilon, \widetilde{u}_0^\varepsilon$ be two initial functions in $\mathcal{X}_\varepsilon$ and let u^ε, $\widetilde{u}^\varepsilon$ be the corresponding solutions of (1.1), respectively.

Let $v^\varepsilon = u^\varepsilon - \widetilde{u}^\varepsilon$. Then function v^ε is a solution of

$$\begin{cases} v_t^\varepsilon + v_x^\varepsilon + v_{xxx}^\varepsilon + \widetilde{u}^\varepsilon v_x^\varepsilon + u_x^\varepsilon v^\varepsilon + a^\varepsilon v^\varepsilon = 0 & \text{in } I \times (0, T^*), \\ v^\varepsilon(0,t) = v^\varepsilon(L,t) = v_x^\varepsilon(L,t) = 0 & \text{in } (0,T^*), \\ v^\varepsilon(x,0) = v_0^\varepsilon(x) & \text{in } I, \end{cases} \tag{3.3}$$

where $v_0^\varepsilon = u_0^\varepsilon - \widetilde{u}_0^\varepsilon \in X_3$. It follows from Proposition 3.1 that $v^\varepsilon \in Y_{3,T^*}$. In order to prove (2.2), it remains to show that

$$\|v^\varepsilon(\cdot,T^*)\|_{H^3(I)} \leq L\|v_0^\varepsilon\|_{L^2(I)} \tag{3.4}$$

for some uniform constant $L > 0$.

Define an operator A by

$$A = -\partial_x - \partial_x^3$$

with the domain X_3. It is proved in [17] that A is the infinitesimal generator of a C_0-semigroup $W(t)$ in $L^2(I)$. Moreover, there exists a constant $C > 0$ such that for any $0 \leq a < b < +\infty$, $\phi \in L^2(I)$, and $f \in L^1(a,b;L^2(I))$, we have

$$\|W(\cdot)\phi + \int_a^\cdot W(\cdot - \tau)f(\tau)\|_{Y_{0,[a,b]}} \leq C(\sqrt{|b-a|}+1)(\|\phi\|_{L^2(I)} + \|f\|_{L^1(a,b;L^2(I))}).$$

Then we can write (3.3) in its integral equation form

$$v^\varepsilon(t) = W(t)v_0^\varepsilon - \int_0^t W(t-\tau)(\widetilde{u}^\varepsilon v_x^\varepsilon + u_x^\varepsilon v^\varepsilon + a^\varepsilon v^\varepsilon)(\tau)d\tau,$$

where the spatial variable is suppressed throughout.

Let N be a positive integer which will be determined later and let $T = T^*/N$. For any integer $n \in [0, N-1]$, we can deduce that

$$\begin{aligned}(3.5)\quad &\|v^\varepsilon\|_{Y_{0,[nT,(n+1)T]}} \\ \le& C(\sqrt{T}+1)(\|v^\varepsilon(\cdot,nT)\|_{L^2(I)} + \|\widetilde{u}^\varepsilon v_x^\varepsilon + u_x^\varepsilon v^\varepsilon + a^\varepsilon v^\varepsilon\|_{L^1(nT,(n+1)T;L^2(I))}) \\ \le& C(T^*)\|v^\varepsilon(\cdot,nT)\|_{L^2(I)} \\ &+ C(T^*)T^{1/4}(\|u^\varepsilon\|_{Y_{0,[nT,(n+1)T]}} + \|\widetilde{u}^\varepsilon\|_{Y_{0,[nT,(n+1)T]}} + 1)\|v^\varepsilon\|_{Y_{0,[nT,(n+1)T]}}.\end{aligned}$$

Here we have used the fact that there exists a constant $C > 0$ such that for any $0 \le a < b < +\infty$ and $u, v \in Y_{0,[a,b]}$,

$$(3.6)\qquad \int_a^b \|uv_x\|_{L^2(I)}dt \le C|b-a|^{1/4}\|u\|_{Y_{0,[a,b]}}\|v\|_{Y_{0,[a,b]}}.$$

Using Proposition 3.1, it is easy to obtain that

$$\|u^\varepsilon\|_{Y_{0,[nT,(n+1)T]}} + \|\widetilde{u}^\varepsilon\|_{Y_{0,[nT,(n+1)T]}} \le \|u^\varepsilon\|_{Y_{0,T^*}} + \|\widetilde{u}^\varepsilon\|_{Y_{0,T^*}} \le C(M,T^*).$$

Combining (3.5), we have

$$\|v^\varepsilon\|_{Y_{0,[nT,(n+1)T]}} \le C(T^*)\|v^\varepsilon(\cdot,nT)\|_{L^2(I)} + C(M,T^*)T^{1/4}\|v^\varepsilon\|_{Y_{0,[nT,(n+1)T]}}.$$

Choosing N large enough such that $C(M,T^*)T^{1/4} < 1/2$, it follows that

$$\|v^\varepsilon\|_{Y_{0,[nT,(n+1)T]}} \le C_0\|v^\varepsilon(\cdot,nT)\|_{L^2(I)} \le C_0\|v^\varepsilon\|_{Y_{0,[(n-1)T,nT]}} \le \cdots \le C_0^{n+1}\|v_0^\varepsilon\|_{L^2(I)},$$

where $C_0 > 0$ is a constant depending on T^* but independent of n and T. Therefore, we can deduce that

$$(3.7)\qquad \|v^\varepsilon\|_{Y_{0,T^*}} \le \sum_{n=0}^{N-1}\|v^\varepsilon\|_{Y_{0,[nT,(n+1)T]}} \le \|v_0^\varepsilon\|_{L^2(I)}\sum_{n=0}^{N-1}C_0^{n+1} = C(M,T^*)\|v_0^\varepsilon\|_{L^2(I)}.$$

Similarly, we can show that for any $v_{01}^\varepsilon, v_{02}^\varepsilon \in L^2(I)$, the corresponding solutions v_1^ε and v_2^ε of (3.3) satisfy

$$(3.8)\qquad \|v_1^\varepsilon - v_2^\varepsilon\|_{Y_{0,T^*}} \le C(M,T^*)\|v_{01}^\varepsilon - v_{02}^\varepsilon\|_{L^2(I)}.$$

Next, we shall prove that the solution v^ε of (3.3) satisfies

$$\|v^\varepsilon\|_{Y_{3,T^*}} \le C(M,T^*)\|v_0^\varepsilon\|_{H^3(I)}.$$

For this purpose, let $w^\varepsilon = v_t^\varepsilon$. Then function w^ε is the solution of

$$\begin{cases} w_t^\varepsilon + w_x^\varepsilon + w_{xxx}^\varepsilon + \widetilde{u}^\varepsilon w_x^\varepsilon + u_x^\varepsilon v^\varepsilon + \widetilde{u}_t^\varepsilon v_x^\varepsilon + \widetilde{u}_{xt}^\varepsilon v^\varepsilon + a^\varepsilon w^\varepsilon = 0 & \text{in } I\times(0,T^*),\\ w^\varepsilon(0,t) = w^\varepsilon(L,t) = w_x^\varepsilon(L,t) = 0 & \text{in } (0,T^*),\\ w^\varepsilon(x,0) = w_0^\varepsilon(x) & \text{in } I,\end{cases}$$

where $w_0^\varepsilon = -(v_0^\varepsilon)' - (v_0^\varepsilon)''' - \widetilde{u}_0^\varepsilon(v_0^\varepsilon)' - (u_0^\varepsilon)'v_0^\varepsilon - a^\varepsilon v_0^\varepsilon \in L^2(I)$. Following the above methods, we can prove that

$$\|w^\varepsilon\|_{Y_{0,T^*}} \le C(M,T^*)(\|w_0^\varepsilon\|_{L^2(I)} + \|\widetilde{u}_t^\varepsilon v_x^\varepsilon\|_{L^1(0,T^*;L^2(I))} + \|\widetilde{u}_{xt}^\varepsilon v^\varepsilon\|_{L^1(0,T^*;L^2(I))}).$$

Taking (3.6), Proposition 3.1, and (3.7) into account, we have

$$\begin{aligned}&\|\widetilde{u}_t^\varepsilon v_x^\varepsilon\|_{L^1(0,T^*;L^2(I))} + \|\widetilde{u}_{xt}^\varepsilon v^\varepsilon\|_{L^1(0,T^*;L^2(I))} \\ \le& C(T^*)(\|u_t^\varepsilon\|_{Y_{0,T^*}} + \|\widetilde{u}_t^\varepsilon\|_{Y_{0,T^*}})\|v^\varepsilon\|_{Y_{0,T^*}} \\ \le& C(M,T^*)\|v_0^\varepsilon\|_{L^2(I)}.\end{aligned}$$

This leads to

$$\|w^\varepsilon\|_{Y_{0,T^*}} \leq C(M,T^*)(\|w_0^\varepsilon\|_{L^2(I)} + \|v_0^\varepsilon\|_{L^2(I)}).$$

Following the methods developed in [18] with minor changes, we can obtain that

$$\|v^\varepsilon\|_{Y_{3,T^*}} \leq C(M,T^*)\|v_0^\varepsilon\|_{H^3(I)}. \tag{3.9}$$

Here we have used the fact that

$$\|v_0^\varepsilon\|_{L^2(I)} \leq \|u_0^\varepsilon\|_{L^2(I)} + \|\widetilde{u}_0^\varepsilon\|_{L^2(I)} \leq M.$$

According to (3.7)–(3.9), system (3.3) defines a continuous nonlinear map K from space X_j to Y_{j,T^*} for $j = 0, 3$. Then, using the nonlinear interpolation theory ([18, 19]), we can obtain that

$$\|v^\varepsilon\|_{Y_{s,T^*}} \leq L_s\|v_0^\varepsilon\|_{H^s(I)}$$

holds for $0 \leq s \leq 3$, where $L_s = L_s(M,T^*) > 0$ is a constant independent of ε. This implies that (3.3) has a strong smoothing property; therefore, we can find a constant $L > 0$ such that

$$\|v^\varepsilon\|_{Y_{3,T^*}} \leq L\|v_0^\varepsilon\|_{L^2(I)}.$$

According to the definitions of v^ε, v_0^ε and Y_{3,T^*}, we have (3.4). The proof of (2.2) is complete.

Step 3. For $\varepsilon \in [0,1]$, let $u_0^\varepsilon, \widetilde{u}_0^\varepsilon$ be two initial functions in $\mathcal{X}_\varepsilon$ and let u^ε, $\widetilde{u}^\varepsilon$ be the corresponding solutions of (1.1), respectively. In this part, consideration is given to the proof of the inequality

$$\|u^\varepsilon(\cdot,t_1) - \widetilde{u}^\varepsilon(\cdot,t_2)\|_{L^2(I)} \leq D(|t_1 - t_2| + \|u_0^\varepsilon - \widetilde{u}_0^\varepsilon\|_{L^2(I)}),\ 0 \leq t_1, t_2 \leq T^*,$$

on the interval $[0,T^*]$ with some uniform constant $D > 0$.

It is easy to deduce that

$$\begin{aligned}
&\|u^\varepsilon(\cdot,t_1) - \widetilde{u}^\varepsilon(\cdot,t_2)\|_{L^2(I)}\\
\leq&\|u^\varepsilon(\cdot,t_1) - u^\varepsilon(\cdot,t_2)\|_{L^2(I)} + \|u^\varepsilon(\cdot,t_2) - \widetilde{u}^\varepsilon(\cdot,t_2)\|_{L^2(I)}\\
\leq&\|u_t^\varepsilon(\cdot,\xi)\|_{L^2(I)}|t_1 - t_2| + \|v^\varepsilon(\cdot,t_2)\|_{L^2(I)}\\
\leq&\|u_t^\varepsilon\|_{Y_{0,T^*}}|t_1 - t_2| + \|v^\varepsilon\|_{Y_{0,T^*}},
\end{aligned}$$

where $\xi \in [t_1,t_2]$ and v^ε is the solution of (3.3). Applying Proposition 3.1 and (3.7), we have

$$\|u^\varepsilon(\cdot,t_1) - \widetilde{u}^\varepsilon(\cdot,t_2)\|_{L^2(I)} \leq C(M,T^*)(|t_1 - t_2| + \|u_0^\varepsilon - \widetilde{u}_0^\varepsilon\|_{L^2(I)}).$$

This ends the proof of (2.3).

Step 4. We intend to prove (2.4). Let $u_0 \in B$ and $z^\varepsilon = u^\varepsilon - u^0$; then z^ε satisfies

$$\begin{cases} z_t^\varepsilon + z_x^\varepsilon + z_{xxx}^\varepsilon + u^\varepsilon z_x^\varepsilon + u_x^0 z^\varepsilon + a^\varepsilon z^\varepsilon + (a^\varepsilon - a^0)u^0 = 0 & \text{in } I \times (0,T^*),\\ z^\varepsilon(0,t) = z^\varepsilon(L,t) = z_x^\varepsilon(L,t) = 0 & \text{in } (0,T^*),\\ z^\varepsilon(x,0) = 0 & \text{in } I. \end{cases} \tag{3.10}$$

Multiplying the first equation in (3.10) by z^ε and integrating in $[0, L]$, we can obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^L |z^\varepsilon(x,t)|^2 dx + \frac{1}{2}|z_x^\varepsilon(0,t)|^2 \\
\le& \int_0^L (|u_x^\varepsilon| + |u_x^0| + |a^\varepsilon|)|z^\varepsilon|^2 dx + \int_0^L |a^\varepsilon - a^0||u^0||z^\varepsilon| dx \\
\le& C(\|u^\varepsilon(\cdot,t)\|_{H^2(I)} + \|u^0(\cdot,t)\|_{H^2(I)} + 1)\int_0^L |z^\varepsilon(x,t)|^2 dx \\
&+ C\|u^0(\cdot,t)\|_{H^1(I)}^2 \int_0^L |a^\varepsilon - a^0|^2 dx \\
\le& C(\|u^\varepsilon\|_{Y_{3,T^*}} + \|u^0\|_{Y_{3,T^*}} + 1)\int_0^L |z^\varepsilon(x,t)|^2 dx + C\|u^0\|_{Y_{3,T^*}}^2 \int_0^L |a^\varepsilon - a^0|^2 dx \\
\le& C(R,T^*)\int_0^L |z^\varepsilon(x,t)|^2 dx + C(R,T^*)\int_0^L |a^\varepsilon - a^0|^2 dx \;\forall\; t \in [0,T^*].
\end{aligned}$$

It follows from Gronwall inequality and assumption (3.1) that

$$\|z^\varepsilon(\cdot,t)\|_{L^2(I)}^2 \le e^{C(R,T^*)t} C(R,T^*)t\|a^\varepsilon - a^0\|_{L^2(I)}^2 \le C(R,T^*)\varepsilon^2 \;\forall\; t \in [0,T^*].$$

This means that

$$\sup_{u_0 \in B} \sup_{0 \le t \le T^*} \|u^\varepsilon(t) - u^0(t)\|_{L^2(I)} \le C(R,T^*)\varepsilon$$

holds for all $0 \le \varepsilon \le 1$. Namely, we obtain (2.4).

Now, we have verified the assumptions in Proposition 3.1; consequently, we complete the proof of Theorem 3.1.

□

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