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# SHARP $L^p \to L^r$ ESTIMATES FOR $k$-PLANE TRANSFORMS IN FINITE FIELDS

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ABSTRACT. We study mapping properties of finite field $k$-plane transforms. Using geometric combinatorics, we do an elaborate analysis to recover the critical endpoint estimate. As a consequence, we obtain optimal $L^p \to L^r$ estimates for all $k$-plane transforms in the finite field setting. In addition, applying Hölder's inequality to our results, we obtain an estimate for multilinear $k$-plane transforms.

## 1. INTRODUCTION

Over the last few decades, finite field analogs of Euclidean harmonic analysis problems have been extensively studied. Tom Wolff [15] initially proposed the finite field Kakeya problem, which was solved by Dvir [5] using the polynomial method. Adapting the method, Ellenberg, Oberlin, and Tao [6] settled the finite field Kakeya maximal conjecture. In 2004, the finite field restriction problem was also initiated by Mockenhaupt and Tao [14]. Like the Euclidean case, the finite field restriction conjectures are still open, although some progress on this problem has been made by researchers (see, for example, [7, 9–13]). We also refer the reader to Wright's lecture notes [16] for finite ring restriction problems.

After Mockenhaupt and Tao, the finite field (maximal) averaging problem was formulated and studied by Carbery, Stones, and Wright [3]. In the paper, they also initially studied mapping properties of finite field $k$-plane transforms. The main purpose of this paper is to give the complete answer to the boundedness problem on $k$-plane transforms in the finite field setting. Let us review the definition and notation related to finite field $k$-plane transforms. Let $\mathbb{F}_q$ be a finite field with $q$ elements. We denote by $\mathbb{F}_q^d, d \geq 2$, a $d$-dimensional vector space over the finite field $\mathbb{F}_q$. We endow $\mathbb{F}_q^d$ with a normalized counting measure $d\mathbf{x}$ so that for $f : \mathbb{F}_q^d \to \mathbb{C}$ we have

$$\|f\|_{L^p(\mathbb{F}_q^d, d\mathbf{x})} = \begin{cases} \left( q^{-d} \sum\limits_{\mathbf{x} \in \mathbb{F}_q^d} |f(\mathbf{x})|^p \right)^{\frac{1}{p}} & \text{if} \quad 1 \leq p < \infty, \\ \max\limits_{\mathbf{x} \in \mathbb{F}_q^d} |f(\mathbf{x})| & \text{if} \quad p = \infty. \end{cases}$$

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Given a fixed dimension $d \geq 2$, let us choose $1 \leq k \leq d-1$ and denote by $M_k$ the set of all $k$-planes in $\mathbb{F}_q^d$, which means affine subspaces in $\mathbb{F}_q^d$ with dimension $k$. From basic linear algebra, we notice that

$$|M_k| \sim q^{(d-k)(k+1)}.$$

Moreover, if $\Pi_{k,s}$ denotes the number of $k$-planes containing a given $s$-plane with $0 \leq s \leq k$, then

$$|\Pi_{k,s}| \sim q^{(d-k)(k-s)}.$$

Throughout this paper, for $X, Y > 0$, we use $X \lesssim Y$ if there is a constant $C > 0$ independent of $q$ such that $X \leq CY$, and $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. We now endow $M_k$ with a normalized counting measure $\lambda_k$ so that for $g : M_k \to \mathbb{C}$, we define its integral as

$$\int_{M_k} g(\omega)\ d\lambda_k(\omega) = \frac{1}{|M_k|} \sum_{\omega \in M_k} g(\omega),$$

where $|M_k|$ denotes the cardinality of the set $M_k$. With the above notation, we define the $k$-plane transform $T_k$ of a function $f : \mathbb{F}_q^d \to \mathbb{C}$ as

$$T_k f(\omega) = \int_\omega f(\mathbf{x})\ d\sigma_\omega(\mathbf{x}) := \frac{1}{|\omega|} \sum_{\mathbf{x} \in \omega} f(\mathbf{x}),$$

where $d\sigma_\omega$ denotes the normalized surface measure on the $k$-plane $\omega \in M_k$. In particular, the operator $T_k$ is called the $X$-ray transform for $k = 1$ and the Radon transform for $k = d-1$. In this finite field setting, the $k$-plane transform problem asks us to determine exponents $1 \leq p, r \leq \infty$ such that the estimate

$$\|T_k f\|_{L^r(M_k, d\lambda_k)} \lesssim \|f\|_{L^p(\mathbb{F}_q^d, d\mathbf{x})} \tag{1.1}$$

holds for every function $f : \mathbb{F}_q^d \to \mathbb{C}$, where the operator norm of $T_k$ must be independent of $q$, which is the size of the underlying finite field $\mathbb{F}_q$. In the Euclidean setting, as a consequence of mixed norm estimates for the $k$-plane transform, this problem was completely solved by M. Christ [2], who improved on results of Drury [4]. On the other hand, Carbery, Stones, and Wright [3] obtained sharp restricted type estimates for all $k$-plane transforms in finite fields. In fact, they proved that for the critical endpoint $(1/p, 1/r) = ((k+1)/(d+1), 1/(d+1))$, the estimate (1.1) holds for all characteristic functions $f = \chi_E$ on any set $E \subset \mathbb{F}_q^d$. More formally, they obtained the following result.

**Theorem 1.1.** *Let $d \geq 2$ and $1 \leq k \leq d-1$. If the estimate*

$$\|T_k f\|_{L^r(G_k, d\lambda_k)} \lesssim \|f\|_{L^p(\mathbb{F}_q^d, d\mathbf{x})} \tag{1.2}$$

*holds for all functions $f$ on $\mathbb{F}_q^d$, then $(1/p, 1/r)$ lies on the convex hull $H$ of*

$$((k+1)/(d+1), 1/(d+1)), (0,0), (1,1), \quad \textit{and} \quad (0,1).$$

*Conversely, if $(1/p, 1/r)$ lies in $H \setminus ((k+1)/(d+1), 1/(d+1))$, then* (1.2) *holds for all functions $f$ on $\mathbb{F}_q^d$. Furthermore, the restricted type inequality*

$$\|T_k f\|_{L^{d+1}(M_k, d\lambda_k)} \lesssim \|f\|_{L^{\frac{d+1}{k+1}, 1}(\mathbb{F}_q^d, d\mathbf{x})} \tag{1.3}$$

*holds for all functions $f$ on $\mathbb{F}_q^d$.*

1.1. **Statement of the main result.** The first part of Theorem 1.1 states the necessary condition for the boundedness of the $k$-plane transform. Note that the necessary condition would be in fact sufficient if the restricted type estimate (1.3) can be extended to a strong type estimate. Therefore, to settle the finite field $k$-plane transform problem, we only need to establish the strong type $L^{(d+1)/(k+1)} \to L^{d+1}$ estimate. In [8], it was already proved that the strong type estimate holds for the $X$-ray transform ($k = 1$) and the Radon transform ($k = d-1$). To obtain the sharp estimate, methods of the discrete Fourier analysis and geometric combinatorics were used for the Radon transform and the $X$-ray transform, respectively. In this paper, we extend the work to other $k$-plane transforms so that we obtain the full solution of the finite field $k$-plane transform problem. The main result we shall prove is as follows.

**Theorem 1.2.** *Let $d \geq 2$ and let $1 \leq k \leq (d-1)$. Then we have*

$$\|T_k f\|_{L^{d+1}(M_k, d\lambda_k)} \lesssim \|f\|_{L^{\frac{d+1}{k+1}}(\mathbb{F}_q^d, d\mathbf{x})} \quad \textit{for all } f \textit{ on } \mathbb{F}_q^d.$$

As mentioned before, this theorem for $k = 1$ and $k = d-1$ was already obtained in [8], and thus our main result is new for the case $2 \leq k \leq d-2$. It is also known that one can deduce the result of Theorem 1.2 for $k = 1$ (the $X$-ray transform) by applying the finite field Kakeya maximal conjecture, which was solved by Ellenberg, Oberlin, and Tao (see Theorem 1.3 and Remark 1.4 in [6]). Likewise one could also derive the results of Theorem 1.2 for $2 \leq k \leq d-1$ if one could prove the conjecture on $k$-plane maximal operator estimates in finite fields (see Conjecture 4.13 in [6]). However, the conjecture has not been solved (see [1] for the best known result on this problem).

By repeatedly using Hölder's inequality, the following estimate for a multilinear $k$-plane transform can be deduced from Theorem 1.2.

**Corollary 1.3.** *With the assumption of Theorem* 1.2*, we have*

$$\left\| \prod_{j=1}^{d+1} T_k f_j \right\|_{L^1(M_k, d\lambda_k)} \lesssim \prod_{j=1}^{d+1} \|f_j\|_{L^{\frac{d+1}{k+1}}(\mathbb{F}_q^d, d\mathbf{x})}$$

*for all functions $f_j, j = 1, 2, \ldots, (d+1)$, on $\mathbb{F}_q^d$.*

*Proof.* Since $1 = \sum_{t=1}^{d+1} \frac{1}{d+1}$, if we repeatedly use Hölder's inequality, we see that

$$\left\| \prod_{j=1}^{d+1} T_k f_j \right\|_{L^1(M_k, d\lambda_k)} \leq \prod_{j=1}^{d+1} \|T_k f_j\|_{L^{d+1}(M_k, d\lambda_k)},$$

and so the statement of the corollary follows immediately from Theorem 1.2. □

Taking $f = f_j$ for $j = 1, 2, \ldots, (d+1)$, Corollary 1.3 also implies Theorem 1.2.

## 2. Proof of the main theorem (Theorem 1.2)

We start proving Theorem 1.2 by making certain reductions. We aim to prove for each integer $1 \leq k \leq (d-1)$ that the estimate

$$\|T_k f\|_{L^{d+1}(M_k, d\lambda_k)} \lesssim \|f\|_{L^{\frac{d+1}{k+1}}(\mathbb{F}_q^d, d\mathbf{x})} = \left( q^{-d} \sum_{\mathbf{x} \in \mathbb{F}_q^d} |f(\mathbf{x})|^{\frac{d+1}{k+1}} \right)^{\frac{k+1}{d+1}} \tag{2.1}$$

holds for all functions $f$ on $\mathbb{F}_q^d$, where we recall that $M_k$ denotes the collection of all affine $k$-planes in $\mathbb{F}_q^d$. Without loss of generality, we may assume that $f$ is a nonnegative real-valued function and

$$\sum_{\mathbf{x}\in\mathbb{F}_q^d} f(\mathbf{x})^{\frac{d+1}{k+1}} = 1. \tag{2.2}$$

Thus we also assume that $\|f\|_\infty \le 1$. Furthermore, we may assume that $f$ is written by a step function

$$f(\mathbf{x}) = \sum_{i=0}^{\infty} 2^{-i} E_i(\mathbf{x}), \tag{2.3}$$

where $E_i'$s are disjoint subsets of $\mathbb{F}_q^d$ and we write $E(\mathbf{x})$ for the characteristic function $\chi_E$ on a set $E \subset \mathbb{F}_q^d$, which allows us to use a simple notation. From (2.2) and (2.3), we also assume that

$$\sum_{j=0}^{\infty} 2^{-\frac{(d+1)j}{k+1}} |E_j| = 1 \quad \text{and so } |E_j| \le 2^{\frac{(d+1)j}{k+1}} \text{ for all } j = 0, 1, \ldots. \tag{2.4}$$

Thus, to prove (2.1), it suffices to prove that

$$\|T_k f\|^{d+1}_{L^{d+1}(M_k, d\lambda_k)} \lesssim q^{-d(k+1)}, \tag{2.5}$$

for all functions $f$ such that the conditions (2.3), (2.4) hold. Since we have assumed that $f \ge 0$, it is clear that $T_k f$ is also a non-negative real-valued function on $M_k$. By expanding the left hand side of the above inequality (2.5) and using the facts that $|\omega| = q^k$ for $\omega \in M_k$ and $|M_k| \sim q^{(d-k)(k+1)}$, we see that

$$\|T_k f\|^{d+1}_{L^{d+1}(M_k, d\lambda_k)} = \frac{1}{|M_k|} \sum_{\omega\in M_k} (T_k f(\omega))^{d+1}$$

$$\sim \frac{1}{q^{k(d+1)}} \frac{1}{q^{(d-k)(k+1)}} \sum_{i_0=0}^{\infty} \cdots \sum_{i_d=0}^{\infty} 2^{-(i_0+\cdots+i_d)} \sum_{\substack{(\mathbf{x}^0,\ldots,\mathbf{x}^d)\\ \in E_{i_0}\times\cdots\times E_{i_d}}} \sum_{\omega\in M_k} \omega(\mathbf{x}^0)\ldots\omega(\mathbf{x}^d)$$

$$\sim \frac{1}{q^{k(d+1)}} \frac{1}{q^{(d-k)(k+1)}} \sum_{\substack{0=i_0\le i_1\\ \le\cdots\le i_d<\infty}} 2^{-(i_0+\cdots+i_d)} \sum_{\substack{(\mathbf{x}^0,\ldots,\mathbf{x}^d)\\ \in E_{i_0}\times\cdots\times E_{i_d}}} \sum_{\omega\in M_k} \omega(\mathbf{x}^0)\ldots\omega(\mathbf{x}^d),$$

where the last line follows from the symmetry of $i_0, \cdots, i_d$. Now, we decompose the sum over $(\mathbf{x}^0, \ldots, \mathbf{x}^d) \in E_{i_0} \times \cdots \times E_{i_d}$ as

$$\sum_{(\mathbf{x}^0,\ldots,\mathbf{x}^d)\in E_{i_0}\times\cdots\times E_{i_d}} = \sum_{s=0}^{\infty} \sum_{(\mathbf{x}^0,\ldots,\mathbf{x}^d)\in\Delta(s,i_0,\ldots,i_d)},$$

where $\Delta(s, i_0, \ldots, i_d) := \{(\mathbf{x}^0, \ldots, \mathbf{x}^d) \in E_{i_0} \times \cdots \times E_{i_d} : [\mathbf{x}^0, \ldots, \mathbf{x}^d] \text{ is an } s\text{-plane}\}$ and $[\mathbf{x}^0, \ldots, \mathbf{x}^d]$ denotes the smallest affine subspace containing the elements $\mathbf{x}^0, \ldots, \mathbf{x}^d$. Now, notice that if $s > k$ and $(\mathbf{x}^0, \ldots, \mathbf{x}^d) \in \Delta(s, i_0, \ldots, i_d)$, then the sum over $\omega \in M_k$ vanishes. On the other hand, if $0 \le s \le k$, then the sum over $\omega \in M_k$ is the same as the number of $k$-planes containing the unique $s$-plane, that is, $\sim q^{(d-k)(k-s)}$.

From these observations and (2.5), our task is to show that for all $E_i, i = 0, 1, \ldots,$ satisfying the condition (2.4),

$$\sum_{i_0=0}^{\infty}\sum_{i_1\geq i_0}^{\infty}\cdots\sum_{i_d\geq i_{d-1}}^{\infty} 2^{-(i_0+i_1+\cdots+i_d)}\sum_{s=0}^{k}|\Delta(s,i_0,\ldots,i_d)|q^{-s(d-k)} \lesssim 1. \tag{2.6}$$

In [8], it was shown that this inequality holds true for a simple case $k = 1$, and so the sharp estimate for the $X$-ray transform was obtained. However, when $k \geq 2$ and the dimension $d$ becomes bigger, it is not a simple problem to prove (2.6), because a lot of complicated cases happen in finding an upper bound of $|\Delta(s, i_0, i_1, \ldots, i_d)|$. In the following subsections, we shall prove (2.6) by making further reductions so that the proof of Theorem 1.2 will be complete.

2.1. **Proof of** (2.6)**.** For each $s = 0, 1, \ldots, k$, it suffices to prove that

$$\sum_{i_0=0}^{\infty}\sum_{i_1\geq i_0}^{\infty}\cdots\sum_{i_d\geq i_{d-1}}^{\infty} 2^{-(i_0+i_1+\cdots+i_d)}|\Delta(s,i_0,\ldots,i_d)|q^{-s(d-k)} \lesssim 1. \tag{2.7}$$

First fix $s = 0, 1, \ldots, k$ and the sets $E_{i_0}, E_{i_1}, \ldots, E_{i_d}$. To find an upper bound of $|\Delta(s, i_0, \ldots, i_d)|$, we shall decompose the set $\Delta(s, i_0, \ldots, i_d)$ as a union of its disjoint subsets. For each $(\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^d) \in \Delta(s, i_0, \ldots, i_d)$ there are unique nonnegative integers $\ell_0, \ell_1, \ldots, \ell_s$ with $0 = \ell_0 < \ell_1 < \ell_2 < \cdots < \ell_{s-1} < \ell_s \leq d$ such that $\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^{\ell_j}$ determine a $j$-plane and $\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^{\ell_j-1}$ determine a $(j-1)$-plane for $j = 1, 2, \ldots, s$, where we define $\ell_0 = 0$. Therefore, we can write

$$\Delta(s,i_0,\ldots,i_d) = \bigcup_{0=\ell_0<\ell_1<\ell_2<\cdots<\ell_s\leq d} L(s,i_0,\ldots,i_d,\ell_0,\ell_1,\ldots,\ell_s),$$

where $L(s, i_0, \ldots, i_d, \ell_0, \ell_1, \ldots, \ell_s)$ consists of those members $(\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^d) \in \Delta(s, i_0, \ldots, i_d)$ such that for every $j = 1, 2, \ldots, s$, the affine span of $\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^{\ell_j}$ is of dimension $j$ and the affine span of $\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^{\ell_j-1}$ is of dimension $j-1$. Now, let us find an upper bound of $|L(s, i_0, \ldots, i_d, \ell_0, \ell_1, \ldots, \ell_s)|$ where $0 = \ell_0 < \ell_1 < \cdots < \ell_s \leq d$. It is clear that if $(\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^d) \in L(s, i_0, \ldots, i_d, \ell_0, \ell_1, \ldots, \ell_s)$, then there are at most $|E_{i_{\ell_j}}|$ choices for $\mathbf{x}^{\ell_j}, j = 0, 1, \ldots, s$. In addition, if $\ell_j < t < \ell_{j+1}$,[1] then there are at most $\min\{|E_{i_t}|, q^j\}$ choices for $\mathbf{x}^t$, because the point $\mathbf{x}^t$ must be contained in the affine $j$-plane of points $\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^{\ell_j}$; otherwise $t$ would be greater than or equal to $\ell_{j+1}$ by the definition of $L(s, i_0, \ldots, i_d, \ell_0, \ell_1, \ldots, \ell_s)$. From these observations, it follows that

$$|L(s,i_0,\ldots,i_d,\ell_0,\ell_1,\ldots,\ell_s)| \leq \prod_{j=0}^{s}\left(|E_{i_{\ell_j}}|\prod_{t=\ell_j+1}^{\ell_{j+1}-1}\min\{|E_{i_t}|,q^j\}\right), \tag{2.8}$$

where we define that if $\ell_{j+1} = \ell_j + 1$, then

$$\prod_{t=\ell_j+1}^{\ell_{j+1}-1}\min\{|E_{i_t}|,q^j\} = 1.$$

Let $A = \{j \in \{0, 1, \ldots, s\} : \ell_{j+1} \neq \ell_j+1\}$. Since $\sum_{j\in A}(\ell_{j+1}-\ell_j-1) = d-s \geq d-k$, the right hand side of (2.8) has at least $(d-k)$ factors each of which takes a form $\min\{|E_{i_t}|, q^j\}$ for some $j \in A$ and $t$ with $\ell_j + 1 \leq t \leq \ell_{j+1} - 1$. Now, we estimate

[1]Throughout this paper we shall assume that $\ell_{s+1} = d + 1$.

that $\min\{|E_{i_t}|, q^j\} \le q^j$ for the $(d-k)$ largest numbers in the set of such $t$, and $\min\{|E_{i_t}|, q^j\} \le |E_{i_t}|$ for the rest of the $(k-s)$ numbers $t$. In this way, we can obtain an upper bound of the right hand side of (2.8), which we shall denote by $U(s, i_0, \ldots, i_d, \ell_0, \ell_1, \ldots, \ell_s)$. For example, if $d = 7, k = 4, s = 2, \ell_0 = 0, \ell_1 = 2, \ell_2 = 5$, then

$$U(s, i_0, \ldots, i_d, \ell_0, \ell_1, \ldots, \ell_s) = |E_{i_0}||E_{i_1}||E_{i_2}||E_{i_3}|q|E_{i_5}|q^2q^2.$$

It is clear that

$$\begin{aligned}|\Delta(s, i_0, \ldots, i_d)| &\lesssim \max_{0=\ell_0<\ell_1<\cdots<\ell_s\le d} |L(s, i_0, \ldots, i_d, \ell_0, \ell_1, \ldots, \ell_s)| \\ &\le \max_{0=\ell_0<\ell_1<\cdots<\ell_s\le d} U(s, i_0, \ldots, i_d, \ell_0, \ell_1, \ldots, \ell_s).\end{aligned}$$

Thus, to prove the estimate (2.7), it is enough to show that for every $s = 0, 1, \ldots, k$ and $0 = \ell_0 < \ell_1 < \cdots < \ell_s \le d$,

(2.9)
$$\sum_{i_0=0}^{\infty} \sum_{i_1\ge i_0}^{\infty} \cdots \sum_{i_d\ge i_{d-1}}^{\infty} 2^{-(i_0+i_1+\cdots+i_d)}\ U(s, i_0, \ldots, i_d, \ell_0, \ell_1, \ldots, \ell_s)\ q^{-s(d-k)} \lesssim 1.$$

We claim that it suffices to prove this estimate (2.9) only for the case when $s = k$. This claim follows by observing from the definition of $U$ that given a value $U(s, i_0, \ldots, i_d, \ell_0, \ell_1, \ldots, \ell_s)$ for $s = 0, 1, \ldots, (k-1)$, we can choose numbers $\ell_1', \ell_2', \ldots, \ell_{s+1}'$ with $1 \le \ell_1' < \ell_2' < \cdots < \ell_{s+1}' \le d$ such that

$$U(s, i_0, \ldots, i_d, \ell_0, \ell_1, \ldots, \ell_s)\ q^{d-k} = U(s+1, i_0, \ldots, i_d, \ell_0, \ell_1', \ell_2', \ldots, \ell_{s+1}').$$

In fact $\{\ell_1', \ell_2', \ldots, \ell_{s+1}'\}$ can be selected by adding one number, say $\ell'$, to $\{\ell_1, \ldots, \ell_s\}$, where $\ell' = \ell_{j_0} + 1$ and $j_0$ is defined by

$$j_0 = \min\{j \in \{0, 1, \ldots, s\} : \ell_{j+1} \ne \ell_j + 1\}.$$

Therefore, our final task is to prove that for every nonnegative integers $\ell_0, \ell_1, \ldots, \ell_k$ with $0 = \ell_0 < \ell_1 < \cdots < \ell_k \le d$, we have

(2.10)
$$S := \sum_{i_0=0}^{\infty} \sum_{i_1\ge i_0}^{\infty} \cdots \sum_{i_d\ge i_{d-1}}^{\infty} 2^{-(i_0+i_1+\cdots+i_d)}\ U(k, i_0, \ldots, i_d, \ell_0, \ell_1, \ldots, \ell_k)\ q^{-k(d-k)} \lesssim 1.$$

This shall be proved in the following subsection.

2.2. **Proof of the estimate** (2.10)**.** We begin with a preliminary lemma.

**Lemma 2.1.** *With the notation above, we have*

$$q^{-k(d-k)}\ U(k, i_0, \ldots, i_d, \ell_0, \ell_1, \ldots, \ell_k) = \left(\prod_{t=0}^{k} |E_{i_{\ell_t}}|\right) \left(q^{-\sum_{t=1}^{k}(\ell_t - t)}\right).$$

*Proof.* By the definition of $U$, we see that

$$\begin{aligned}U(k, i_0, \ldots, i_d, \ell_0, \ell_1, \ldots, \ell_k) &= \prod_{t=0}^{k} |E_{i_{\ell_t}}|\ q^{t(\ell_{t+1}-\ell_t-1)} \\ &= \left(\prod_{t=0}^{k} |E_{i_{\ell_t}}|\right) \left(q^{\sum_{t=0}^{k} t(\ell_{t+1}-\ell_t-1)}\right),\end{aligned}$$

where $\ell_0 = 0$ and $\ell_{k+1} = d+1$. It follows that

$$q^{-k(d-k)}\, U(k, i_0, \dots, i_d, \ell_0, \ell_1, \dots, \ell_k) = \left(\prod_{t=0}^{k} |E_{i_{\ell_t}}|\right)\left(q^{-k(d-k)+\sum_{t=1}^{k} t(\ell_{t+1}-\ell_t-1)}\right).$$

Thus the proof of Lemma 2.1 will be complete if we show that

$$-k(d-k) + \sum_{t=1}^{k} t(\ell_{t+1} - \ell_t - 1) = -\sum_{t=1}^{k}(\ell_t - t).$$

To prove this equality, observe that

$$\sum_{t=1}^{k} t(\ell_{t+1} - \ell_t) = -\ell_1 - \ell_2 - \cdots - \ell_k + k\ell_{k+1} = \left(-\sum_{t=1}^{k}\ell_t\right) + k(d+1).$$

Then we obtain that

$$\begin{aligned} -k(d-k) + \sum_{t=1}^{k} t(\ell_{t+1} - \ell_t - 1) &= -k(d-k) - \sum_{t=1}^{k}\ell_t + k(d+1) - \frac{k(k+1)}{2} \\ &= \frac{k(k+1)}{2} - \sum_{t=1}^{k}\ell_t = \sum_{t=1}^{k} t - \sum_{t=1}^{k}\ell_t \\ &= -\sum_{t=1}^{k}(\ell_t - t). \end{aligned}$$

□

We shall give the complete proof of the estimate (2.10). From Lemma 2.1, we aim to prove that

$$S = \sum_{i_0=0}^{\infty}\sum_{i_1\ge i_0}^{\infty}\cdots\sum_{i_d\ge i_{d-1}}^{\infty} 2^{-(i_0+i_1+\cdots+i_d)}\left(\prod_{t=0}^{k}|E_{i_{\ell_t}}|\right)\left(q^{-\sum_{t=1}^{k}(\ell_t-t)}\right) \lesssim 1.$$

Write the term $S$ as

$$S = \sum_{i_0=0}^{\infty}\sum_{i_1=0}^{\infty}\cdots\sum_{i_d=0}^{\infty} 1_{i_0\le i_1\le\cdots\le i_d}\, 2^{-(i_0+i_1+\cdots+i_d)}\left(\prod_{t=0}^{k}|E_{i_{\ell_t}}|\right)\left(q^{-\sum_{t=1}^{k}(\ell_t-t)}\right),$$

where we define that $1_{i_0\le i_1\le\cdots\le i_d} = 1$ if $i_0 \le i_1 \le \cdots \le i_d$, and 0 otherwise. By Fubini's theorem, we can decompose the sums as follows:[2]

$$(2.11)\qquad S = \sum_{i_0=0}^{\infty}\sum_{i_1=0}^{\infty}\cdots\sum_{i_d=0}^{\infty} = \sum_{i_0=0}^{\infty}\sum_{i_{\ell_1}=0}^{\infty}\sum_{i_{\ell_2}=0}^{\infty}\cdots\sum_{i_{\ell_k}=0}^{\infty}\sum_{i_{j_1}=0}^{\infty}\sum_{i_{j_2}=0}^{\infty}\cdots\sum_{i_{j_{d-k}}=0}^{\infty},$$

where $j_1, j_2, \dots, j_{d-k}$ denote natural numbers such that $\{j_1, j_2, \dots, j_{d-k}\} = \{1, 2, \dots, d\} \setminus \{\ell_1, \ell_2, \dots, \ell_k\}$ and $1 \le j_1 < j_2 < \cdots < j_{d-k} \le d$. For each $j_t,\ t = 1, 2,$

[2]To simplify notation, the general term is omitted.

$\dots, d-k$, let us denote by $\langle j_t\rangle$ the greatest element of $\{\ell_0, \ell_1, \dots, \ell_k\}$ less than $j_t$. From the definition of $1_{i_0\le i_1\le\cdots\le i_d}$, it is clear that

$$S \le \sum_{i_0=0}^{\infty}\sum_{i_{\ell_1}\ge i_0}^{\infty}\cdots\sum_{i_{\ell_k}\ge i_{\ell_{k-1}}}^{\infty}\sum_{i_{j_1}\ge i_{\langle j_1\rangle}}^{\infty}\sum_{i_{j_2}\ge i_{\langle j_2\rangle}}^{\infty}\cdots$$
$$\sum_{i_{j_{d-k}}\ge i_{\langle j_{d-k}\rangle}}^{\infty} 2^{-(i_0+i_1+\cdots+i_d)}\left(\prod_{t=0}^{k}|E_{i_{\ell_t}}|\right)\left(q^{-\sum_{t=1}^{k}(\ell_t-t)}\right)$$
$$= \sum_{i_0=0}^{\infty}\sum_{i_{\ell_1}\ge i_0}^{\infty}\cdots\sum_{i_{\ell_k}\ge i_{\ell_{k-1}}}^{\infty}\left(\prod_{t=0}^{k}|E_{i_{\ell_t}}|\right)\left(q^{-\sum_{t=1}^{k}(\ell_t-t)}\right)$$
$$\times\left(\sum_{i_{j_1}\ge i_{\langle j_1\rangle}}^{\infty}\cdots\sum_{i_{j_{d-k}}\ge i_{\langle j_{d-k}\rangle}}^{\infty} 2^{-(i_0+i_1+\cdots+i_d)}\right).$$

Inner sums can be computed by using a simple fact that the value of a convergent geometric series is similar to the first term of the series. In addition, use the definition of $\langle j_t\rangle,\ t=1,2,\dots,d-k$, and a simple fact that there are $(\ell_{t+1}-\ell_t-1)$ natural numbers between $\ell_t$ and $\ell_{t+1}$ for $t=0,1,\dots,k$. We are led to the estimate

$$\sum_{i_{j_1}\ge i_{\langle j_1\rangle}}^{\infty}\sum_{i_{j_2}\ge i_{\langle j_2\rangle}}^{\infty}\cdots\sum_{i_{j_{d-k}}\ge i_{\langle j_{d-k}\rangle}}^{\infty} 2^{-(i_0+i_1+\cdots+i_d)} \sim \prod_{t=0}^{k} 2^{-(\ell_{t+1}-\ell_t)i_{\ell_t}}.$$

It follows that

$$S \lesssim \sum_{i_0=0}^{\infty}\sum_{i_{\ell_1}\ge i_0}^{\infty}\cdots\sum_{i_{\ell_k}\ge i_{\ell_{k-1}}}^{\infty}\left(\prod_{t=0}^{k}|E_{i_{\ell_t}}|\right)\left(q^{-\sum_{t=1}^{k}(\ell_t-t)}\right)\left(\prod_{t=0}^{k}2^{-(\ell_{t+1}-\ell_t)i_{\ell_t}}\right)$$
$$= \sum_{i_0=0}^{\infty}\sum_{i_{\ell_1}\ge i_0}^{\infty}\cdots\sum_{i_{\ell_k}\ge i_{\ell_{k-1}}}^{\infty}\left(|E_{i_0}|\,2^{-\ell_1 i_0}\right)\left(\prod_{t=1}^{k}|E_{i_{\ell_t}}|\,q^{-\ell_t+t}\,2^{-(\ell_{t+1}-\ell_t)i_{\ell_t}}\right). \tag{2.12}$$

Now, we shall observe that for each $t=1,2,\dots,k$,

$$|E_{i_{\ell_t}}|\,q^{-\ell_t+t} \le 2^{\frac{(d+1)(d-\ell_t+t)}{d(k+1)}i_{\ell_t}}. \tag{2.13}$$

Since $|E_{i_{\ell_t}}|\le q^d$ (namely, $|E_{i_{\ell_t}}|^{1/d}\le q$) and $\ell_t-t\ge 0$, it is obvious that

$$|E_{i_{\ell_t}}|^{\frac{\ell_t-t}{d}} \le q^{\ell_t-t} \quad\text{or}\quad |E_{i_{\ell_t}}|^{\frac{\ell_t-t}{d}}q^{-\ell_t+t}\le 1.$$

On the other hand, we see from (2.4) that

$$|E_{i_{\ell_t}}| \le 2^{\frac{(d+1)}{k+1}i_{\ell_t}}.$$

Then (2.13) is easily shown by observing that

$$|E_{i_{\ell_t}}|\,q^{-\ell_t+t} = |E_{i_{\ell_t}}|^{1-\frac{(\ell_t-t)}{d}}|E_{i_{\ell_t}}|^{\frac{(\ell_t-t)}{d}}q^{-\ell_t+t}$$
$$\le |E_{i_{\ell_t}}|^{\frac{d-\ell_t+t}{d}} \le 2^{\frac{(d+1)(d-\ell_t+t)}{d(k+1)}i_{\ell_t}}.$$

From (2.12) and (2.13), we have
(2.14)
$$S \lesssim \sum_{i_0=0}^{\infty} \sum_{i_{\ell_1} \geq i_0}^{\infty} \sum_{i_{\ell_2} \geq i_{\ell_1}}^{\infty} \cdots \sum_{i_{\ell_k} \geq i_{\ell_{k-1}}}^{\infty} \left(|E_{i_0}|\, 2^{-\ell_1 i_0}\right) \left(\prod_{t=1}^{k} 2^{\left(\frac{(d+1)(d-\ell_t+t)}{d(k+1)} - \ell_{t+1} + \ell_t\right) i_{\ell_t}}\right).$$
Using a simple fact that the value of a convergent geometric series is similar as the first term of the series, we shall repeatedly compute the inner sums

$$\mathrm{I} := \sum_{i_{\ell_1} \geq i_0}^{\infty} \sum_{i_{\ell_2} \geq i_{\ell_1}}^{\infty} \cdots \sum_{i_{\ell_k} \geq i_{\ell_{k-1}}}^{\infty} \left(\prod_{t=1}^{k} 2^{\left(\frac{(d+1)(d-\ell_t+t)}{d(k+1)} - \ell_{t+1} + \ell_t\right) i_{\ell_t}}\right) \tag{2.15}$$

from the variable $i_{\ell_k}$ to the variable $i_{\ell_1}$. However, to repeatly compute the inner sums we must make sure that each geometric series converges. To assert that each series is convergent, it will be enough to show that for every $r = 1, 2, \ldots, k$,

$$\sum_{t=r}^{k} \left(\frac{(d+1)(d-\ell_t+t)}{d(k+1)} - \ell_{t+1} + \ell_t\right) < 0. \tag{2.16}$$

Now let us see why (2.16) holds. Multiplying (2.16) by the factor $d(k+1)$, we see that the statement (2.16) is the same as

$$\sum_{t=r}^{k} ((d+1)(d-\ell_t+t) + d(k+1)(\ell_t - \ell_{t+1})) < 0.$$

Since $\sum\limits_{t=r}^{k} d = d(k-r+1)$ and $\sum\limits_{t=r}^{k} (\ell_t - \ell_{t+1}) = \ell_r - \ell_{k+1} = \ell_r - (d+1)$, the above condition is equivalent to

$$(d+1)\left(d(k-r+1) + \sum_{t=r}^{k} (t-\ell_t)\right) + d(k+1)\ell_r - d(d+1)(k+1) < 0.$$

Write $d(k+1)\ell_r = d(k+1)(\ell_r - r) + dr(k+1)$ and try to simplify the left hand side of the above inequality. Then, for $r = 1, 2, \ldots, k$, we can easily see that the above inequality becomes

$$dr(k-d) + (1-dk)(r-\ell_r) + (d+1)\sum_{t=r+1}^{k} (t-\ell_t) < 0,$$

where we assume that $\sum_{t=r+1}^{k}(t-\ell_t) = 0$ if $k = 1$. This condition is clearly the same as

$$(dk-1)(\ell_r - r) < dr(d-k) + (d+1)\sum_{t=r+1}^{k} (\ell_t - t). \tag{2.17}$$

To prove this equality, let $\alpha = \ell_r - r \geq 0$. Since $\ell_0, \ell_1, \ldots, \ell_k$ are nonnegative integers with $0 = \ell_0 < \ell_1 < \ell_2 < \cdots < \ell_k \leq d$, it is clear that $\alpha = \ell_r - r \leq \ell_t - t$ for all $t \geq r$. Therefore, to prove (2.17), it will be enough to show that for $r = 1, 2, \ldots, k$,

$$(dk-1)\alpha < dr(d-k) + (d+1)(k-r)\alpha. \tag{2.18}$$

Solving for $\alpha$, this inequality is equivalent to

$$\alpha < \frac{dr(d-k)}{dr-k+r-1}. \tag{2.19}$$

Now, observe that for each $r = 1, 2, \ldots, k$, the maximum value of $\ell_r$ happens in the case when $\ell_k = d$, $\ell_{k-1} = d-1$, $\ell_{k-2} = d-2, \ldots, \ell_r = d-k+r$. This implies that $\alpha = \ell_r - r \le d-k$. Hence, to prove (2.18), it suffices to show that for $r = 1, 2, \ldots, k < d$,

$$d - k < \frac{dr(d-k)}{dr - k + r - 1},$$

which is equivalent to the inequality $r < k+1$. Since $r = 1, 2, \ldots, k$, this inequality clearly holds. This proves (2.16), which implies that each of the inner sums (2.15) is a convergent geometric series whose value is similar to its first term. Computing the sum I in (2.15) by this fact, we have

$$\mathrm{I} \sim 2^{i_0\left(\sum_{t=1}^{k} \frac{(d+1)(d-\ell_t+t)}{d(k+1)} - \ell_{t+1} + \ell_t\right)}.$$

As before, since $\ell_{k+1} = d+1$ and $\sum_{t=1}^{k}(\ell_t - \ell_{t+1}) = \ell_1 - \ell_{k+1}$, we can check that

$$\sum_{t=1}^{k}\left(\frac{(d+1)(d-\ell_t+t)}{d(k+1)} - \ell_{t+1} + \ell_t\right) = -\frac{(d+1)}{k+1} + \ell_1 + \frac{d+1}{d(k+1)} \sum_{t=1}^{k}(t-\ell_t)$$
$$\le -\frac{(d+1)}{k+1} + \ell_1.$$

Hence we see that

$$\mathrm{I} \lesssim 2^{i_0\left(-\frac{(d+1)}{k+1} + \ell_1\right)}.$$

Recall the definition of I in (2.15). Then combining this estimate with (2.14) yields

$$S \lesssim \sum_{i_0=0}^{\infty} |E_{i_0}| 2^{-\frac{(d+1)}{k+1} i_0} = 1,$$

where the equality follows from (2.4). Thus we finish the proof.

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