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A NATURAL LINEAR EQUATION IN AFFINE GEOMETRY: THE AFFINE QUASI-EINSTEIN EQUATION

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ABSTRACT. We study the affine quasi-Einstein equation, a second order linear homogeneous equation, which is invariantly defined on any affine manifold. We prove that the space of solutions is finite-dimensional, and its dimension is a strongly projective invariant. Moreover the maximal dimension is shown to be achieved if and only if the manifold is strongly projectively flat.

1. INTRODUCTION

An affine manifold is a pair $\mathcal{M} = (M, \nabla)$ where M is a smooth manifold of dimension m and ∇ is a torsion free connection on the tangent bundle of M. Adopt the *Einstein convention* and sum over repeated indices. Expand $\nabla_{\partial_{x^i}}\partial_{x^j} = \Gamma_{ij}{}^k\partial_{x^k}$ in a system of local coordinates $\vec{x} = (x^1, \ldots, x^m)$ to define the *Christoffel symbols* of the connection $\Gamma = (\Gamma_{ij}{}^k)$; the condition that ∇ is torsion free is then equivalent to the symmetry $\Gamma_{ij}{}^k = \Gamma_{ji}{}^k$. If $f \in C^\infty(M)$, then the *Hessian* $\mathcal{H}_\nabla f$ is the symmetric $(0,2)$-tensor defined by setting

$$\mathcal{H}_\nabla f := \nabla^2 f = (\partial_{x^i}\partial_{x^j} f - \Gamma_{ij}{}^k \partial_{x^k} f)\, dx^i \otimes dx^j\,.$$

Let $\rho_\nabla(x,y) := \operatorname{Tr}\{z \to R_\nabla(z,x)y\}$ be the *Ricci tensor*. Since in general this need not be a symmetric 2-tensor, we introduce the symmetric and anti-symmetric Ricci tensors:

$$\rho_{s,\nabla}(x,y) := \tfrac{1}{2}\{\rho_\nabla(x,y) + \rho_\nabla(y,x)\},$$
$$\rho_{a,\nabla}(x,y) := \tfrac{1}{2}\{\rho_\nabla(x,y) - \rho_\nabla(y,x)\}\,.$$

1.1. **The affine quasi-Einstein equation.** The *affine quasi-Einstein operator* $\mathfrak{Q}_{\mu,\nabla}$ is the linear second order partial differential operator

$$\mathfrak{Q}_{\mu,\nabla} f := \mathcal{H}_\nabla f - \mu f \rho_{s,\nabla} \text{ mapping } C^\infty(M) \text{ to } C^\infty(S^2M) \tag{1.a}$$

where the eigenvalue μ is a parameter of the theory. For fixed μ, this operator is natural in the category of affine manifolds and this family of operators parametrizes, modulo scaling, all the natural second order differential operators from $C^\infty(M)$ to $C^\infty(S^2M)$. We study the *affine quasi-Einstein equation* $\mathfrak{Q}_{\mu,\nabla} f = 0$, i.e.

$$\mathcal{H}_\nabla f = \mu f \rho_{s,\nabla}\,. \tag{1.b}$$

We denote the space of all solutions to Equation (1.b) by

$$E(\mu, \nabla) := \ker(\mathfrak{Q}_{\mu,\nabla}) = \{f \in \mathcal{C}^2(M) : \mathcal{H}_\nabla f = \mu f \rho_{s,\nabla}\}\,.$$

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Similarly, if $P \in M$, let $E(P, \mu, \nabla)$ be the linear space of all germs of smooth functions based at P satisfying Equation (1.b). Note that if $\rho_{s,\nabla} = 0$, then $E(\mu, \nabla) = E(0, \nabla)$ for any μ. Also observe that $E(0, \nabla)$ is the space of *affine Yamabe solitons* [3].

The operator $\mathfrak{Q}_{\mu,\nabla}$ of Equation (1.a) and the associated affine quasi-Einstein Equation (1.b) are of sufficient interest in their own right in affine geometry to justify a foundational paper of this nature. However Equation (1.b) also appears in the study of the pseudo-Riemannian quasi–Einstein equation using the Riemannian extension; we postpone until the end of the introduction a further discussion of this context to avoid interrupting the flow of our present discussion and to establish the necessary notational conventions.

1.2. **Foundational results.** We will establish the following result in Section 2:

Theorem 1.1. *Let $\mathcal{M} = (M, \nabla)$ be an affine manifold. Let $f \in E(P, \mu, \nabla)$.*

1. *One has $f \in C^\infty(M)$. If $\mathcal{M}$ is real analytic, then f is real analytic.*
2. *If X is the germ of an affine Killing vector field based at P, then $Xf \in E(P, \mu, \nabla)$.*
3. *If $f(P) = 0$, and if $df(P) = 0$, then $f \equiv 0$ near P.*
4. *One has $\dim\{E(P, \mu, \nabla)\} \leq m + 1$.*
5. *If $\mathcal{M}$ is simply connected and if $\dim\{E(P, \mu, \nabla)\}$ is constant on M, then f extends uniquely to an element of $E(\mu, \nabla)$.*

1.3. **Projective equivalence.**

Definition 1.2. We say that ∇ and $\tilde\nabla$ are *projectively equivalent* if there exists a 1-form ω so that $\nabla_X Y = \tilde\nabla_X Y + \omega(X)Y + \omega(Y)X$ for all X and Y. In this setting, we say that ω *provides a projective equivalence from ∇ to $\tilde\nabla$*; $-\omega$ then provides a projective equivalence from $\tilde\nabla$ to ∇. If ω is closed, we say that ∇ and $\tilde\nabla$ are *strongly projectively equivalent.*

If two projectively equivalent connections have symmetric Ricci tensors, then the 1-form ω giving the projective equivalence is closed and the two connections are, in fact, strongly projectively equivalent (see [8, 14, 17] for more information).

We say that ∇ is *projectively flat* if ∇ is projectively equivalent to a flat connection. Note that ∇ is projectively flat if and only if it is possible to choose a coordinate system so that the unparametrized geodesics of ∇ are straight lines. Strongly projectively flat surfaces are characterized as follows (see [8, 14]).

Lemma 1.3. *Let $\mathcal{M}$ be an affine surface.*

1. *Let ω provide a projective equivalence between ∇ and a flat connection.*
   - (a) *If $\rho_{a,\nabla} = 0$, then $d\omega = 0$ so ∇ is strongly projectively flat.*
   - (b) *If $d\omega = 0$, then ρ_∇ and $\nabla\rho_\nabla$ are totally symmetric.*
2. *If ρ_∇ and $\nabla\rho_\nabla$ are totally symmetric, then ∇ is strongly projectively flat.*

Two projectively equivalent connections are said to be *Liouville projectively equivalent* if their Ricci tensors coincide (see, for example, [11, 14]). We will establish the following result in Section 3.1. It shows that $\dim\{E(-\frac{1}{m-1}, \nabla)\}$ is a strong projective invariant and that $\dim\{E(\mu, \nabla)\}$ for arbitrary μ is a strong Liouville projective invariant.

Theorem 1.4. *Let $\mathcal{M}$ be an affine manifold of dimension m. Let $\mu_m = -\frac{1}{m-1}$. Let $\omega = dg$ provide a strong projective equivalence from ∇ to $\tilde{\nabla}$.*

(1) *The map $f \to e^g f$ is an isomorphism from $E(P, \mu_m, \nabla)$ to $E(P, \mu_m, \tilde{\nabla})$.*
(2) *The following assertions are equivalent:*
  (a) $\rho_{s,\tilde{\nabla}} = \rho_{s,\nabla}$.
  (b) $\mathcal{H}_\nabla g - dg \otimes dg = 0$.
  (c) $e^{-g} \in E(P, 0, \nabla)$.
(3) *If any of the assertions in (2) hold, then the map $f \to e^g f$ is an isomorphism from $E(P, \mu, \nabla)$ to $E(P, \mu, \tilde{\nabla})$ for any μ.*

Remark 1.5. If ∇ and $\tilde{\nabla}$ are strongly projectively equivalent, then the alternating Ricci tensors coincide, i.e. $\rho_{a,\tilde{\nabla}} = \rho_{a,\nabla}$ (see [17]). However, if ∇ and $\tilde{\nabla}$ are only projectively equivalent, then the alternating Ricci tensors can differ and Theorem 1.4 can fail. Let ∇ be the usual flat connection on $\mathbb{R}^2$ and let $\omega = x^2 dx^1$ define a projective equivalence from ∇ to $\tilde{\nabla}$. It is a straightforward computation to see that $\dim\{E(P, -1, \nabla)\} = 3$ and $\dim\{E(P, -1, \tilde{\nabla})\} = 0$. Thus Theorem 1.4 fails if we replace strong projective equivalence by projective equivalence. Although the geodesic structure is unchanged, $\rho_{a,\tilde{\nabla}} \neq 0$ in this instance and consequently the alternating Ricci tensor is not preserved by projective equivalence either.

We can say more about the geometry if $\dim\{E(P, \mu, \nabla)\} = m+1$ is extremal for some μ. The eigenvalue $\mu_m := -\frac{1}{m-1}$ plays a distinguished role. We will establish the following result in Section 3.2:

Theorem 1.6. *Let $\mathcal{M}$ be an affine manifold of dimension m. Let $\mu_m := -\frac{1}{m-1}$.*

(1) *$\mathcal{M}$ is strongly projectively flat if and only if $\dim\{E(\mu_m, \nabla)\} = m + 1$.*
(2) *If $\dim\{E(\mu, \nabla)\} = m + 1$ for any μ, then $\mathcal{M}$ is strongly projectively flat.*
(3) *If $\dim\{E(\mu, \nabla)\} = m + 1$ for $\mu \neq \mu_m$, then $\mathcal{M}$ is Ricci flat.*
(4) *Suppose $\dim\{E(P, \mu_m, \nabla)\} = m+1$. One may choose a basis $\{\phi_0, \dots, \phi_m\}$ for $E(P, \mu_m, \nabla)$ so that $\phi_0(P) \neq 0$ and so that $\phi_i(P) = 0$ for $i > 0$. Set $z^i := \phi_i/\phi_0$. Then $\vec{z} = (z^1, \dots, z^m)$ is a system of coordinates defined near P such that the unparametrized geodesics of $\mathcal{M}$ are straight lines.*

Remark 1.7. The coordinates of Assertion (4) are very much in the spirit of the Weierstrass preparation theorem for minimal surfaces; geometrically meaningful local coordinates arise from the underlying analysis.

We will prove the following result in Section 3.3:

Theorem 1.8. *Let $\mathcal{M}$ be an affine manifold of dimension m. Let $\mu_m := -\frac{1}{m-1}$.*

(1) *If $\mathcal{M}$ is strongly projectively equivalent to a connection $\tilde{\nabla}$ with $\rho_{s,\tilde{\nabla}} = 0$, then $E(\mu_m, \nabla) \neq 0$.*
(2) *If there exists $f \in E(P, \mu_m, \nabla)$ with $f(P) \neq 0$, then $\mathcal{M}$ is strongly projectively equivalent to a connection $\tilde{\nabla}$ with $\rho_{s,\tilde{\nabla}} = 0$ near P.*

Surface geometry is particularly tractable since the geometry is carried by the Ricci tensor. In this setting, $\mu_2 = -1$ and one has

Theorem 1.9. *Let $\mathcal{M}$ be an affine surface. Then $\dim\{E(-1, \nabla)\} \neq 2$.*

In the appendix, we will discuss some results concerning surface geometry in more detail. We will use Theorem A.7 to show there are affine connections ∇_i on

$\mathbb{R}^+ \times \mathbb{R}$ such that

$$\dim\{E(-1,\nabla_1)\} = 0, \quad \dim\{E(-1,\nabla_2)\} = 1, \quad \dim\{E(-1,\nabla_3)\} = 3\,.$$

Thus the remaining values can all be attained. In Example A.5, we will discuss a family of 3-dimensional affine manifolds where $\dim\{E(-\frac{1}{2},\nabla)\}$ can be 0, 1, 2, and 4 but is never 3. This suggests that for general m one could show that $\dim\{E(P,-\frac{1}{m-1},\nabla)\} \neq m$ so this value is forbidden. Our research continues on this problem.

1.4. **Riemannian extensions.** The Riemannian extension provides a procedure to transfer information from affine geometry into neutral signature geometry in a natural way. Let $\mathcal{M} = (M,\nabla)$ be an affine manifold. If $(x^1,\dots,x^m)$ are local coordinates on M, let $(y_1,\dots,y_m)$ be the corresponding dual coordinates on the cotangent bundle T^*M; if ω is a 1-form, we can express $\omega = y_i dx^i$. Let Φ be an auxiliary symmetric $(0,2)$-tensor field in M. Let ${\Gamma_{ij}}^k$ be the Christoffel symbols of the connection ∇. The *deformed Riemannian extension* is the neutral signature metric on T^*M which is defined by setting [1, 16]:

$$g_{\nabla,\Phi} = dx^i \otimes dy_i + dy^i \otimes dx_i + \left\{\Phi_{ij} - 2y_k\,{\Gamma_{ij}}^k\right\} dx^i \otimes dx^j\,. \tag{1.c}$$

This is invariantly defined [2]. Let π be the canonical projection from T^*M to M. If $f \in C^\infty(M)$, then

$$\mathcal{H}_{g_{\nabla,\Phi}}\pi^* f = \pi^*\mathcal{H}_\nabla f, \quad \rho_{g_{\nabla,\Phi}} = 2\pi^*\rho_{s,\nabla}, \quad \|d\pi^* f\|^2_{g_{\nabla,\Phi}} = 0\,.$$

Let $\mathcal{N} := (N, g_N, \Psi, \mu)$ where (N, g_N) is a pseudo-Riemannian manifold of dimension n, where $\Psi \in C^\infty(N)$, and where $\mu \in \mathbb{R}$. We say that $\mathcal{N}$ is a *quasi-Einstein manifold* if

$$\mathcal{H}_{g_N}\Psi + \rho_{g_N} - \mu\, d\Psi \otimes d\Psi = \lambda\, g_N \text{ for some } \lambda \in \mathbb{R}\,. \tag{1.d}$$

One has the following link between deformed Riemannian extensions and quasi-Einstein structures [4]:

Theorem 1.10.

(1) *Let $\mathcal{M} = (M,\nabla)$ be an affine surface, let $\psi \in C^\infty(M)$, and let $\mu \in \mathbb{R}$. If $\mathcal{H}_\nabla\psi + 2\rho_{s,\nabla} - \mu\, d\psi \otimes d\psi = 0$, then $(T^*M, g_{\nabla,\Phi}, \pi^*\psi, \mu)$ is a self-dual quasi-Einstein manifold with $\|d\pi^*\psi\|^2_{g_{\nabla,\Phi}} = 0$ and $\lambda = 0$, for any Φ.*

(2) *Let (N, g_N, Ψ, μ) be a self-dual quasi-Einstein manifold of signature $(2,2)$ with $\mu \neq -\frac{1}{2}$ and $\|d\Psi\|^2_{g_N} = 0$ which is not Ricci flat. Then $\lambda = 0$ and (N, g_N, Ψ, μ) is locally isometric to a manifold which has the form given in Assertion (1).*

We suppose $\mu \neq 0$ and make the change of variables $f = e^{-\frac{1}{2}\mu\,\psi}$. The equation $\mathcal{H}_\nabla\psi + 2\rho_{s,\nabla} - \mu\, d\psi \otimes d\psi = 0$ then becomes $\mathcal{H}_\nabla f = \mu\, f\, \rho_{s,\nabla}$. This is the *affine quasi-Einstein equation* given in (1.b). Let $f > 0$ be a smooth function on M. Express $\pi^* f = e^{-\mu\, F}$ for some $F \in C^\infty(T^*M)$ and $\mu \neq 0$. Then F solves Equation (1.d) in $(T^*M, g_{\nabla,\Phi})$ if and only if $f \in E(2\mu,\nabla)$.

Remark 1.11. The eigenvalue $\mu_m = -\frac{1}{m-1}$, which plays a role in the projective structure of (M,∇), is linked to some geometric properties of the deformed Riemannian extensions $\mathcal{N} := (T^*M, g_{\nabla,\Phi})$: if $\dim\{E(\mu_m,\nabla)\} \geq 1$, then $\mathcal{N}$ is conformally Einstein [4].

Afifi [1] showed that if a deformed Riemannian extension $g_{\nabla,\Phi}$ given by Equation (1.c) is locally conformally flat, then ∇ is projectively flat with symmetric Ricci tensor. Theorem A.7 shows the existence of surfaces with $\dim\{E(-1,\nabla)\}=1$. The corresponding deformed Riemannian extensions $(T^*M, g_{\nabla,\Phi})$ are conformally Einstein but not conformally flat for any deformation tensor Φ by Theorem 1.6.

Remark 1.12. Two metrics in the same conformal class are said to be *Liouville equivalent* if their Ricci tensors coincide (see [11, 12]). Let ∇ and ∇^{dg} be strongly projectively equivalent connections. Then the corresponding Riemannian extensions g_∇ and $g_{\nabla^{dg}}$ are conformally equivalent (just considering the transformation $(x^k, y_k)\mapsto(x^k, e^{2g}y_k)$). Moreover, g_∇ and $g_{\nabla^{dg}}$ are Liouville equivalent if and only if $\mathcal{H}_\nabla g = dg\otimes dg$. Therefore, from the pseudo-Riemannian point of view, affine Yamabe solitons $\phi\in E(0,\nabla)$ determine Liouville transformations of the Riemannian extension g_∇ (see Assertion (2) in Theorem 1.4).

The projective deformations in Example A.2 and Example A.3 induce Liouville equivalent Riemannian extensions [11]. Therefore it follows from [12, Corollary 2] that none of the Riemannian extensions g_∇ and g_{∇^ω} is geodesically complete.

Remark 1.13. There is a close connection between quasi-Einstein structures and warped product Einstein metrics (see [4] and references therein). The warping function of any Einstein warped product is a solution of Equation (1.d) with $\mu=\frac{1}{k}$, $k\in\mathbb{N}$. Conversely, if $f\in E(\frac{1}{2k},\nabla)$ for some positive integer k and if $\mathcal{E}$ is a Ricci flat manifold of dimension k, then the warped product $\mathcal{N}\times_{\pi^*f}\mathcal{E}$ with base manifold $\mathcal{N}:=(T^*M, g_{\nabla,\Phi})$ is Ricci flat. Theorem A.1-(3) and Theorem A.8 show that there exist homogeneous surfaces with $\dim\{E(\mu,\nabla)\}\geq 1$ for arbitrary $\mu=\frac{1}{2k}$.

2. The proof of Theorem 1.1

We establish the assertions of Theorem 1.1 seriatim.

2.1. Smoothness properties of solutions to Equation (1.b).

Introduce local coordinates $x=(x^1,\dots,x^m)$. Let $\mathfrak{Q}_{\mu,\nabla,ij}$ be the components of the quasi-Einstein operator of Equation (1.a). Let

$$D_\mu := \operatorname{Tr}\{\mathfrak{Q}_{\mu,\nabla}\} = \sum_{i=1}^m \mathfrak{Q}_{\mu,\nabla,ii} = \sum_{i=1}^m \partial_{x^i x^i} + \sum_{i=1}^m\sum_{j=1}^m \Gamma_{ii}{}^j\partial_{x^j} - \mu\sum_{i=1}^m \rho_{ii}\,.$$

The operator D_μ is then an elliptic second order partial differential operator. Let $f\in C^2(M)$ satisfy $\mathfrak{Q}_{\mu,\nabla}f=0$. One then has $D_\mu f=0$ and standard elliptic theory shows $f\in C^\infty(M)$. Suppose in addition that the underlying structure is real analytic. It then follows that D_μ is analytic-hypoelliptic and hence $D_\mu f=0$ implies f is real analytic, see, for example, the discussion in [6, 18].

2.2. Affine Killing vector fields.

Let Φ_t^X be the 1-parameter flow associated with an affine Killing vector field X. Then Φ_t^X commutes with ∇ and hence with $\mathfrak{Q}_{\mu,\nabla}$ for all t. Thus if $f\in E(\mu,\nabla)$, then $(\Phi_t^X)^*f\in E(\mu,\nabla)$ for any t. Differentiating this relation with respect to t and setting $t=0$ then shows $Xf\in E(\mu,\nabla)$ as desired.

2.3. **Initial conditions.** We wish to show that if $f \in E(P,\mu,\nabla)$, if $f(P)=0$, and if $df(P)=0$, then $f\equiv 0$. In the real analytic category, this is immediate as we can use Equation (1.b) to show all the higher derivatives vanish. Our task is to give a different derivation in the C^∞ context. To simplify the discussion, we shall assume $m=2$. Introduce local coordinates (x^1,x^2) on M centered at P. Let $B_\varepsilon(0)$ be the ball of radius ε about the origin. Assume that $f\in E(P,\mu,\nabla)$ satisfies $f(0)=df(0)=0$. We will show there exists $\varepsilon>0$ so that $f\equiv 0$ on $B_\varepsilon(0)$. Choose T and ε so that

$$\text{(2.a)}\quad \tfrac{1}{3}<T,\quad |\Gamma_{ij}{}^k(x)|\le T,\quad |\mu\,\rho_{s,\nabla,ij}(x)|\le T \text{ for all } x\in B_\varepsilon(0),\quad \varepsilon<\tfrac{1}{12T}\,.$$

Let

$$\|f\|_1 := \sup_{x\in B_\varepsilon(0)}\{|\partial_{x^1}f(x)|,|\partial_{x^2}f(x)|,|f(x)|\}\,.$$

Let $\vec{x}=(a,b)\in B_\varepsilon(0)$. Let $\gamma(t)=t\vec{x}$. We use Equation (2.a) to estimate:

$$\begin{aligned}
|\partial_t\partial_{x^1}f|\,(t\vec{x}) &= |a\partial_{x^1x^1}f+b\partial_{x^1x^2}f|\,(t\vec{x})\le |a\partial_{x^1x^1}f|\,(t\vec{x})+|b\partial_{x^1x^2}f|\,(t\vec{x})\\
&= |a|\cdot\left|\Gamma_{11}{}^1\partial_{x^1}f+\Gamma_{11}{}^2\partial_{x^2}f+f\mu\rho_{s\nabla,11}\right|(t\vec{x})\\
&\quad +|b|\cdot\left|\Gamma_{12}{}^1\partial_{x^1}f+\Gamma_{12}{}^2\partial_{x^2}f+f\mu\rho_{s\nabla,12}\right|(t\vec{x})\\
&\le 3(|a|+|b|)T\|f\|_1\le 6\,\varepsilon\,T\|f\|_1\,.
\end{aligned}$$

As $\partial_{x^1}f(0)=0$, we may use the Fundamental Theorem of Calculus to estimate:

$$|\partial_{x^1}f(\vec{x})|\le\int_{t=0}^1|\partial_t\partial_{x^1}f(t\vec{x})|\,dt\le\int_{t=0}^1 6\,\varepsilon\,T\,\|f\|_1dt=6\,\varepsilon\,T\|f\|_1\,.$$

We show similarly that $|\partial_{x^2}f(\vec{x})|\le 6\,\varepsilon\,T\|f_1\|$. Finally, since $\frac{1}{3}<T$ and since $f(0)=0$, we estimate

$$|f(\vec{x})|\le\int_{t=0}^1|\partial_t f(t\vec{x})|\,dt\le\int_{t=0}^1(|a|+|b|)\|f\|_1dt\le 2\,\varepsilon\,\|f\|_1\le 6\,\varepsilon\,T\|f\|_1\,.$$

Consequently, $\|f\|_1\le 6\,\varepsilon\,T\|f\|_1$. Since $6\,\varepsilon\,T<\frac{1}{2}$, $\|f\|_1\le\frac{1}{2}\|f\|_1$. This implies $\|f\|_1=0$ on $B_\varepsilon(0)$ and proves Theorem 1.1 (3).

2.4. **Estimating the dimension of $E(P,\mu,\nabla)$.** Let $f\in E(P,\mu,\nabla)$. By Theorem 1.1 (3), f is determined by $f(P)$ and $df(P)$. Assertion (4) now follows.

2.5. **Extending solutions to Equation (1.b).** The final assertion of Theorem 1.1 follows using exactly the same arguments of "analytic continuation" that were used to prove similar assertions for Killing vector fields or affine Killing vector fields (see [13]). □

3. Projective equivalence

In what follows, it will be convenient to work with just one component. Suppose that Φ is a symmetric $(0,2)$-tensor defined on some vector space V and suppose that one could show that $\Phi_{11}=0$ relative to any basis. It then follows that $\Phi=0$; this process is called *polarization.* If $\mathcal{M}=(M,\nabla)$ is an affine manifold, then

$$\begin{aligned}
R_{\nabla,ijk}{}^l&=\partial_{x^i}\Gamma_{jk}{}^l-\partial_{x^j}\Gamma_{ik}{}^l+\Gamma_{in}{}^l\Gamma_{jk}{}^n-\Gamma_{jn}{}^l\Gamma_{ik}{}^n,\\
\rho_{\nabla,jk}&=\partial_{x^i}\Gamma_{jk}{}^i-\partial_{x^j}\Gamma_{ik}{}^i+\Gamma_{in}{}^i\Gamma_{jk}{}^n-\Gamma_{jn}{}^i\Gamma_{ik}{}^n\,.
\end{aligned}$$

Lemma 3.1. *Let $\omega=dg$ provide a strong projective equivalence from ∇ to $\tilde\nabla$.*

(1) $\rho_{s,\tilde\nabla}=\rho_{s,\nabla}-(m-1)\{\mathcal{H}_\nabla g-dg\otimes dg\}$.

(2) *If* $\mu=-\frac{1}{m-1}$ *or if* $\mathcal{H}_\nabla g-dg\otimes dg=0$, *then* $\mathfrak{Q}_{\mu,\nabla}=e^{-g}\,\mathfrak{Q}_{\mu,\tilde\nabla}\,e^{g}$.

Proof. Assume $\tilde\nabla_XY=\nabla_XY+dg(X)Y+dg(Y)X$, i.e.

$$\tilde\Gamma_{ij}{}^k=\Gamma_{ij}{}^k+\delta_i^k\,\partial_{x^j}g+\delta_j^k\,\partial_{x^i}g\,.$$

Fix a point P of M. Since we are working in the category of connections without torsion, we can choose a coordinate system so $\Gamma(P)=0$. We compute at the point P and set $\Gamma_{ij}{}^k(P)=0$ to see

$$\begin{aligned}\rho_{\tilde\nabla,11}(P)&=\{\partial_{x^i}\tilde\Gamma_{11}{}^i-\partial_{x^1}\tilde\Gamma_{i1}{}^i+\tilde\Gamma_{in}{}^i\tilde\Gamma_{11}{}^n-\tilde\Gamma_{1n}{}^i\tilde\Gamma_{i1}{}^n\}(P)\\&=\{\partial_{x^i}\Gamma_{11}{}^i-\partial_{x^1}\Gamma_{i1}{}^i+(1-m)\partial_{x^1x^1}g\\&\qquad+2(m+1)(\partial_{x^1}g)^2-(m+3)(\partial_{x^1}g)^2\}(P)\\&=\{\rho_{\nabla,11}+(m-1)((\partial_{x^1}g)^2-\partial_{x^1x^1}g)\}(P)\\&=\{\rho_\nabla-(m-1)(\mathcal{H}_\nabla g-dg\otimes dg)\}_{11}(P)\,.\end{aligned}$$

Polarizing this identity establishes Assertion (1). To prove Assertion (2), we examine $\mathfrak{Q}_{\mu,\nabla,11}$ and $\{e^{-g}\,\mathfrak{Q}_{\mu,\tilde\nabla}\,e^g\}_{11}$ at P. We compute:

$$\begin{aligned}&\{e^{-g}\mathcal{H}_{\tilde\nabla,11}e^gf\}(P)=\{e^{-g}\partial_{x^1x^1}(fe^g)-\tilde\Gamma_{11}{}^ke^{-g}\partial_{x^k}(fe^g)\}(P)\\&\quad=\{\partial_{x^1x^1}f+2\partial_{x^1}f\partial_{x^1}g+f\partial_{x^1x^1}g+f(\partial_{x^1}g)^2-2\partial_{x^1}g(\partial_{x^1}f+f\partial_{x^1}g)\}(P)\\&\quad=\{\mathcal{H}_{\nabla,11}f+f(\partial_{x^1x^1}g-(\partial_{x^1}g)^2)\}(P).\end{aligned}$$

We complete the proof by polarizing the resulting identity:

$$\begin{aligned}&\{e^{-g}\mathfrak{Q}_{\mu,\tilde\nabla}\,e^gf-\mathfrak{Q}_{\mu,\nabla}f\}_{11}(P)\\&\quad=\{e^{-g}(\mathcal{H}_{\tilde\nabla}e^gf-\mu\rho_{s,\tilde\nabla}e^gf)_{11}-(\mathcal{H}_\nabla f-\mu\rho_{s,\nabla}f)_{11}\}(P)\\&\quad=\{f(1+(m-1)\mu)(\partial_{x^1x^1}g-(\partial_{x^1}g)^2)\}(P).\end{aligned}$$ □

3.1. **Proof of Theorem 1.4.** Theorem 1.4 (1) is immediate from the intertwining relation of Lemma 3.1 (2). The equivalence of Assertion (2a) and Assertion (2b) follows from Lemma 3.1 (1). The equivalence of Assertion (2b) and Assertion (2c) follows by noting

$$\mathfrak{Q}_{0,\nabla}(e^{-g})=\mathcal{H}_\nabla(e^{-g})=-e^{-g}\{\mathcal{H}_\nabla(g)-dg\otimes dg\}\,.$$

Assertion (3) now follows from Assertion (2b) and Lemma 3.1 (2). □

3.2. **Proof of Theorem 1.6.** Let $\mu_m:=-\frac{1}{m-1}$. To prove Assertion (1), we suppose that $\mathcal{M}$ is strongly projectively flat, i.e. ∇ is strongly projectively equivalent to a flat connection $\tilde\nabla$. Under this assumption, there are local coordinates around $P\in M$ so that the Christoffel symbols $\tilde\Gamma_{ij}{}^k$ vanish identically. Thus,

$$E(P,\mu_m,\tilde\nabla)=\operatorname{Span}\{1,x^1,\dots,x^m\}\,.$$

Consequently, by Theorem 1.4, $\dim\{E(\mu_m,\nabla)\}=\dim\{E(\mu_m,\tilde\nabla)\}=m+1$. Next, assume that $\dim\{E(P,\mu,\nabla)\}=m+1$ for some μ. If $\phi\in E(P,\mu,\nabla)$, let

$$\Theta(\phi):=(\phi,\partial_{x^1}\phi,\dots,\partial_{x^m}\phi)(P)\in\mathbb{R}^{m+1}\,.$$

This vanishes if and only if $\phi\equiv0$. For dimensional reasons, Θ must be an isomorphism. Let e_i be the standard basis for $\mathbb{R}^{m+1}$ and let $\phi_i=\Theta^{-1}(e_i)$ be the corresponding basis for $E(P,\mu,\nabla)$. Since $\Theta(\phi_i)=e_i$, we have

$$\begin{array}{lllll}
\phi_0(P)=1, & \partial_{x^1}\phi_0(P)=0, & \partial_{x^2}\phi_0(P)=0, & \dots, & \partial_{x^m}\phi_0(P)=0,\\
\phi_1(P)=0, & \partial_{x^1}\phi_1(P)=1, & \partial_{x^2}\phi_1(P)=0, & \dots, & \partial_{x^m}\phi_1(P)=0,\\
\phi_2(P)=0, & \partial_{x^1}\phi_2(P)=0, & \partial_{x^2}\phi_2(P)=1, & \dots, & \partial_{x^m}\phi_2(P)=0,\\
\dots\dots & \dots\dots & \dots\dots & \dots & \dots\dots,\\
\phi_m(P)=0, & \partial_{x^1}\phi_m(P)=0, & \partial_{x^2}\phi_m(P)=0, & \dots, & \partial_{x^m}\phi_m(P)=1.
\end{array}$$

Set $z^1:=\phi_1/\phi_0$, $\dots$, $z^m:=\phi_m/\phi_0$. We then have

$$\begin{array}{lllll}
z^1(P)=0, & \partial_{x^1}z^1(P)=1, & \partial_{x^2}z^1(P)=0, & \dots, & \partial_{x^m}z^1(P)=0,\\
z^2(P)=0, & \partial_{x^1}z^2(P)=0, & \partial_{x^2}z^2(P)=1, & \dots, & \partial_{x^m}z^2(P)=0,\\
\dots\dots & \dots\dots & \dots\dots & \dots & \dots\dots\\
z^m(P)=0, & \partial_{x^1}z^m(P)=0, & \partial_{x^2}z^m(P)=0, & \dots, & \partial_{x^m}z^m(P)=1.
\end{array}$$

Thus $\vec z(P)=0$ and $d\vec z(P)=\text{id}$. Hence this is an admissible change of coordinates centered at P. Set $g=\log(\phi_0)$. We then obtain

$$E(P,\mu,\nabla)=e^g\,\text{Span}\{1,z^1,\dots,z^m\}\,. \tag{3.a}$$

We have that

$$\mathfrak{Q}_{\mu,\nabla}(e^g)=\mathfrak{Q}_{\mu,\nabla}(\phi_0)=0 \text{ and } \mathfrak{Q}_{\mu,\nabla}(z^ke^g)=\mathfrak{Q}_{\mu,\nabla}(\phi_k)=0\,.$$

We set $e^{-g}\{z^k\mathfrak{Q}_{\mu,\nabla}(e^g)-\mathfrak{Q}_{\mu,\nabla}(z^ke^g)\}=0$ and examine the resulting relations. Fix i, j, and k. We compute:

$$\begin{aligned}
&e^{-g}\left\{\partial_{z^i}\partial_{z^j}(z^ke^g)-z^k\partial_{z^i}\partial_{z^j}(e^g)\right\}=\delta_j^k\,\partial_{z^i}g+\delta_i^k\,\partial_{z^j}g,\\
&-e^{-g}\Gamma_{ij}{}^\ell\left\{\partial_{z^\ell}(z^ke^g)-z^k\partial_{z^\ell}(e^g)\right\}=-\Gamma_{ij}{}^k,\\
&e^{-g}\left\{\mu\rho_{s,\nabla}(z^ke^g)-z^k\mu\rho_{s,\nabla}e^g\right\}=0,\\
&0=e^g\left\{\mathfrak{Q}_{\mu,\nabla}(z^ke^g)-z^k\mathfrak{Q}_{\mu,\nabla}(e^g)\right\}_{ij}=\delta_j^k\,\partial_{z^i}g+\delta_i^k\,\partial_{z^j}g-\Gamma_{ij}{}^k.
\end{aligned}$$

Let $\tilde\Gamma_{ij}{}^k=0$ define a flat connection $\tilde\nabla$. We have $\Gamma_{ij}{}^k=\tilde\Gamma_{ij}{}^k+\delta_j^k\,\partial_{z^i}g+\delta_i^k\,\partial_{z^j}g$ so dg provides a strong projective equivalence from $\tilde\nabla$ to ∇. Consequently, ∇ is strongly projectively flat. This establishes Assertions (1) and (2).

Furthermore, by Theorem 1.4, $\tilde f\to e^g\tilde f$ is an isomorphism from $E(P,\mu_m,\tilde\nabla)$ to $E(P,\mu_m,\nabla)$. Since $1\in E(P,\mu_m,\tilde\nabla)$, $e^g\in E(P,\mu_m,\nabla)$. By Equation (3.a), $e^g\in E(P,\mu,\nabla)$. This means $\mathcal{H}_\nabla e^g=e^g\mu_m\rho_{s,\nabla}$ and $\mathcal{H}_\nabla e^g=e^g\mu\rho_{s,\nabla}$. Since $\mu\neq\mu_m$, this implies $\rho_{s,\nabla}=0$. Since $\tilde\nabla$ is flat, $\rho_{a,\tilde\nabla}=0$. By Remark 1.5, the alternating Ricci tensor is preserved by strong projective equivalence. Consequently, $\rho_{a,\nabla}=0$ as well. This implies ∇ is Ricci flat which establishes Assertion (3). Assertion (4) follows from the discussion given above. □

3.3. **The proof of Theorem 1.8.** Suppose dg provides a strong projective equivalence from ∇ to a connection $\tilde\nabla$ with $\rho_{s,\tilde\nabla}=0$. We use Lemma 3.1 to see that $\mathcal{H}_\nabla g-dg\otimes dg=\frac{1}{m-1}\rho_{s,\nabla}$. Set $f=e^{-g}$. Then

$$\mathcal{H}_\nabla f=e^{-g}\{-\mathcal{H}_\nabla g+dg\otimes dg\}=-\tfrac{1}{m-1}f\rho_{s,\nabla}$$

so $f\in E(\mu_m,\nabla)$ is non-trivial. This establishes Assertion (1) of Theorem 1.8. Conversely, of course, if $f\in E(P,\mu_m,\nabla)$ satisfies $f(P)\neq0$, then we may assume $f(P)>0$ and set $g=-\log(f)$. Reversing the argument then establishes Assertion (2) of Theorem 1.8. □

3.4. **The proof of Theorem 1.9.** Let $m = 2$ and $\mu_2 = -1$. Suppose to the contrary that $\dim\{E(P, -1, \nabla)\} = 2$; we argue for a contradiction. Suppose first that $f(P) > 0$ for some $f \in E(P, -1, \nabla)$. Express $f = e^g$ near P. Let $-dg$ provide a strong projective equivalence from ∇ to $\tilde{\nabla}$. By Theorem 1.4, $1 = e^{-g} f \in E(P, -1, \tilde{\nabla})$. It now follows that $\rho_{s,\tilde{\nabla}} = 0$. By Remark 1.5, $\rho_{a,\nabla} = \rho_{a,\tilde{\nabla}}$. Thus if $\rho_{a,\nabla} = 0$, then $\tilde{\nabla}$ is Ricci flat and hence, since $m = 2$, $\tilde{\nabla}$ is flat. Consequently, we apply Theorem 1.6 to conclude $\dim\{E(P, -1, \nabla)\} = 3$ contrary to our assumption.

We suppose therefore that $\rho_{a,\tilde{\nabla}} = \rho_{a,\nabla}$ is non-trivial and $\rho_{s,\tilde{\nabla}} = 0$. Hence $\rho_{a,\tilde{\nabla}}$ defines a nonzero two-form, which shows that the curvature tensor is recurrent. Thus $(M, \tilde{\nabla})$ is locally described by the work of Wong [19, Theorem 4.2]. Recently, Derdzinski [7, Theorem 6.1] has shown that local coordinates can be specialized so that the only non-zero Christoffel symbols are ${\Gamma_{11}}^1 = -\partial_{x^1}\phi$ and ${\Gamma_{22}}^2 = \partial_{x^2}\phi$. We have $E(P, \mu_2, \tilde{\nabla}) = \ker(\mathcal{H}_{\tilde{\nabla}})$. Since this is, by assumption, 2-dimensional, we can apply Theorem 1.1 to choose $\tilde{f} \in \ker(\mathcal{H}_{\tilde{\nabla}})$ so that $d\tilde{f}(P) \neq 0$. We compute

$$0 = \mathcal{H}_{\tilde{\nabla},11}\tilde{f} = \partial_{x^1x^1}\tilde{f} + \partial_{x^1}\phi\,\partial_{x^1}\tilde{f}, \quad 0 = \mathcal{H}_{\tilde{\nabla},22}\tilde{f} = \partial_{x^2x^2}\tilde{f} - \partial_{x^2}\phi\,\partial_{x^2}\tilde{f},$$
$$0 = \mathcal{H}_{\tilde{\nabla},12}\tilde{f} = \partial_{x^1x^2}\tilde{f}\,.$$

The relation $\partial_{x^1x^2}\tilde{f} = 0$ implies $\tilde{f}(x^1, x^2) = a(x^1) + b(x^2)$. Differentiating the remaining relations with respect to x^2 and x^1, respectively, yields

$$\partial_{x^1x^2}\phi\, a'(x^1) = 0 \text{ and } -\partial_{x^1x^2}\phi\, b'(x^2) = 0\,.$$

By assumption, $d\tilde{f}(P) \neq 0$ and thus $(a'(0), b'(0)) \neq (0, 0)$. Thus $\partial_{x^1x^2}\phi$ vanishes identically at P. This implies the geometry is flat and $\rho_{a,\tilde{\nabla}} = 0$ contrary to our assumption.

Suppose $f(P) = 0$ for every $f \in E(P, -1, \nabla)$ and $\dim\{E(P, -1, \nabla)\} = 2$. Let $\{f_1, f_2\}$ be a basis for $E(P, -1, \nabla)$. Since $f_i(P) = 0$, we may apply Theorem 1.1 to see $df_1(P)$ and $df_2(P)$ are linearly independent. Thus we can choose local coordinates centered at P so that $E(P, -1, \nabla) = \operatorname{Span}\{x^1, x^2\}$. If $Q \neq P$, then $\dim\{E(Q, -1, \nabla)\} \geq 2$ and there exists a non-vanishing element f_Q of $E(Q, -1, \nabla)$ with $f_Q(Q) \neq 0$. The argument given above shows that $\dim\{E(Q, -1, \nabla)\} = 3$. Thus ∇ is strongly projectively flat near Q so by Lemma 1.3 (1), ρ_∇ and $\nabla\rho_\nabla$ are totally symmetric at Q. Thus, by continuity, the same holds at P. Thus by Lemma 1.3 (2), we can conclude that ∇ is strongly projectively flat on a neighborhood of P and $\dim\{E(P, -1, \nabla)\} = 3$ contrary to our assumption.

Appendix A. Locally homogeneous affine surfaces

We say that $\mathcal{M} = (M, \nabla)$ is *locally homogeneous* if, given any two points of M, there is the germ of a diffeomorphism T taking one point to another with $T^*\nabla = \nabla$. Locally homogeneous affine surfaces have been classified by Opozda [15]. Let $\mathcal{M}$ be a locally homogeneous affine surface which is not flat, i.e. has non-vanishing Ricci tensor. Then at least one of the following three possibilities holds, which are not exclusive, and which describe the local geometry:

Type $\mathcal{A}$: There exist local coordinates (x^1, x^2) so that ${\Gamma_{ij}}^k$ are constant.

Type $\mathcal{B}$: There exist local coordinates (x^1, x^2) so that ${\Gamma_{ij}}^k = (x^1)^{-1}{C_{ij}}^k$ where ${C_{ij}}^k$ are constant.

Type $\mathcal{C}$: ∇ is the Levi-Civita connection of a metric of constant sectional curvature.

In Section A.1 and Section A.2, we present 2 and 3-dimensional solutions to the affine quasi-Einstein equation (1.b) which are Type $\mathcal{A}$ and Type $\mathcal{B}$ geometries. Our account here is purely expository to illustrate some of the phenomena which occur; we shall postpone the proofs of these results for a subsequent paper [5]. In each case we consider the essentially different eigenvalues $\mu = 0$, $\mu_m = -\frac{1}{m-1}$, and $\mu \neq 0, -\frac{1}{m-1}$ separately.

A.1. **Type $\mathcal{A}$ surfaces.** Let $\mathcal{M} = (\mathbb{R}^2, \nabla)$ be a Type $\mathcal{A}$ surface model which is not flat; the Christoffel symbols satisfy ${\Gamma_{ij}}^k = {\Gamma_{ji}}^k \in \mathbb{R}$. Any Type $\mathcal{A}$ surface is projectively flat with symmetric Ricci tensor [3], thus strongly projectively flat.

Theorem A.1. *Let $\mathcal{M}$ be a Type $\mathcal{A}$ surface model.*

(1) *Let $\mu = 0$. Then $E(0, \nabla) = \operatorname{Span}\{1\}$ or, up to linear equivalence, one of the following holds:*
   (a) ${\Gamma_{11}}^1 = 1$, ${\Gamma_{12}}^1 = 0$, ${\Gamma_{22}}^1 = 0$, *and* $E(0, \nabla) = \operatorname{Span}\{1, e^{x^1}\}$.
   (b) ${\Gamma_{11}}^1 = {\Gamma_{12}}^1 = {\Gamma_{22}}^1 = 0$, *and* $E(0, \nabla) = \operatorname{Span}\{1, x^1\}$.

(2) *Let $\mu = -1$. Then* $\dim\{E(-1, \nabla)\} = 3$.

(3) *Let $\mu \neq 0, -1$. Then* $\dim\{E(\mu, \nabla)\} = \left\{ \begin{array}{l} 2 \text{ if } \operatorname{Rank}\{\rho_\nabla\} = 1 \\ 0 \text{ if } \operatorname{Rank}\{\rho_\nabla\} = 2 \end{array} \right\}$.

We can use Theorem 1.4 to construct non-trivial projective deformations.

Example A.2. We set ${\Gamma_{11}}^1 = 1$, ${\Gamma_{12}}^1 = 0$, ${\Gamma_{22}}^1 = 0$ as in Theorem A.1-(1.a). Then $\phi(x^1, x^2) = a + e^{x^1} \in E(0, \nabla)$. Following Theorem 1.4, set $g = -\log \phi(x^1, x^2)$ and consider the strongly projectively equivalent connection $\tilde\nabla$ determined by the 1-form $\omega = dg$. We have $\rho_\nabla = \rho_{\tilde\nabla} = \rho_{\nabla,11} dx^1 \otimes dx^1$; both $\nabla\rho_\nabla$ and $\tilde\nabla\rho_{\tilde\nabla}$ are multiples of $dx^1 \otimes dx^1 \otimes dx^1$. Thus $\alpha := \nabla\rho_{111}^2 \cdot \rho_{11}^{-3}$ is an affine invariant (see [3]) and we have $\alpha_\nabla = 4({\Gamma_{12}}^2 - ({\Gamma_{12}}^2)^2 + {\Gamma_{11}}^2{\Gamma_{22}}^2)^{-1}$ and $\alpha_{\tilde\nabla} = \alpha_\nabla \cdot (a - e^{x^1})^2 (a + e^{x^1})^{-2}$. Since $\alpha_{\tilde\nabla}$ is non-constant for $a \neq 0$, we are getting affine inequivalent surfaces which are strongly Liouville equivalent. If $a = 0$, we obtain an isomorphic Type $\mathcal{A}$ structure.

Example A.3. We set ${\Gamma_{11}}^1 = 0$, ${\Gamma_{12}}^1 = 0$, ${\Gamma_{22}}^1 = 0$, as in Theorem A.1-(1.b). We then have $\rho_\nabla = \{{\Gamma_{11}}^2{\Gamma_{22}}^2 - ({\Gamma_{12}}^2)^2\} dx^1 \otimes dx^1$ and $\nabla\rho_\nabla = 0$. Since $x^1 \in E(0, \nabla)$, we follow Theorem 1.4 and consider the strongly Liouville equivalent connection $\tilde\nabla$ determined by the 1-form $\omega = -d\log x^1$. We verify that

$$\rho_{\tilde\nabla} = \rho_\nabla \quad \text{and} \quad \tilde\nabla\rho_{\tilde\nabla} = 4(x^1)^{-1}\rho_{\nabla,11} dx^1 \otimes dx^1 \otimes dx^1$$

so this is not a symmetric space if we choose ${\Gamma_{11}}^2{\Gamma_{22}}^2 - ({\Gamma_{12}}^2)^2 \neq 0$. Hence ∇ is not locally isomorphic to $\tilde\nabla$.

A.1.1. *Higher dimensional examples of Type $\mathcal{A}$.* Let $\mathcal{M} = (\mathbb{R}^3, \nabla)$ be a Type $\mathcal{A}$ geometry, so the Christoffel symbols ${\Gamma_{ij}}^k$ are constant. We shall only list the non-zero Christoffel symbols in what follows and omit the details of the computation. Here $\mu_3 = -\frac{1}{2}$. The following is an example where ρ is non-degenerate and $\mu \neq -\frac{1}{2}$.

Example A.4. Set the non-zero Christoffel symbols ${\Gamma_{12}}^3 = 1$, ${\Gamma_{13}}^1 = 3$, ${\Gamma_{23}}^2 = 4$, ${\Gamma_{33}}^3 = 5$. Then $\rho_\nabla = 5dx^1 \otimes dx^2 + 5dx^2 \otimes dx^1 + 10dx^3 \otimes dx^3$. We have

$$E(\mu, \nabla) = \left\{ \begin{array}{ll} \operatorname{Span}\{e^{3x^3}, x^1 e^{3x^3}\} & \text{if } \mu = -\frac{3}{5} \\ \operatorname{Span}\{1\} & \text{if } \mu = 0 \\ \{0\} & \text{otherwise} \end{array} \right\}.$$

The Ricci tensor in the following example is degenerate, but non-zero, and there are an infinite number of non-trivial eigenvalues; this is a genuinely new phenomena not present for Type $\mathcal{A}$ surface models.

Example A.5. Let $\mathcal{M}_{x,y,z,w} := (\mathbb{R}^3, \nabla)$ be a 3-dimensional Type $\mathcal{A}$ model where the (possibly) non-zero Christoffel symbols are:

$$\Gamma_{11}{}^1 = z,\quad \Gamma_{12}{}^1 = 1,\quad \Gamma_{13}{}^1 = x,\quad \Gamma_{22}{}^2 = 1,\quad \Gamma_{23}{}^1 = x,\quad \Gamma_{33}{}^2 = y,\quad \Gamma_{33}{}^3 = w.$$

The Ricci tensor is

$$\rho_\nabla = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x(z-1) \\ 0 & x(z-1) & wx - x^2 + 2y \end{pmatrix}.$$

Depending on the values of x, y, z and w, $\dim\{E(-\frac{1}{2}, \nabla)\}$ is as follows:

(1) $\dim\{E(-\frac{1}{2}, \nabla)\} = 0$ if and only if $x \neq 0$ and either $z = 0$ or $z \notin \{0, 1\}$ and $w \neq \frac{x+2xz-xz^2}{2z}$.
(2) $\dim\{E(-\frac{1}{2}, \nabla)\} = 1$ if and only if $x \neq 0$, $z \notin \{0, 1\}$ and $w = \frac{x+2xz-xz^2}{2z}$.
(3) $\dim\{E(-\frac{1}{2}, \nabla)\} = 2$ if and only if $x \neq 0$, $z = 1$, $w \neq x$.
(4) $\dim\{E(-\frac{1}{2}, \nabla)\} = 4$ if and only if $x = 0$ or $w = x$ and $z = 1$.

A.2. **Type $\mathcal{B}$ surface models.** Let $\mathcal{M} = (\mathbb{R}^+ \times \mathbb{R}, \nabla)$ be a Type $\mathcal{B}$ affine surface model; the Christoffel symbols are given by $\Gamma_{ij}{}^k = (x^1)^{-1} C_{ij}{}^k$ where $C_{ij}{}^k$ are constant. We assume $\rho_\nabla \neq 0$ to ensure the geometry is not flat. We have $\mathcal{M}$ is also Type $\mathcal{A}$ if and only if $(C_{12}{}^1, C_{22}{}^1, C_{22}{}^2) = (0, 0, 0)$; the Ricci tensor has rank 1 in this instance (see [3]). We first examine the Yamabe solitons, working modulo linear equivalence:

Theorem A.6. *Let $\mathcal{M}$ be a Type $\mathcal{B}$ surface. Then $E(0, \nabla) = \operatorname{Span}\{1\}$ except in the following cases where we also require $\rho_\nabla \neq 0$.*

(1) $(C_{11}{}^1, C_{12}{}^1, C_{22}{}^1) = (-1, 0, 0)$, *and* $E(0, \nabla) = \operatorname{Span}\{1, \log(x^1)\}$.
(2) $(C_{11}{}^1, C_{12}{}^1, C_{22}{}^1) = \kappa(-1, 0, 0)$, $E(0, \nabla) = \operatorname{Span}\{1, (x^1)^{C_{11}{}^1+1}\}$, *and* $\kappa \neq 1$.
(3) $(C_{11}{}^2, C_{12}{}^2, C_{22}{}^2) = (0, 0, 0)$, *and* $E(0, \nabla) = \operatorname{Span}\{1, x^2\}$.
(4) $(C_{11}{}^1, C_{12}{}^1, C_{22}{}^1) = c(C_{11}{}^2, C_{12}{}^2, C_{22}{}^2)$, *and* $E(0, \nabla) = \operatorname{Span}\{1, x^1 - cx^2\}$.

Any Type $\mathcal{B}$ surface which is also Type $\mathcal{A}$ is strongly projectively flat. There are, however, strongly projectively flat surfaces of Type $\mathcal{B}$ which are not of Type $\mathcal{A}$. Moreover, there exist Type $\mathcal{B}$ surfaces where $\dim\{E(-1, \nabla)\} = 1$.

Theorem A.7. *Let $\mathcal{M}$ be a Type $\mathcal{B}$ surface. Let $\mu = -1$. Then one of the following holds*

(1) $\dim\{E(-1, \nabla)\} = 1$ *if and only if $\mathcal{M}$ is linearly equivalent to:*
   (a) $C_{22}{}^1 = 0$, $C_{22}{}^2 = C_{12}{}^1 \neq 0$, *or*
   (b) $C_{22}{}^1 = \pm 1$, $C_{12}{}^1 = 0$, $C_{22}{}^2 = \pm 2C_{11}{}^2 \neq 0$, $C_{11}{}^1 = 1 + 2C_{12}{}^2 \pm (C_{11}{}^2)^2$.
(2) $\dim\{E(-1, \nabla)\} = 3$ *if and only if $\mathcal{M}$ is strongly projectively flat. In this case $\mathcal{M}$ is linearly equivalent to one of the surfaces:*
   (a) $C_{12}{}^1 = C_{22}{}^1 = C_{22}{}^2 = 0$ *(i.e. $\mathcal{M}$ is also of Type $\mathcal{A}$).*
   (b) $C_{11}{}^1 = 1 + 2C_{12}{}^2$, $C_{11}{}^2 = 0$, $C_{12}{}^1 = 0$, $C_{12}{}^2 \neq 0$, $C_{22}{}^1 = \pm 1$, $C_{22}{}^2 = 0$.

Let $\mu \neq 0$ and $\mu \neq -1$. In the Type $\mathcal{A}$ setting, Theorem A.1 shows that $\dim\{E(\mu,\nabla)\} = 0$ or $\dim\{E(\mu,\nabla)\} = 2$. The situation is quite different in the Type $\mathcal{B}$ setting as there are examples where $\dim\{E(\mu,\nabla)\} = 1$.

Theorem A.8. *Let $\mathcal{M}$ be a Type $\mathcal{B}$ model which is not of Type $\mathcal{A}$ with $\rho_{s,\nabla} \neq 0$ and let $\mu \neq 0, -1$.*

(1) $\dim\{E(\mu,\nabla)\} \geq 1$ *if and only if $\mathcal{M}$ is linearly equivalent to a surface given by $C_{22}{}^1 = \pm 1$, $C_{12}{}^1 = 0$, $C_{22}{}^2 = \pm 2 C_{11}{}^2$, where μ is determined by $\mu = \Delta^{-2}\{1 + 2C_{12}{}^2 \pm 2(C_{11}{}^2)^2 - (C_{11}{}^1 - C_{12}{}^2)^2\}$, for $\Delta := 1 - C_{11}{}^1 + C_{12}{}^2 \neq 0$.*

(2) $\dim\{E(\mu,\nabla)\} = 2$ *if and only if $\mathcal{M}$ is linearly equivalent to one of the following two surfaces:*

(a) $C_{11}{}^1 = -1 + C_{12}{}^2$, $C_{11}{}^2 = 0$, $C_{12}{}^1 = 0$, $C_{22}{}^1 = \pm 1$, $C_{22}{}^2 = 0$, *where* $\mu = \frac{1}{2} C_{12}{}^2 \neq 0$.

(b) $C_{11}{}^1 = -\frac{1}{2}(5 \pm 16(C_{11}{}^2)^2)$, $C_{12}{}^1 = 0$, $C_{12}{}^2 = -\frac{1}{2}(3 \pm 8(C_{11}{}^2)^2)$, $C_{22}{}^1 = \pm 1$, $C_{22}{}^2 = \pm 2 C_{11}{}^2$, *where* $\mu = -\frac{3 \pm 8(C_{11}{}^2)^2}{4 \pm 8(C_{11}{}^2)^2}$. *and where* $C_{11}{}^2 \neq 0, \pm\frac{1}{\sqrt{2}}$.

Remark A.9. The existence of examples where $\dim\{E(0,\nabla)\}$ is either 0, 1 or 2 was shown in Theorem A.6. The surfaces of Theorem A.7 provide examples where one has $\dim\{E(-1,\nabla)\} = 1$ and $\dim\{E(-1,\nabla)\} = 3$. In Theorem A.8, we gave examples of homogeneous affine surfaces where $\dim\{E(\mu,\nabla)\} = 1$ and $\dim\{E(\mu,\nabla)\} = 2$, for arbitrary $\mu \neq 0, -1$. Thus all values for $\dim\{E(\mu,\nabla)\}$ are permissible.

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