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ON THE EXISTENCE OF PERIODIC SOLUTIONS FOR A FRACTIONAL SCHRÖDINGER EQUATION

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ABSTRACT. We present an elementary proof of the existence of a nontrivial weak periodic solution for a nonlinear fractional problem driven by a relativistic Schrödinger operator with periodic boundary conditions and involving a periodic continuous subcritical nonlinearity satisfying a more general Ambrosetti-Rabinowitz condition.

1. INTRODUCTION

In this note we deal with the existence of nontrivial weak periodic solutions for the following nonlinear fractional problem:

$$\begin{cases} [(-\Delta+m^2)^s - m^{2s} + \mu]u = f(x,u) & \text{in } (-\pi,\pi)^N, \\ u(x+2\pi e_i) = u(x) & \text{for all } x\in\mathbb{R}^N, i=1,\dots,N, \end{cases} \tag{1.1}$$

where $m>0$, $\mu>0$, $s\in(0,1)$, $N>2s$, and $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a continuous function with $f(\cdot,0)=0$ and satisfying suitable growth assumptions.

Here, the fractional Schrödinger operator $(-\Delta+m^2)^s$ is a nonlocal operator which can be defined for any $u=\sum_{k\in\mathbb{Z}^N} c_k \frac{e^{\imath k\cdot x}}{(2\pi)^{\frac{N}{2}}}\in\mathcal{C}^\infty_{2\pi}(\mathbb{R}^N)$, that is, u is infinitely differentiable in $\mathbb{R}^N$ and 2π-periodic in each variable, by setting

$$(-\Delta+m^2)^s u(x) = \sum_{k\in\mathbb{Z}^N} c_k(|k|^2+m^2)^s \,\frac{e^{\imath k\cdot x}}{(2\pi)^{\frac{N}{2}}}, \tag{1.2}$$

where $c_k := \frac{1}{(2\pi)^{\frac{N}{2}}}\int_{(-\pi,\pi)^N} u(x)e^{-\imath k\cdot x}dx$ $(k\in\mathbb{Z}^N)$ are the Fourier coefficients of the function u.

This operator can be extended by density on the Hilbert space

$$\mathbb{H}^s_{2\pi} := \Big\{u=\sum_{k\in\mathbb{Z}^N} c_k\frac{e^{\imath k\cdot x}}{(2\pi)^{\frac{N}{2}}}\in L^2(-\pi,\pi)^N : \sum_{k\in\mathbb{Z}^N}(|k|^2+m^2)^s\,|c_k|^2<+\infty\Big\}$$

endowed with the norm $|u|_{\mathbb{H}^s_{2\pi}} := \left(\sum_{k\in\mathbb{Z}^N}(|k|^2+m^2)^s|c_k|^2\right)^{1/2}$.

When $m=0$, the operator in (1.2) arises in models with periodic boundary conditions; see [11, 12, 17]. In $\mathbb{R}^N$, one of the reasons for studying the operator $(-\Delta+m^2)^s$ is related to its physical meaning. Indeed, when $s=1/2$, the operator $(-\Delta+m^2)^{1/2}-m$ corresponds to the free Hamiltonian of a relativistic particle

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with mass m; see [18]. There is also a deep connection between $(-\Delta+m^2)^s-m^{2s}$ and the theory of Lévy processes; such operator is the infinitesimal generator of a relativistic $2s$-stable process $\{X_t^m\}_{t\geq 0}$, that is, a Lévy process with characteristic function given by

$$\mathbb{E}(e^{i\xi\cdot X_t^m}) := e^{-t[(m^2+|\xi|^2)^s-m^{2s}]} \quad (\xi\in\mathbb{R}^N);$$

further details can be found in [7]. However, as observed in [25], the operators $(-\Delta+m^2)^s$ on $\mathbb{R}^N$ and (1.2) are not the same, but coincide in the case in which the domain is a torus.

More generally, the study of fractional and nonlocal operators has received a lot of interest in the last decade in both pure and applied mathematical research. In fact, these operators appear in several concrete real-world applications, such as phase transitions, flames propagation, chemical reaction in liquids, population dynamics, American options in finance, and crystal dislocation; see [13, 19].

Nonlinear problems involving nonlocal operators are currently actively studied. Servadei and Valdinoci in [24] investigated the existence of nontrivial solutions for equations driven by a nonlocal integro-differential operator with homogeneous Dirichlet boundary conditions and in the presence of a subcritical nonlinearity satisfying the Ambrosetti-Rabinowitz condition. Felmer et al. [15] proved existence, regularity, and symmetry properties of positive solutions for a fractional Schrödinger equation in $\mathbb{R}^N$ with a subcritical nonlinearity. Barrios et al. [6] dealt with the existence of positive solutions for a critical problem under the effect of lower order perturbations and involving the spectral Laplacian in a bounded domain. Dipierro et al. [14] considered a fractional evolution equation arising in the Peierls-Nabarro model for crystal dislocation. Pucci et al. [20] studied the existence of multiple solutions for nonhomogeneous fractional p-Laplacian equations of Schrödinger-Kirchhoff type with a nonlinearity verifying the Ambrosetti-Rabinowitz condition. Roncal and Stinga [22, 23] established Harnack's inequalities, regularity estimates, and pointwise formulas for the fractional Laplacian on a torus. In [2–4], the author obtained some existence results for superlinear fractional equations with periodic boundary conditions via suitable versions of the Linking Theorem. We also mention [5], in which the author and Molica Bisci proved the existence of multiple periodic solutions for a nonlocal periodic problem by combining a variant of the Mountain Pass Theorem and a local minimum result for differentiable functionals.

Motivated by the interest shared by the mathematical community in this topic, the main goal of this paper is to investigate the existence of nontrivial periodic solutions to (1.1) under the assumptions that the nonlinear term f satisfies growth hypotheses weaker than those commonly used in the literature. More precisely, we suppose that $f(x,t):\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a continuous function, 2π-periodic in x, $f(\cdot,0)=0$, and verifies the following conditions:

($f1$) There exist $M>0$ and $q\in(2,2_s^*)$ with $2_s^*:=\frac{2N}{N-2s}$, such that

$$|f(x,t)|\leq M(1+|t|^{q-1}) \text{ for any } x\in(-\pi,\pi)^N \text{ and for all } t\in\mathbb{R}.$$

($f2$) $\displaystyle\lim_{t\to 0}\frac{F_+(x,t)}{|t|^2}=0$ uniformly for $x\in[-\pi,\pi]^N$, where $F(x,t)=\displaystyle\int_0^t f(x,\tau)\,d\tau$ and $F_+:=\max\{F,0\}$.

($f3$) There is a compact set $K\subset(-\pi,\pi)^N$ with nonempty interior such that F is bounded below on $K\times[0,\infty)$, and there exists a sequence $t_n\to\infty$ such that

$$\lim_{n\to\infty} \frac{F(x,t_n)}{t_n^2} = \infty \qquad \text{for a.e. } x \in K.$$

($f4$) $f(x,t)t \geq 2F(x,t)$.

($f5$) There are a constant $\alpha > 2$ and a function $\sigma \in L^1(-\pi,\pi)^N$ such that

$$f(x,t)t \geq \alpha F(x,t)$$

whenever $F(x,t) \geq \sigma(x)$.

Condition ($f5$) was first introduced in [21] to study the existence of nontrivial solutions for semilinear elliptic problems with homogeneous Dirichlet boundary conditions.

Let us point out that the standard Ambrosetti-Rabinowitz condition [1]

$$f(\cdot,t)t \geq \alpha F(\cdot,t) \quad \text{if } |t| \geq R \tag{1.3}$$

with $\alpha > 2$ and $R \geq 0$ is a special case of assumption ($f5$) when σ is constant. Indeed, it implies ($f5$) with any $\sigma > \max_{|t|\leq R, x\in[-\pi,\pi]^N} F(x,t)$. We also note that ($f5$) is more general than (1.3), even when σ is constant. For instance, if we consider the function $\tilde{F}(\tau) = \tau^6\left(\sin\left(\frac{1}{\tau^2}\right)+1\right) + \tau^4$, and we define the function $F(x,t) := \tilde{F}(xt)$ for $(x,t) \in (-\pi,\pi)\times\mathbb{R}$, then F satisfies conditions ($f2$), ($f3$), ($f4$), and ($f5$) with $\alpha = 3$ and any large constant σ such that $\tilde{F}(\tau) \geq \sigma$ implies that $\tilde{F}'(\tau)\tau \geq 3\tilde{F}(\tau)$. On the other hand, (1.3) fails for every $\alpha > 2$ and $R \geq 0$.

Now we state our main result.

Theorem 1. *Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a continuous function, 2π-periodic in x, with $f(\cdot,0) = 0$ and verifying the assumptions ($f1$)-($f5$). Then there exists a nontrivial weak periodic solution $u \in \mathbb{H}^s_{2\pi}$ to (1.1).*

As customary in many fractional problems set in $\mathbb{R}^N$ or in bounded domain (see [8–10, 26]), we will work in an extended space, which reduces the fractional operator to a local one, confining the nonlocal feature to a boundary reaction problem. As proved in [2, 3] (see also [22, 26]), such a procedure works also in a periodic setting.

Therefore, instead of (1.1), we investigate the following degenerate elliptic problem:

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla U) + m^2 y^{1-2s} U = 0 & \text{in } \mathcal{S}_{2\pi}, \\ U_{|\{x_i=-\pi\}} = U_{|\{x_i=\pi\}} & \text{on } \partial_L \mathcal{S}_{2\pi}, \\ \partial_\nu^{1-2s} U = \kappa_s[m^{2s}U - \mu U + f(x,U)] & \text{on } \partial^0 \mathcal{S}_{2\pi}, \end{cases} \tag{1.4}$$

where

$$\kappa_s = 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)} \qquad \text{and} \qquad \partial_\nu^{1-2s} U := -\lim_{y\to 0^+} y^{1-2s}\frac{\partial U}{\partial y}(x,y)$$

is the conormal exterior derivative of U.

Taking into account the variational structure of (1.4), we seek critical points of the energy functional $\mathcal{J}$ given by

$$\mathcal{J}(U) = \frac{1}{2}\left[\|U\|^2_{\mathbb{X}^s_{2\pi}} - (m^{2s}-\mu)\kappa_s |\mathrm{Tr}(U)|^2_{L^2(-\pi,\pi)^N}\right] - \kappa_s \int_{\partial^0 \mathcal{S}_{2\pi}} F(x, \mathrm{Tr}(U))\, dx,$$

defined on the Hilbert space $\mathbb{X}^s_{2\pi}$, which is the closure of the set $\mathcal{C}^\infty_{2\pi}(\overline{\mathbb{R}^{N+1}_+})$ of smooth and 2π-periodic (in x) functions in $\mathbb{R}^{N+1}_+$ with respect to the norm

$$\|U\|_{\mathbb{X}^s_{2\pi}} := \Big(\iint_{\mathcal{S}_{2\pi}} y^{1-2s}(|\nabla U|^2 + m^{2s}U^2)\, dxdy\Big)^{1/2}.$$

In order to prove the existence of a nontrivial weak solution to (1.4), we first verify that $\mathcal{J}$ satisfies the assumptions of an abstract critical point result [21] (see also [16, 27]), which guarantees the existence of a bounded Palais-Smale sequence (U_n) for functionals having a mountain pass geometry, and then we show that, up to a subsequence, (U_n) converges to a nontrivial solution U of (1.4).

The structure of the paper is the following: Section 2 contains some preliminary results concerning the fractional periodic Sobolev spaces and the extension method in a periodic setting, and in Section 3 we give the proof of Theorem 1.

2. Preliminaries

Throughout this paper, we denote the upper half-space in $\mathbb{R}^{N+1}$ by

$$\mathbb{R}^{N+1}_+ = \{(x, y) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N, y > 0\}.$$

With $\mathcal{S}_{2\pi} := (-\pi, \pi)^N \times (0, \infty)$ we denote the half-cylinder in $\mathbb{R}^{N+1}_+$ with basis $\partial^0\mathcal{S}_{2\pi} := (-\pi, \pi)^N \times \{0\}$ and lateral boundary $\partial_L\mathcal{S}_{2\pi} := \partial(-\pi, \pi)^N \times [0, +\infty)$. We also use the notation $|u|_r$ to denote the norm of any function $u : \mathbb{R}^N \to \mathbb{R}$ in $L^r(-\pi, \pi)^N$.

Now, we recall the following fundamental results concerning the spaces $\mathbb{X}^s_{2\pi}$ and $\mathbb{H}^s_{2\pi}$.

Theorem 2 ([2,3]). *There exists a surjective linear operator* $\mathrm{Tr} : \mathbb{X}^s_{2\pi} \to \mathbb{H}^s_{2\pi}$ *such that:*

(i) $\mathrm{Tr}(U) = U|_{\partial^0\mathcal{S}_{2\pi}}$ *for all* $U \in \mathcal{C}^\infty_{2\pi}(\overline{\mathbb{R}^{N+1}_+}) \cap \mathbb{X}^s_{2\pi}$;

(ii) Tr *is bounded and*

$$\sqrt{\kappa_s}|\mathrm{Tr}(U)|_{\mathbb{H}^s_{2\pi}} \le \|U\|_{\mathbb{X}^s_{2\pi}}, \tag{2.1}$$

for every $U \in \mathbb{X}^s_{2\pi}$. *In particular, equality holds in* (2.1) *for some* $v \in \mathbb{X}^s_{2\pi}$ *if and only if* v *weakly solves the equation*

$$-\operatorname{div}(y^{1-2s}\nabla U) + m^2 y^{1-2s}U = 0 \quad \text{in } \mathcal{S}_{2\pi}.$$

Theorem 3 ([2, 3]). *Let* $N > 2s$. *Then* $\mathrm{Tr}(\mathbb{X}^s_{2\pi})$ *is continuously embedded in* $L^q(-\pi, \pi)^N$ *for any* $1 \le q \le 2^\sharp_s$. *Moreover,* $\mathrm{Tr}(\mathbb{X}^s_{2\pi})$ *is compactly embedded in* $L^q(-\pi, \pi)^N$ *for any* $1 \le q < 2^\sharp_s$.

Taking into account Theorems 2 and 3, it is possible to introduce the notion of extension for a function $u \in \mathbb{H}^s_{2\pi}$. More precisely, we have the following result.

Theorem 4 ([2,3]). *Let* $u \in \mathbb{H}^s_{2\pi}$. *Then, there exists a unique* $U \in \mathbb{X}^s_{2\pi}$ *such that*

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla U) + m^2 y^{1-2s}U = 0 & \text{in } \mathcal{S}_{2\pi}, \\ U_{|\{x_i=-\pi\}} = U_{|\{x_i=\pi\}} & \text{on } \partial_L\mathcal{S}_{2\pi}, \\ U(\cdot, 0) = u & \text{on } \partial^0\mathcal{S}_{2\pi}, \end{cases}$$

and

$$-\lim_{y\to 0^+} y^{1-2s}\frac{\partial U}{\partial y}(x, y) = \kappa_s(-\Delta + m^2)^s u(x) \quad \text{in } \mathbb{H}^{-s}_{2\pi}.$$

We call $U \in \mathbb{X}^s_{2\pi}$ *the extension of* $u \in \mathbb{H}^s_{2\pi}$*, and we denote it by* $\mathrm{Ext}(u)$*. Moreover,* $\mathrm{Ext}(u)$ *satisfies the following properties:*

(i) $\mathrm{Ext}(u)$ *is smooth for* $y > 0$ *and* 2π*-periodic in* x*;*
(ii) $\|\mathrm{Ext}(u)\|_{\mathbb{X}^s_{2\pi}} \leq \|V\|_{\mathbb{X}^s_{2\pi}}$ *for any* $V \in \mathbb{X}^s_{2\pi}$ *such that* $\mathrm{Tr}(V) = u$*;*
(iii) $\|\mathrm{Ext}(u)\|_{\mathbb{X}^s_{2\pi}} = \sqrt{\kappa_s}|u|_{\mathbb{H}^s_{2\pi}}$.

Therefore, we can reformulate the nonlocal problem (1.1) with periodic boundary conditions in a local way according to the following definition.

Definition 1. We say that $u \in \mathbb{H}^s_{2\pi}$ is a weak solution to (1.1) if and only if $u = \mathrm{Tr}(U)$ and $U \in \mathbb{X}^s_{2\pi}$ satisfies

$$\iint_{\mathcal{S}_{2\pi}} y^{1-2s}(\nabla U \nabla V + m^2 UV)\, dxdy$$
$$= \kappa_s \int_{(-\pi,\pi)^N} [(m^{2s} - \lambda)\mathrm{Tr}(U) + f(x, \mathrm{Tr}(U))]\mathrm{Tr}(V)\, dx$$

for every $V \in \mathbb{X}^s_{2\pi}$.

Remark 1. For simplicity, we will assume that $\kappa_s = 1$. Moreover, with abuse of notation, we denote the trace of a function $U : \mathbb{R}^{N+1}_+ \to \mathbb{R}$ by u; that is, $u = \mathrm{Tr}(U)$.

3. Periodic solutions to (1.4)

In this section we provide the proof of Theorem 1. To do this, we will exploit the following useful result:

Theorem 5 ([21])**.** *Let* X *be a Banach space, let* $\Phi : X \to \mathbb{R}$*, and let* $\Psi : X \to \mathbb{R}$ *be* $\mathcal{C}^1$*-functionals. Set*

$$\mathcal{J}(U) = \Phi(U) - \Psi(U) \quad \textit{for all } U \in X.$$

Assume that

(i) Φ *is homogeneous of degree* $p > 1$*;*
(ii) *there exist* $U_1, U_2 \in X$ *such that*
$$\max\{\mathcal{J}(U_1), \mathcal{J}(U_2)\} < c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{J}(\gamma(t)),$$
where $\Gamma = \{\gamma \in \mathcal{C}^0([0,1], X) : \gamma(0) = U_1, \gamma(1) = U_2\}$*;*
(iii) $\lim_{\|u\|\to+\infty} \Phi(U) = \infty$*;*
(iv) $\Psi'(U)U - p\Psi(U) \geq 0$ *for every* $u \in X$*;*
(v) $\lim_{\Psi(U)\to\infty} \Psi'(U)U - p\Psi(U) = \infty$.

Then $\mathcal{J}$ *possesses a bounded Palais-Smale sequence at level* c*.*

Let us introduce the functional $\mathcal{J} : \mathbb{X}^s_{2\pi} \to \mathbb{R}$ defined by setting

$$\mathcal{J}(U) := \Phi(U) - \Psi(U) \quad \text{for all } U \in \mathbb{X}^s_{2\pi}$$

where

$$\Phi(U) := \frac{1}{2}\left[\|U\|^2_{\mathbb{X}^s_{2\pi}} - (m^{2s} - \mu)|u|_2^2\right] \quad \text{and} \quad \Psi(U) := \int_{(-\pi,\pi)^N} F(x, u)dx.$$

In view of Theorem 3 and the assumptions on f, it is easy to see that $\Phi, \Psi \in \mathcal{C}^1(\mathbb{X}^s_{2\pi}, \mathbb{R})$. Clearly Φ is homogeneous of degree 2. Moreover $\|U\|^2_{\mathbb{X}^s_{2\pi}} - (m^{2s} - \mu)|u|_2^2$ is an equivalent norm to $\|\cdot\|_{\mathbb{X}^s_{2\pi}}$.

Indeed, by using $\|U\|^2_{\mathbb{X}^s_{2\pi}} \geq m^{2s}|u|_2^2$ for any $U \in \mathbb{X}^s_{2\pi}$ (see (2.1)), we have

$$\min\left\{\frac{\mu}{m^{2s}},1\right\}\|U\|^2_{\mathbb{X}^s_{2\pi}} \leq \|U\|^2_{\mathbb{X}^s_{2\pi}} - (m^{2s}-\mu)|u|_2^2 \leq \max\left\{\frac{\mu}{m^{2s}},1\right\}\|U\|^2_{\mathbb{X}^s_{2\pi}}, \tag{3.1}$$

so we can define

$$\|U\|_e^2 := \|U\|^2_{\mathbb{X}^s_{2\pi}} - (m^{2s}-\mu)|u|_2^2.$$

Our claim is to prove that Φ and Ψ satisfy the assumptions of Theorem 5. Obviously (i), (iii) are verified, and (iv) is satisfied in view of $(f4)$. We check that condition (ii) holds. Firstly, we note that by using $(f1)$, we have

$$0 \leq F_+(x,t) \leq M(|t|+|t|^q) \text{ for any } x \in (-\pi,\pi)^N, \qquad t \in \mathbb{R}; \tag{3.2}$$

that is, the Nemytskii operator $F_+(\cdot,u)$ maps $\mathbb{X}^s_{2\pi}$ into $L^1(-\pi,\pi)^N$.

Now, we show that there exists $\delta > 0$ such that

$$\mathcal{J}(U) \geq \delta\|U\|^2_{\mathbb{X}^s_{2\pi}} \tag{3.3}$$

for U in some neighborhood of zero in $\mathbb{X}^s_{2\pi}$. Since $\mathcal{J}(0)=0$ and

$$\mathcal{J}(U) \geq \mathcal{J}_+(U) := \frac{1}{2}\|U\|_e^2 - \int_{(-\pi,\pi)^N} F_+(x,u)\,dx,$$

we can prove (3.3) for $\mathcal{J}_+$ instead of $\mathcal{J}$. By using $(f1)$, $(f2)$, (3.1), (3.2), and Theorem 3, we deduce that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}
\mathcal{J}_+(U) &= \frac{1}{2}\|U\|_e^2 - \int_\Omega F_+(x,u)\,dx \\
&\geq B\|U\|^2_{\mathbb{X}^s_{2\pi}} - \varepsilon|u|_2^2 - C_\varepsilon|u|_q^q \\
&\geq B\|U\|^2_{\mathbb{X}^s_{2\pi}} - A\varepsilon\|U\|^2_{\mathbb{X}^s_{2\pi}} - A\,C_\varepsilon\|U\|^q_{\mathbb{X}^s_{2\pi}} \\
&= \|U\|^2_{\mathbb{X}^s_{2\pi}}\left[(B-A\varepsilon) - A\,C_\varepsilon\|U\|^{q-2}_{\mathbb{X}^s_{2\pi}}\right],
\end{aligned}$$

where $2B = \min\{\frac{\mu}{m^{2s}},1\} > 0$ and A is a positive constant depending only on q, s, N, and m.

Choosing $\varepsilon = \frac{B}{4A}$, we can deduce that $\mathcal{J}_+(U) \geq \frac{B}{4}\|U\|^2_{\mathbb{X}^s_{2\pi}}$ provided that $\|U\|_{\mathbb{X}^s_{2\pi}} < (B/2AC_{\frac{B}{4A}})^{\frac{1}{q-2}}$.

This completes the proof of relation (3.3). Now, we fix $\varphi \in C^\infty_{2\pi}(\mathbb{R}^N)$ such that $0 \leq \varphi \leq 1$, $\operatorname{supp}\varphi \subset K$, and $\varphi = 1$ on Ω_1, where Ω_1 is some nonempty open subset of the interior of K, and we take $\eta \in C^\infty(\mathbb{R}_+)$ such that $0 \leq \eta \leq 1$, $\eta = 1$ in $[0,1]$, and $\eta = 0$ in $[2,\infty)$. Then $V(x,y) := \varphi(x)\eta(y) \in \mathbb{X}^s_{2\pi}$ and $\mathrm{Tr}(V) = \varphi$. Therefore, we have the estimate

$$\Psi(t_nV) = \int_{\Omega_1} F(x,t_nv)\,dx + \int_{K\setminus\Omega_1} F(x,t_nv)\,dx \geq \int_{\Omega_1} F(x,t_n)\,dx + \lambda|K\setminus\Omega_1|,$$

where λ is a lower bound for F on $K\times[0,\infty)$, and the sequence $\{t_n\} \subset \mathbb{R}$ is defined as in $(f3)$. Since $|\Omega_1| > 0$ and $t_n^{-2}F(x,t_n) \to +\infty$ a.e. on Ω_1, by Fatou's Lemma it follows that $\lim_{n\to\infty}\frac{\Psi(t_nV)}{t_n^2} = +\infty$.

As a consequence

$$\Phi(t_nV) - \Psi(t_nV) = t_n^2\left[\frac{\|V\|_e^2}{2} - t_n^{-2}\Psi(t_nV)\right] < 0 \tag{3.4}$$

for n big enough.

In view of (3.3) and (3.4), and taking $U_1 = 0$ and $U_2 = t_{\bar{n}} V$ (with $\bar{n}$ large enough), we can infer that condition (ii) is satisfied.

At this point, we only need to check (v). Let $\{U_n\} \subset \mathbb{X}^s_{2\pi}$ be a sequence such that $\lim_{n\to\infty} \Psi(U_n) = +\infty$.
Since

$$\begin{aligned}\Psi(U_n) &= \int_{\{F(x,u_n)<\sigma(x)\}} F(x,u_n)\,dx + \int_{\{F(x,u_n)\geq\sigma(x)\}} F(x,u_n)\,dx \\ &\leq |\sigma|_1 + \int_{\{F(x,u_n)\geq\sigma(x)\}} F(x,u_n)\,dx\end{aligned}$$

we get

$$\lim_{n\to\infty} \int_{\{F(x,u_n)\geq\sigma(x)\}} F(x,u_n)\,dx = +\infty. \tag{3.5}$$

Taking into account $(f4)$ and $(f5)$, we obtain

$$\begin{aligned}\Psi'(U_n)U_n - 2\Psi(U_n) &= \int_{\{F(x,u_n)<\sigma(x)\}} [f(x,u_n)u_n - 2F(x,u_n)]\,dx \\ &\quad + \int_{\{F(x,u_n)\geq\sigma(x)\}} [f(x,u_n)u_n - 2F(x,u_n)]\,dx \\ &\geq (\alpha-2)\int_{\{F(x,u_n)\geq\sigma(x)\}} F(x,u_n)\,dx, \qquad (3.6)\end{aligned}$$

which together with (3.5) yields

$$\lim_{n\to\infty} \Psi'(U_n)U_n - 2\Psi(U_n) = +\infty$$

being $\alpha > 2$.

Now, we are in the position to apply Theorem 5. Thus there exists a bounded Palais-Smale sequence $\{U_n\} \subset \mathbb{X}^s_{2\pi}$ such that

$$\mathcal{J}(U_n) \to c > 0 \quad \text{and} \quad \mathcal{J}'(U_n) \to 0 \qquad \text{as } n \to \infty,$$

where

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{J}(\gamma(t)) \qquad \text{and} \qquad \Gamma := \{\gamma \in \mathcal{C}^0([0,1], \mathbb{X}^s_{2\pi}) : \gamma(0) = 0, \gamma(1) = t_{\bar{n}} V\}.$$

Since $\{U_n\}$ is bounded in $\mathbb{X}^s_{2\pi}$, from Theorem 3 we may assume that as $n \to \infty$,

$$\begin{aligned}&U_n \rightharpoonup U \text{ in } \mathbb{X}^s_{2\pi}, \\ &u_n \to u \text{ in } L^q(-\pi,\pi)^N, \\ &u_n \to u \text{ a.e. in } (-\pi,\pi)^N.\end{aligned} \tag{3.7}$$

By using the continuity of f, $(f1)$, (3.7), and the Dominated Convergence Theorem we get

$$\int_{(-\pi,\pi)^N} f(x,u_n)u_n\,dx \to \int_{(-\pi,\pi)^N} f(x,u)u\,dx \quad \text{as } n \to \infty$$

and

$$\int_{(-\pi,\pi)^N} f(x,u_n)u\,dx \to \int_{(-\pi,\pi)^N} f(x,u)u\,dx \quad \text{as } n \to \infty,$$

which imply that

$$\int_{(-\pi,\pi)^N} f(x,u_n)(u_n-u)\,dx \to 0 \quad \text{as } n\to\infty. \tag{3.8}$$

Putting together $\mathcal{J}'(U_n)\to 0$, $\{U_n\}$ is bounded, (3.7), and (3.8), we deduce that $\|U_n\|_{\mathbb{X}^s_{2\pi}} \to \|U\|_{\mathbb{X}^s_{2\pi}}$ as $n\to\infty$. This, together with (3.7) and the fact that $\mathbb{X}^s_{2\pi}$ is a Hilbert space, gives that $U_n\to U$ in $\mathbb{X}^s_{2\pi}$ as $n\to\infty$. Then we can infer that $\mathcal{J}(U)=c$ and $\mathcal{J}'(U)=0$; that is, U is a nontrivial weak solution to (1.4). This concludes the proof of Theorem 1.

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