PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 9, September 2018, Pages 3661–3677
http://dx.doi.org/10.1090/proc/13891
Article electronically published on June 1, 2018

CLASSICALLY INTEGRAL QUADRATIC FORMS EXCEPTING AT MOST TWO VALUES

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(Communicated by Kathrin Bringmann)

Abstract. Let $S \subseteq \mathbb{N}$ be finite. Is there a positive definite quadratic form that fails to represent only those elements in S? For $S = \emptyset$, this was solved (for classically integral forms) by the 15-Theorem of Conway-Schneeberger in the early 1990s and (for all integral forms) by the 290-Theorem of Bhargava-Hanke in the mid-2000s. In 1938 Halmos attempted to list all weighted sums of four squares that failed to represent $S = \{m\}$; of his 88 candidates, he could provide complete justifications for all but one. In the same spirit, we ask, "For which $S = \{m, n\}$ does there exist a quadratic form excepting only the elements of S?" Extending the techniques of Bhargava and Hanke, we answer this question for quaternary forms. In the process, we provide a new proof of the original outstanding conjecture of Halmos, namely, that $x^2 + 2y^2 + 7z^2 + 13w^2$ represents all positive integers except 5. We develop new strategies to handle forms of higher dimensions, yielding an enumeration of and proofs for the 73 possible pairs that a classically integral positive definite quadratic form may except.

1. Introduction and statement of results

Determining which integers are represented by a positive definite quadratic form has been of long-standing mathematical interest. One particular phrasing of the question is as follows: given a finite subset $S \subseteq \mathbb{N}$, do there exist quadratic forms (and if so, how many) that represent $n \in \mathbb{N}$ if and only if $n \notin S$?

When $S = \emptyset$, the answer is known; this is the case of classifying *universal* positive definite quadratic forms. In 1916, Ramanujan [14] proved the existence of 55 universal quaternary diagonal positive definite quadratic forms. While it was later discovered that one of those forms fails to represent $n = 15$, the remaining 54 do give the complete list. Ramanujan's idea for generating candidate forms was similar to that of escalation, a crucial tool in later universality results. For classically integral forms, there is the Conway-Schneeberger 15-Theorem. First announced in 1993, it states that a classically integral positive definite quadratic form is universal if and only if it represents all positive integers up to 15; moreover, provided was a list of the 204 quaternary forms with this property. For the more general case, there is the 2005 result of Bhargava-Hanke: the 290-Theorem. This states that an integral positive definite quadratic form is universal if and only if it represents all positive integers up to 290; again, provided was a list of 6,436 such forms in four variables [3].

Received by the editors April 22, 2017, and, in revised form, June 28, 2017.
2010 *Mathematics Subject Classification.* Primary 11E25; Secondary 11E20, 11E45.

A natural next question is to classify those positive definite quadratic forms representing all positive integers with a single exception (i.e., the case when $S = \{m\}$). Using Ramanujan's idea of escalation, in [9] Halmos provided a list of 135 diagonal quaternary candidate forms that could fail to represent at most one integer. Using known results about representation by ternary subforms, Halmos was able to narrow that list down to 88 candidate forms. He provided proofs that 87 of those did indeed fail to represent exactly one integer, though like Ramanujan there is some overcounting (two of Halmos's 88 except exactly *two* integers). Halmos suspected the remaining form $x^2 + 2y^2 + 7z^2 + 13w^2$ only failed to represent 5, but he could not provide a proof. Pall [13] did supply a proof in 1940, using genus theory and basic modular arithmetic. Continuing the study of forms representing all but finitely many integers, in 2009 Bochnak and Oh [4] provided necessary and sufficient local conditions for when a classically integral positive definite quadratic form excepts finitely many values.

In this paper we consider quadratic forms which except precisely two values (see Theorem 1, stated below and proved in section 5). Previous results in this area concentrate on fixing a particular quadratic form and verifying that it fails to represent only two values, as opposed to our result of listing with verification all possible pairs of excepted values. As a recent example of former results, in [10] Hanke solved an outstanding conjecture of Kneser by showing that $x^2 + 3y^2 + 5z^2 + 7w^2$ represents precisely those $m \in \mathbb{N} - \{2, 22\}$. Additionally, we offer (as our Corollary 2) a different set of conditions, analogous to those of Bhargava-Hanke, for the case where a quadratic form excepts precisely one value.

We now state our main results.

Theorem 1. *There are exactly* 73 *sets* $S = \{m, n\}$ *(with* $1 \leq m < n$*) for which there exists a classically integral positive definite quadratic form representing precisely* $\mathbb{N} - S$.

Remark. A more precise version of Theorem 1 explicitly listing those 73 sets will be given in section 5.

Both the 15- and 290-Theorems supply more specific lists of **critical integers** whose representability by a form Q is necessary and sufficient for universality. The criticality of such an integer m is proven by explicitly constructing an almost universal quadratic form excepting only $\{m\}$.

Remark. With the knowledge that $m \leq 15$, our escalation process independently verifies each of the 9 critical integers $\{1, 2, 3, 5, 6, 7, 10, 14, 15\}$ given by the 15-Theorem. Furthermore, for all excepted pairs $\{m, n\}$, note that m must be a critical integer.

The following corollary gives criteria for proving that a given form excepts precisely one critical integer.

Corollary 2. *Given any* $m \in \{1, 2, 3, 5, 6, 7, 10, 14, 15\}$*, let* n_{max} *denote the largest* n *such that* $\{m, n\}$ *is one of the* 73 *pairs from Theorem* 1*. If a form* Q *excepts* m *but represents all other* $k \leq n_{max}$*, then* m *is the only exception for* Q.

We are also able to provide the proof that Halmos could not, using a method different from that of Pall:

Corollary 3. *The only positive integer m that is not represented by $x^2 + 2y^2 + 7z^2 + 13w^2$ is $m = 5$.*

We further note the following errors in Halmos's original results:

Corollary 4. *The forms $x^2 + y^2 + 2z^2 + 22w^2$ and $x^2 + 2y^2 + 4z^2 + 22w^2$ represent precisely $\mathbb{N} - \{14, 78\}$, in contradiction to Halmos's conjecture of $\mathbb{N} - \{14\}$.*

The paper is organized as follows. Section 2 provides general background on the relevant theory of quadratic forms and modular forms needed for our proof (specifically in the quaternary case). The next section describes the computational methods that were used (again, in the quaternary case). Section 4 explains how the theoretical and computational techniques of the previous two sections generalize to higher-dimensional forms, with section 5 providing the proofs of the main results. Lastly, we include an example in section 6.2 of a detailed proof of Corollary 3.

The source code and log files used in the proofs of our results are located at `https://github.com/almostuniversals/two-vals`. We provide the full suite of Magma files that carried out the almost universality computations, a sample SageMath worksheet for generating candidate forms, and one log file (from running the Magma suite) per possible exception pair verifying that a form excepting only the pair exists. Our GitHub directory also contains a .zip file holding logs for all candidate forms.

2. Background

Let $n \in \mathbb{N}$. An **n-ary integral quadratic form** is a homogeneous integral polynomial of degree two of the form

$$Q(\vec{x}) = Q(x_1, \ldots, x_n) = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j \in \mathbb{Z}[x_1, \ldots, x_n].$$

An equivalent way to represent quadratic forms is by symmetric matrices in $\mathcal{M}_n\left(\frac{1}{2}\mathbb{Z}\right)$; that is, for each n-ary integral quadratic form there exists a unique symmetric matrix $A_Q \in \mathcal{M}_n\left(\frac{1}{2}\mathbb{Z}\right)$ such that

$$Q(\vec{x}) = \vec{x}^{\,t} A_Q \vec{x}.$$

When $A_Q \in \mathcal{M}_n(\mathbb{Z})$ (equivalently, when all cross-terms of Q are even) we say that Q is **classically integral**. We say that an integral n-ary quadratic form is **positive definite** if $Q(\vec{x}) \ge 0$, with equality if and only if $\vec{x} = \vec{0}$. Note that $\det(A_Q) > 0$ for any positive definite quadratic form Q. For the remainder of this paper, assume that any "form" is a "classically integral positive definite quadratic form".

Given two n-ary forms Q_1 and Q_2 with respective matrices A_{Q_1} and A_{Q_2}, we say that Q_1 and Q_2 are **equivalent (over $\mathbb{Z}$)** if and only if there exists a matrix $M \in GL_n(\mathbb{Z})$ such that $A_{Q_1} = M A_{Q_2} M^t$.

Let $m \in \mathbb{N}$ and let Q be an n-ary form. We say that m is **represented** by Q if there exists $\vec{x} \in \mathbb{Z}^n$ so that

$$Q(\vec{x}) = m.$$

Let

$$r_Q(m) := \#\{\vec{x} \in \mathbb{Z}^n \mid Q(\vec{x}) = m\}$$

denote the **representation number of m by Q.** As mentioned earlier, a form Q is said to be **universal** if $r_Q(m) > 0$ for all $m \in \mathbb{N}$. In [9], Halmos used the term **almost universal** to denote a form Q which failed to represent a single $m \in \mathbb{N}$. In the same spirit, we take **almost universal** to denote a form representing all $m \in \mathbb{N} - S$, where $S \subseteq \mathbb{N}$ is finite.

2.1. **Escalation.** Let Q be an n-ary form and A_Q be its matrix representation. If L is an n-dimensional lattice endowed with inner product $\langle \cdot, \cdot \rangle$ such that $Q(\vec{x}) = \langle \vec{x}, \vec{x} \rangle$, we say that L is the **associated lattice** for Q. We define the **truant** of Q to be the least $n \in \mathbb{N} - S$ such that n is not represented by Q. It should be noted that this generalizes the definition appearing in earlier literature, most notably in [3], where $S = \emptyset$.

An **escalation** of L is any lattice generated by L and a vector whose norm is the truant. An **escalator lattice** is any lattice that can be obtained by successive escalation of the trivial lattice.

Given a form Q of $n-1$ variables with associated matrix A_Q and truant t, any escalation by t will result in an n-ary form $\widehat{Q}$ with associated matrix

$$A_{\widehat{Q}} = \left[\begin{array}{ccc|c} & & & a_{1n}/2 \\ & A_Q & & \vdots \\ & & & a_{(n-1)n}/2 \\ \hline a_{1n}/2 & \cdots & a_{(n-1)n}/2 & t \end{array}\right].$$

The restrictions that $\det(A_{\widehat{Q}}) > 0$ and that $a_{in} \in 2\mathbb{Z}$ for $1 \leq i \leq n-1$ imply that there will be finitely many possible escalations of a quadratic form Q by its truant t. Note that the resulting escalation lattice does not depend on the order in which basis vectors are appended.

2.2. **Modular forms.** For details and additional background we refer the reader to [8], [10], and [19]. Any additional specific references will be provided in context.

Throughout, let Q be a *quaternary* form with associated matrix A_Q. We define the **level** $N = N_Q$ of Q to be the smallest integer such that $N(2(A_Q))^{-1}$ is an integral matrix with even diagonal entries.

We define the **determinant** $D = D_Q$ of Q to be the determinant of A_Q. For all primes $p \nmid 2N$, we define the following quadratic character χ_Q:

$$\chi_Q(p) = \left(\frac{D}{p}\right).$$

Recall that the local normalized form (also called the Jordan decomposition) of Q is

$$Q(\vec{x}) \equiv_{\mathbb{Z}_p} \sum_j p^{v_j} Q_j(x_j),$$

with $\dim(Q_j) \leq 2$ (and in fact, for $p \neq 2$, $\dim(Q_j) = 1$). We then define

$$\mathbb{S}_0 = \{j \mid v_j = 0\}, \qquad \mathbb{S}_1 = \{j \mid v_j = 1\},$$
$$\mathbb{S}_2 = \{j \mid v_j \geq 2\},$$

and we let $s_i = \sum_{j \in \mathbb{S}_i} \dim(Q_j)$.

We then define the **theta series** associated to Q as

$$\Theta_Q(z) = 1 + \sum_{m \geq 1} r_Q(m) e^{2\pi i m z}.$$

Theorem 5. $\Theta_Q(z)$ *is a modular form of weight* 2 *over* $\Gamma_0(N_Q)$ *with associated character* χ_Q.

Proof. See [1, Theorem 2.2, pg. 61]. ☐

Each space of modular forms of fixed weight, level, and character decomposes into the space of cusp forms and the space of Eisenstein series. Therefore, we write $\Theta_Q(z) = E_Q(z) + C_Q(z)$, where $E_Q(z)$ is Eisenstein and $C_Q(z)$ is a cusp form. We then consider

$$r_Q(m) = a_E(m) + a_C(m),$$

where $r_Q(m)$, $a_E(m)$, and $a_C(m)$ respectively denote the m^{th} Fourier coefficient for $\Theta_Q(z)$, $E_Q(z)$, and $C_Q(z)$.

The Eisenstein coefficients are well understood; they are nonnegative, rational, and can be computed explicitly. The main tool is due to Siegel [18]. Note that throughout, p denotes a place:

Theorem 6 (Siegel)**.**

$$a_E(m) = \prod_{p\leq\infty} \beta_p(m),$$

where

$$\beta_p(m) = \lim_{U\to\{m\}} \frac{\mathrm{Vol}(Q^{-1}(U))}{\mathrm{Vol}(U)},$$

where U *is an open neighborhood of* m *in* $\mathbb{Q}_p$.

Specifically for $p \neq \infty$*, we have that*

$$\beta_p(m) = \lim_{v\to\infty} \frac{\#\{\vec{x} \in (\mathbb{Z}/p^v\mathbb{Z})^4 \mid Q(\vec{x}) \equiv m \pmod{p^v}\}}{p^{3v}}.$$

To compute $\beta_\infty(m)$, the following result applies:

Theorem 7 (Siegel)**.**

$$\beta_\infty(m) = \frac{\pi^2 m}{\sqrt{\det(A_Q)}},$$

Proof. This is simply a special case of Hilfssatz 72 of [18]. ☐

Now we recall some terminology from Hanke [10] relevant specifically to local density computations at finite places.

Let $R_{p^v}(m) := \left\{\vec{x} \in (\mathbb{Z}/p^v\mathbb{Z})^4 \mid Q(\vec{x}) \equiv m \pmod{p^v}\right\}$ and set $r_{p^v}(m) := \#R_{p^v}(m)$. We say $\vec{x} \in R_{p^v}(m)$

- is of **Zero type** if $\vec{x} \equiv \vec{0} \pmod{p}$, in which case we say $\vec{x} \in R_{p^v}^{\mathrm{Zero}}(m)$ with $r_{p^v}^{\mathrm{Zero}}(m) := \#R_{p^v}^{\mathrm{Zero}}(m)$;
- is of **Good type** if $p^{v_j}x_j \not\equiv 0 \pmod{p}$ for some $j \in \{1,2,3,4\}$, in which case we say $\vec{x} \in R_{p^v}^{\mathrm{Good}}(m)$ with $r_{p^v}^{\mathrm{Good}}(m) := \#R_{p^v}^{\mathrm{Good}}(m)$;
- and is of **Bad type** otherwise, in which case we say $\vec{x} \in R_{p^v}^{\mathrm{Bad}}(m)$ with $r_{p^v}^{\mathrm{Bad}}(m)$.

If $r_{p^v}(m) > 0$ for all primes p and for all $v \in \mathbb{N}$, we say that m is **locally represented**. If $Q(\vec{x}) = 0$ has only the trivial solution over $\mathbb{Z}_p$, we say that p is an **anisotropic prime** for Q.

In the following theorems, we discuss reduction maps that allow for explicit calculation of local densities. Let the **multiplicity of a map** $f : X \to Y$ **at a**

given $y \in Y$ be $\#\{x \in X \mid f(x) = y\}$. If all $y \in Y$ have the same multiplicity M, then we say that the map has multiplicity M.

Theorem 8. *We have*

$$r_{p^{k+\ell}}^{\text{Good}}(m) = p^{3\ell} r_{p^k}^{\text{Good}}(m)$$

for $k \geq 1$ for p odd and for $k \geq 3$ for $p = 2$.

Proof. See [10, Lemma 3.2]. ☐

Theorem 9. *The map*

$$\begin{array}{rcl} \pi_Z : R_{p^k}^{\text{Zero}}(m) & \to & R_{p^{k-2}}\left(\dfrac{m}{p^2}\right) \\ \vec{x} & \mapsto & p^{-1}\vec{x} \pmod{p^{k-2}} \end{array}$$

is a surjective map with multiplicity p^4.

Proof. See [10, pg. 359]. ☐

Theorem 10. *There are two forms of bad-type solutions. Accounting for the number of such solutions will require the local density of forms related to Q (whose specific diagonalization is given below). For even more detailed descriptions of these supplementary forms and concrete examples of such computations, we refer the reader to* [19, Chapters 3, 4].

- *Bad-Type-I solutions occur when $\mathbb{S}_1 \neq \emptyset$ and $\vec{x}_{\mathbb{S}_1} \not\equiv \vec{0}$. The map*

$$\pi_{B'} : R_{p^k,Q}^{\text{Bad-1}}(m) \quad \to \quad R_{p^{k-1},Q'}^{\text{Good}}\left(\frac{m}{p}\right),$$

  which is defined for each index j by

$$\begin{array}{lll} x_j \mapsto p^{-1}x_j & v_j' = v_j + 1, & j \in \mathbb{S}_0, \\ x_j \mapsto x_j & v_j' = v_j - 1, & j \notin \mathbb{S}_0, \end{array}$$

  is surjective with multiplicity $p^{s_1+s_2}$.
- *Bad-Type-II solutions can only occur when $\mathbb{S}_2 \neq \emptyset$ and involves either $\mathbb{S}_1 = \emptyset$ or $\vec{x}_{\mathbb{S}_1} \equiv \vec{0}$. The map*

$$\pi_{B''} : R_{p^k,Q}^{\text{Bad-II}}(m) \quad \to \quad R_{p^{k-2},Q''}^{\vec{x}_{\mathbb{S}_2} \not\equiv \vec{0}}\left(\frac{m}{p^2}\right),$$

  which is defined for each index j by

$$\begin{array}{lll} x_j \mapsto p^{-1}x_j & v_j'' = v_j, & j \in \mathbb{S}_0 \cup \mathbb{S}_1, \\ x_j \mapsto x_j & v_j'' = v_j - 2, & j \notin \mathbb{S}_2, \end{array}$$

  is surjective with multiplicity $p^{8-s_0-s_1}$.

Proof. Again, see [10, pg. 360]. ☐

In practice, instead of exact formulas for $a_E(m)$, we employ effective lower bounds on $a_E(m)$. Given a form Q with level N_Q and associated character χ_Q, for all m locally represented by Q we have

$$a_E(m) \geq C_E d_m \left(\prod_{\substack{p \nmid N_Q, p|m \\ \chi(p)=-1}} \frac{p-1}{p+1} \right),$$

where d_m is a particular divisor of m and where C_E is a highly technical constant achieved from reasonable lower bounds for all $\beta_p(m)$. An exact formula for C_E is given in Theorem 5.7(b) of [10].

We similarly compute an upper bound on $|a_C(m)|$, due to Deligne [7]. For a given Q, there exists a constant C_f, determined by writing $C_Q(z)$ as a linear combination of normalized Hecke eigenforms (and shifts thereof). Specifically, if $g_i(z)$ is such a normalized Hecke eigenform and $C_Q(z) = \sum_i \gamma_i g_i(z)$, then $C_f = \sum_i |\gamma_i|$. Then for all $m \in \mathbb{N}$,

$$|a_C(m)| \leq C_f \sqrt{m}\ \tau(m),$$

where $\tau(m)$ counts the number of positive divisors of m.

3. Computational methods

Following the ideas of [3] and [10] with the notation of [3], we know that any $m \in \mathbb{N}$ is represented by Q if it is locally represented, has bounded divisibility at all anisotropic primes, and if the following inequality holds:

$$\frac{\sqrt{m'}}{\tau(m)} \prod_{\substack{p|m, p \nmid N \\ \chi(p)=-1}} \frac{p-1}{p+1} \quad > \quad \frac{C_f}{C_E}, \tag{1}$$

where m' is the largest divisor of m with no anisotropic prime factors. Define $B(m)$ to be the left side of (1). Note that $B(m)$ is multiplicative. Hence, as m becomes divisible by more primes, $B(m)$ is an increasing function. (It is not true in general that B is an increasing function; it is possible for primes $p < q$ to have $B(p) > B(q)$, as described below in Lemma 11.) However, there are only finitely many $m \in \mathbb{N}$ where

$$B(m) \quad \leq \quad \frac{C_f}{C_E}. \tag{2}$$

We call locally represented $m \in \mathbb{N}$ satisfying (2) **eligible**.

3.1. Generating eligible numbers.

Since $\frac{C_f}{C_E}$ can be quite large in comparison to $B(m)$, we require a faster method of computing eligible numbers than sequentially calculating all numbers such that inequality (2) holds. Therefore, we introduce the concept of an eligible prime. Define

$$C_B := \prod_{\substack{p \\ B(p)<1}} B(p).$$

Note that the only primes that contribute to this product are anisotropic or less than 11, since $B(p) \geq 1$ for all primes $p \geq 11$, unless p is anisotropic. An **eligible**

prime is then a prime p such that

$$B(p) \quad \leq \quad \frac{C_f}{C_E C_B}. \tag{3}$$

Observe that not all eligible primes are eligible numbers. To generate a list of eligible primes, we enumerate through all primes until (3) no longer holds. We note, however, that we must also check the next prime after that for which (3) fails due to the following result (which was not mentioned in [3]):

Lemma 11. *For any primes $p < q$, if $B(p) > B(q)$, then $q - p \leq 2$.*

Proof. The only case needing consideration is if p and q are not anisotropic with $\chi(p) \neq -1$ and $\chi(q) = -1$. Then

$$B(p) = \frac{\sqrt{p}}{2} \qquad \text{and} \qquad B(q) = \frac{\sqrt{q}}{2}\frac{q-1}{q+1}.$$

Note that if $q \leq p + 2$,

$$\left(\frac{B(q)}{B(p)}\right)^2 \leq \frac{p^3 + 4p^2 + 5p + 2}{p^3 + 6p^2 + 9p}$$

is always less than 1; therefore, $B(p) > B(q)$ when $q - p \leq 2$.

On the other hand, if $q \geq p + 3$, then

$$\left(\frac{B(q)}{B(p)}\right)^2 \geq \frac{p^3 + 9p^2 + 25p + 21}{p^3 + 8p^2 + 16p},$$

which is always greater than 1; so $B(p) < B(q)$ if $q - p \geq 3$. □

Once our list of eligible primes is generated, we sort by $B(p)$ in order to implement the following algorithm, which is also outlined in [3, section 4.3.1].

Since any squarefree eligible number is the product of a finite number of distinct eligible primes, we take the product of the smallest eligible primes p_ℓ until $p_1 \cdots p_{\ell+1}$ is not eligible. We then know that any squarefree eligible number will be the product of at most ℓ distinct eligible primes.

Set $a := p_1 \cdots p_r$ to be the product of the first r eligible primes, for each $1 \leq r \leq \ell$. While a is eligible, we replace p_r by p_{r+1} and continue replacing this single prime until a is no longer eligible. We then repeat the process, replacing the last two primes p_{r-1} and p_r with p_r and p_{r+1}, respectively. We continue this until we either have run out of eligible primes or until we can no longer increment.

Once we have a set of squarefree eligible numbers (including 1), we determine their representability by the form Q using techniques described in more detail in section 3.2. This results in a finite set S_1 of squarefree numbers which fail to be represented by Q. However, as S_1 is comprised only of squarefree integers, we do not yet have a complete list of possible exceptions; for instance, if Q excepts 2 and 8, then $S_1 = \{2\}$. Thus, we construct a new set S_2 of possible exceptions of the form sp^2, where $s \in S_1$, p is an eligible prime, and $B(sp^2) \leq \frac{C_f}{C_E}$. We repeat this process, continuing until $S_h = \emptyset$ for some h. We then take $\bigcup_{1 \leq i \leq h-1} S_i$ to be the entire set of possible exceptions. Note that we need not check $B\left(sp^2\right)$ for p larger than our max eligible prime since $B\left(p^j\right) > B\left(p\right)$ for $j \geq 2$, unless $p = 2$ or p is anisotropic (and our largest anisotropic prime is always less than our largest eligible prime). This means that if sp is not eligible, then neither is sp^2.

3.2. **Computing representability.** Given a set of eligible numbers, we proceed to check their representability by the form Q. The naive approach of simply computing the theta series up to the largest eligible number is infeasible for most forms because of the time and storage involved. We instead first check representability using a **split local cover** of Q, a form Q' that locally represents the same numbers as Q and can be written as $Q' = dx^2 \oplus T$ for a minimal $d \in \mathbb{N}$ and ternary subform T. Furthermore, Q' has the property that global representation by Q' implies global representation by Q. Therefore, we check the global representability of each eligible number a by Q' and hence by Q by checking if $a - dx^2$ is globally represented by T. We thus compute the theta series of T up to precision $Y = \lceil 2dc\sqrt{X} \rceil$, where X is the largest eligible number and c is a constant to allow for multiple attempts at checking $a - dx^2$. For the code in [3], $c = 5$ sufficed; for our purposes, the typical range for c was $5 \le c \le 10$. Any numbers that fail to be found to be represented by T are then handled by computing the theta series of Q to that precision.

Despite the improvements gained by the split local cover, we still encounter memory and speed issues on many forms. To deal with these, we use an **approximate boolean theta function**, also described in [3, section 4.3.2]. This stores a single bit for each number up to Y indicating whether or not it is represented by T. Additionally, rather than computing the entirety of the theta series, we evaluate Q only at the vectors in the intersection of an appropriately chosen small rectangular prism and the ellipsoid $T(\vec{y}) \le Y$. This gives far fewer vectors to check, at the expense of giving potential false negatives for numbers that are represented by vectors outside the prism.

The combined use of the split local cover and approximate boolean theta function significantly improves runtime speed and memory usage. According to [3], it requires storing $\sqrt{X}$ bits and has a runtime of $O\left(X^{1/4}\right)$. This is a substantial improvement over the naive method, which stores X bits and takes $O\left(X^2\right)$ time. We saw such improvement firsthand: the form

$$3x^2 - 2xy + 4y^2 - 4xz - 2yz + 6z^2 - 2xw + 8yw - 2zw + 7z^2,$$

which fails to represent the pair $\{1, 2\}$, took approximately 31 minutes to run using a split local cover and approximate boolean theta function. By contrast, our systems were unable to handle the memory requirements for computing the form without a boolean theta function. Even using a split local cover and boolean theta function without approximation took 4 hours and 25 minutes.

4. Higher escalations

In order to fully address Theorem 1 we must move beyond four variables to find any additional pairs $\{m, n\}$ for which there exists a classically integral positive definite quadratic form representing exactly the set $\mathbb{N} - \{m, n\}$. We begin by considering the quaternary forms obtained from our previous escalations which except more than 2 numbers. We classify these into three types:

- Type A: Forms that except only a finite set of numbers;
- Type B: Forms that except infinitely many numbers but have no local obstructions; and
- Type C: Forms that have local obstructions.

For a given form of Type A, let $L := \{m, n_1, \ldots, n_k\}$ be its ordered set of exceptions. To determine if there is a higher escalation excepting precisely the pair

$\{m, n_i\}$ for $1 \leq i \leq k$, we escalate by the truant and check that $\{m, n_i\}$ still fails to be represented. We repeat this process until either n_i is represented, making this pair an impossibility for escalations of this form, or $L - \{m, n_i\}$ is represented.

Forms of Type B are due to unbounded divisibility by an anisotropic prime p at integers not represented by the form. (All forms we encountered have at most one anisotropic prime.) Consequently, we conjecture that all but finitely many exceptions are contained in

$$\mathcal{F} := \bigcup_{k \in S} F_k,$$

where S is a finite subset of $\mathbb{N}$ and $F_k = \left\{kp^j \mid j \in \mathbb{N}\right\}$. We then compute the representability of all eligible numbers not in $\mathcal{F}$ by using the methods of section 3.2, resulting in a finite set of exceptions not in $\mathcal{F}$. For each $k \in S$, we observe that escalations by k and kp will suffice for representation of all eligible numbers in F_k. However, in practice, a single escalation usually results in a finite set of exceptions, to which we can then apply the methods of Type A.

The methods of Type B forms do not directly extend to Type C forms, as it is not immediately clear whether local obstructions only cause finitely many square-free exceptions. Instead, we seek to generalize the notion of the "10-14 switch", employed by Bhargava and Hanke in [3, section 5.2]. This technique exploits the commutativity of the escalation process by escalating a ternary form first by the truant(s) of the quaternaries with local obstructions that it generates and then by the truant of said ternary. In their case, this removes all local obstructions. However, a single switch is not sufficient for our case; therefore, we generalize to multiple potential switches. First, we escalate the quaternary forms as usual; then for each resulting quinary form we search for a quaternary subform with no local obstructions. We find such subforms for all quinary Type C escalations, to which we can then apply the methods of Type A or Type B.

We provide the following example to illustrate our need for generalizing the "10-14" switch on Type C forms. The quaternary form Q with

$$A_Q = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 3 & -2 & 0 \\ 0 & -2 & 4 & 0 \\ -1 & 0 & 0 & 7 \end{pmatrix}$$

is of Type C, having a local obstruction at 17. Two quinary escalations of this form, by a truant of 9, are Q_1 and Q_2, where

$$A_{Q_1} = \begin{pmatrix} 2 & 0 & 0 & -1 & -3 \\ 0 & 3 & -2 & 0 & 0 \\ 0 & -2 & 4 & 0 & 0 \\ -1 & 0 & 0 & 7 & 0 \\ -3 & 0 & 0 & 0 & 9 \end{pmatrix} \quad \text{and} \quad A_{Q_2} = \begin{pmatrix} 2 & 0 & 0 & -1 & 0 \\ 0 & 3 & -2 & 0 & -1 \\ 0 & -2 & 4 & 0 & 2 \\ -1 & 0 & 0 & 7 & -3 \\ 0 & -1 & 2 & -3 & 9 \end{pmatrix}.$$

The subform of Q_1 obtained by removing the first row and column from A_{Q_1} leads to a quaternary with no local obstructions. However, that of Q_2 obtained in the same way simply leads back to a form of Type C (having a local obstruction at 5). Consideration of these subforms corresponds to what we may refer to as a $2-\{3, 4, 7, 9\}$ switch occurring at the beginning of the escalation process. The 2 here refers to the truant by which we do not immediately escalate, while the elements of the set are the new truants by which we do escalate to achieve quaternary forms.

Note that a set, as opposed to an ordered list, is sufficient by commutativity of the escalation process.

The only quaternary subform of Q_2 with no local obstruction is that obtained by removal of the third row and column. This consideration corresponds to a $4-\{7,9\}$ switch during escalation at the binary form stage. Removal of the third row and column for Q_1, however, yields a quaternary form that does have a local obstruction. There is therefore no single escalation switch in the case of this Type C form that suffices.

5. Proofs of main results

We now state with more detail our main results.

Theorem 1. *There are exactly* 73 *sets* $S = \{m, n\}$ (*with* $m < n$) *for which there exists a classically integral positive definite quadratic form that represents precisely those natural numbers outside* S.

Table 1. Possible excepted pairs

m	n	Min. Required Variables
1	$2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 19, 21, 23, 25, 26, 30, 41$	4
	55	5
2	$3, 5, 6, 8, 10, 11, 15, 18, 22, 30, 38$	4
	$14, 50$	5
3	$6, 7, 11, 12, 19, 21, 27, 30, 35, 39$	4
5	$7, 10, 13, 14, 20, 21, 29, 30, 35$	4
	$37, 42, 125$	5
6	15	4
	54	5
7	$10, 15, 23, 28, 31, 39, 55$	4
10	$15, 26, 40, 58$	4
	250	5
14	$30, 56, 78$	4

To generate our candidate forms excepting all pairs $\{m, n\}$, we borrow from the theory of escalator lattices. By the 15-Theorem [2], we know that $m \leq 15$, so it remains to find the possible values of n.

For quaternary forms, we find the maximum value of n by fixing an m and pursuing our standard method of escalating by a vector with a norm equal to the truant. We implement the escalation process outlined in section 2.1 using the free and open-source computer algebra system SageMath [16] and the `QuadraticForm` class in particular. Once we reach four-dimensional forms, we use the maximal truant of the forms on our list as the upper bound for n.

When $m = 6$, we find that none of the four-variable candidates generated with our usual method except any additional values. Thus we approach this case by explicitly fixing each $n \leq 15$ (because 15 is the smallest truant of any ternary escalator excepting 6) to determine whether any forms excepting $\{6, n\}$ exist. For all other m, we begin the escalation process anew, this time with a bounded range for n given m. Along the way, we remove any forms that represent m or n and any which are not positive definite. At each dimension we also iterate through our list to remove those forms equivalent to another in the list.

These techniques generate exhaustive lists of four-variable candidate forms failing to represent exactly two $x < 10000$, as well as candidate forms excepting precisely one $x < 10000$. To prove that these forms represent all $x \geq 10000$, we implement the methods described in section 3 using the Magma computer algebra system [5]. This proves that there are 65 possible excepted pairs by quaternary forms.

To determine a full list of all possible pairs $\{m, n\}$, we next consider higher-dimensional forms. We begin by sorting the quaternary forms (resulting from our prior escalations) that except more than two values into the categories listed in section 4 and applying the corresponding techniques. Although some quinary forms do except more than two values, escalating to six or more variables yields no new pairs. Hence, we complete our list with 8 additional pairs $\{m, n\}$ for which there is an almost universal quinary quadratic form.

6. An example

6.1. **Escalation for $m = 5$.** We escalate to construct candidates for quadratic forms that except $\{5, n\}$. This process will also provide a list of all forms which could only fail to represent 5, and therefore will include Halmos's form. The escalation of the trivial lattice results in the lattice defined by $\begin{bmatrix}1\end{bmatrix}$, which is simply the quadratic form x^2. Since the truant here is 2, any escalation must be of the form

$$A_{(a)} = \begin{bmatrix} 1 & a \\ a & 2 \end{bmatrix},$$

where $a \in \mathbb{Z}$ and $\det\left(A_{(a)}\right) > 0$. This forces $a \in \{0, \pm 1\}$. Noting that

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} A_{(-1)} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^t = A_{(1)},$$

we need only consider the two escalators $A_{(0)}$ and $A_{(1)}$. However, as $A_{(1)}$ represents 5, we in fact proceed only with the escalator $A_{(0)}$.

The truant of $A_{(0)}$ is 7, so the three-dimensional escalators are of the form

$$A_{(0,b,c)} = \begin{bmatrix} 1 & 0 & b \\ 0 & 2 & c \\ b & c & 7 \end{bmatrix}$$

for $b, c \in \mathbb{Z}$ and $\det\left(A_{(0,b,c)}\right) > 0$. This yields 31 escalator matrices. Up to equivalence, however, there are only the 6 below:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -3 \\ -1 & -3 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & -3 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 7 \end{bmatrix},$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix},$$

which have the respective truants

$$\begin{array}{ccc} 13, & 20, & 13, \\ 10, & 13, & 14. \end{array}$$

Hence, we use these truants to escalate once again, obtaining 166 quaternary forms up to equivalence.

6.2. **Halmos's form.** We use techniques different from those of Pall [13] to prove the previously mentioned conjecture of Halmos [9].

Corollary 12. *The diagonal quadratic form* $Q(\vec{x}) = x^2+2y^2+7z^2+13w^2$ *represents all positive integers except for* 5.

We first note that our form has level $N_Q = 728 = 2^3 \cdot 7 \cdot 13$ and character $\chi_Q(p) = \left(\frac{182}{p}\right)$. Finding all $m \in \mathbb{N}$ to be locally represented, using Siegel's product formula and Theorem 7, we see that

$$\begin{aligned} a_E(m) &= \frac{\pi^2 m}{\sqrt{182}} \left(\prod_{p<\infty} \beta_p(m) \right) \\ &= \frac{\pi^2 m}{\sqrt{182}} \beta_2(m)\beta_7 m \beta_{13}(m) \left(\prod_{2,7,13 \neq p \mid m} \beta_p(m) \right) \left(\prod_{2,7,13 \neq q \nmid m} \beta_q(m) \right). \end{aligned}$$

Lemma 13. *For primes* $q \neq 2, 7, 13$ *with* $q \nmid m$,

$$\beta_q(m) = \left(1 - \frac{\chi_Q(q)}{q^2}\right).$$

Proof. This follows from [10, Lemma 3.3.2]. □

Thus

$$a_E(m) = \frac{\pi^2 m}{\sqrt{182}} \beta_2(m)\beta_7 m \beta_{13}(m) (L_{\mathbb{Q}}(2, \chi_Q))^{-1} \left(\prod_{2,7,13 \neq p \mid m} \frac{\beta_p(m) p^2}{p^2 - \chi_Q(p)} \right).$$

Lemma 14. *Let* $L_{\mathbb{Q}}(2, \chi_Q) = \dfrac{213\sqrt{182}\pi^2}{33124}$.

Proof. This follows from techniques outlined in [11, pg. 104]. □

This now means that

$$a_E(m) = \frac{182m}{213} \beta_2(m)\beta_7(m)\beta_{13}(m) \left(\prod_{2,7,13 \neq p \mid m} \frac{\beta_p(m) p^2}{p^2 - \chi_Q(p)} \right).$$

Lemma 15. *Let*

$$\beta_2(m)=\begin{cases}\dfrac{3}{4}\displaystyle\sum_{i=0}^{k-1}\frac{1}{2^{2i}}+\frac{1}{2^{2k}}\begin{cases}3/4 & \textit{if } \operatorname{ord}_2(m)=2k\ ,\ m/2^{2k}\equiv 1,3 \pmod 8,\\ 5/4 & \textit{if } \operatorname{ord}_2(m)=2k\ ,\ m/2^{2k}\equiv 5,7 \pmod 8,\end{cases}\\[2ex] \dfrac{3}{4}\displaystyle\sum_{i=0}^{k}\frac{1}{2^{2i}}+\frac{1}{2^{2k+1}}\begin{cases}3/4 & \textit{if } \operatorname{ord}_2(m)=2k+1\ ,\ m/2^{2k+1}\equiv 1,3 \pmod 8,\\ 1/4 & \textit{if } \operatorname{ord}_2(m)=2k+1\ ,\ m/2^{2k+1}\equiv 5,7 \pmod 8,\end{cases}\end{cases}$$

$$\beta_7(m)=\begin{cases}\dfrac{48}{49}\displaystyle\sum_{i=0}^{k-1}\frac{1}{7^{2i}}+\frac{1}{7^{2k}}\begin{cases}8/7 & \textit{if } \operatorname{ord}_7(m)=2k\ ,\ m/7^{2k}\equiv 1,2,4 \pmod 7,\\ 6/7 & \textit{if } \operatorname{ord}_7(m)=2k\ ,\ m/7^{2k}\equiv 3,5,6 \pmod 7,\end{cases}\\[2ex] \dfrac{48}{49}\displaystyle\sum_{i=0}^{k}\frac{1}{7^{2i}}+\frac{1}{7^{2k+1}}\begin{cases}2/7 & \textit{if } \operatorname{ord}_7(m)=2k+1\ ,\ m/7^{2k+1}\equiv 1,2,4 \pmod 7,\\ 0 & \textit{if } \operatorname{ord}_7(m)=2k+1\ ,\ m/7^{2k+1}\equiv 3,5,6 \pmod 7,\end{cases}\end{cases}$$

$$\beta_{13}(m)=\begin{cases}\dfrac{168}{169}\displaystyle\sum_{i=0}^{k-1}\frac{1}{13^{2i}}+\frac{1}{13^{2k}}\begin{cases}14/13 & \textit{if } \operatorname{ord}_{13}(m)=2k\ ,\ m/13^{2k}\equiv 1,3,4,9,10,12 \pmod{13},\\ 12/13 & \textit{if } \operatorname{ord}_{13}(m)=2k\ ,\ m/13^{2k}\equiv 2,5,6,7,8,11 \pmod{13},\end{cases}\\[2ex] \dfrac{168}{169}\displaystyle\sum_{i=0}^{k}\frac{1}{13^{2i}}+\frac{1}{13^{2k+1}}\begin{cases}2/13 & \textit{if } \operatorname{ord}_{13}(m)=2k+1\ ,\ m/13^{2k+1}\equiv 1,3,4,9,10,12 \pmod{13},\\ 0 & \textit{if } \operatorname{ord}_{13}(m)=2k+1\ ,\ m/13^{2k+1}\equiv 2,5,6,7,8,11 \pmod{13},\end{cases}\end{cases}$$

for $k\in\mathbb{N}\cup\{0\}$.

Proof. We provide details for the claim regarding $\beta_2(m)$ with $\operatorname{ord}_2(m)$ odd. The remaining proofs behave similarly. Additional examples of these computations can be found in [19].

Suppose that $\operatorname{ord}_2(m)=2k+1$ for $k\in\mathbb{N}\cup\{0\}$. Then there are solutions of the Good, Zero, and Bad-I types:

$$\beta_2(m)=\lim_{v\to\infty}\frac{r_{2^v}^{\text{Good}}(m)}{2^{3v}}+\lim_{v\to\infty}\frac{r_{2^v}^{\text{Zero}}(m)}{2^{3v}}+\lim_{v\to\infty}\frac{r_{2^v}^{\text{Bad-I}}(m)}{2^{3v}}.$$

We compute these individually, beginning with the Bad types:

Let Q' be the quadratic form $Q'(\vec{x})=2x^2+y^2+14z^2+26w^2$. Then

$$\begin{aligned}\lim_{v\to\infty}\frac{r_{2^v}^{\text{Bad-I}}(m)}{2^{3v}}&=\lim_{v\to\infty}\frac{2r_{2^{v-1},Q'}^{\text{Good}}\left(\dfrac{m}{2}\right)}{2^{3v}}\\ &=\frac{r_{2^7,Q'}^{\text{Good}}\left(\dfrac{m}{2}\right)}{2^{23}}.\end{aligned}$$

Note that $r_{2^7,Q'}^{\text{Good}}\left(\frac{m}{2}\right)$ is nonzero only if $\operatorname{ord}_2(m)=1$. Now, for the Good types:

$$\lim_{v\to\infty}\frac{r_{2^v}^{\text{Good}}(m)}{2^{3v}}=\frac{r_{2^7}^{\text{Good}}(m)}{2^{21}}=\frac{3}{4},$$

and for the Zero types:

$$\begin{aligned}\lim_{v\to\infty}\frac{r_{2^v}^{\text{Zero}}(m)}{2^{3v}} &= \lim_{v\to\infty}\frac{1}{2^{3v}}\left[\sum_{i=1}^{k}2^{4i}r_{2^{v-2i}}^{\text{Good}}\left(\frac{m}{2^{2i}}\right)+\sum_{i=1}^{k}2^{4i}r_{2^{v-2i}}^{\text{Bad-I}}\left(\frac{m}{2^{2i}}\right)\right]\\ &= \sum_{i=1}^{k}\frac{r_{2^7}^{\text{Good}}\left(m/2^{2i}\right)}{2^{2i+21}}+\sum_{i=1}^{k}\frac{r_{2^7,Q'}^{\text{Good}}\left(m/2^{2i+1}\right)}{2^{2i+23}}\ .\end{aligned}$$

Simplifying the sum of Good, Bad, and Zero type solutions yields the above claim. ☐

Lemma 16. *For $k\in\mathbb{N}\cup\{0\}$ and primes $p\neq 2,7,13$ such that $p\mid m$,*

$$\beta_p(m)\cdot\frac{p^2}{p^2-\chi_Q(p)}=\begin{cases}\dfrac{1}{p^{2k}}\left(\dfrac{p^{2k+1}-1}{p-1}\right) & \textit{if } \operatorname{ord}_p(m)=2k\ ,\ \chi_Q(p)=1,\\ \dfrac{1}{p^{2k}}\left(\dfrac{p^{2k+1}+1}{p+1}\right) & \textit{if } \operatorname{ord}_p(m)=2k\ ,\ \chi_Q(p)=-1,\\ \dfrac{1}{p^{2k+1}}\left(\dfrac{p^{2k+2}-1}{p-1}\right) & \textit{if } \operatorname{ord}_p(m)=2k+1\ ,\ \chi_Q(p)=1,\\ \dfrac{1}{p^{2k+1}}\left(\dfrac{p^{2k+2}-1}{p+1}\right) & \textit{if } \operatorname{ord}_p(m)=2k+1\ ,\ \chi_Q(p)=-1.\end{cases}$$

Proof. See [19, Lemma 3.3.6]. ☐

Lemmas 14 through 16, along with Theorem 7, provide a means to calculate $a_E(m)$ for any $m\in\mathbb{N}$.

Given the above formula for $a_E(m)$, as well as values for $r_Q(m)$, we are able to calculate $a_C(m)=r_Q(m)-a_E(m)$ for any m. Using [5], we determine that the cuspidal subspace of $\mathcal{M}_2\left(\Gamma_0(728),\chi_Q\right)$ has a dimension of 108. Computing a basis of normalized Hecke eigenforms for this subspace to determine our cuspidal bound as in [3, section 4.2.2], we find that

$$C_f=\sum_i|\gamma_i|\approx 13.4964\ .$$

Additionally, for the Eisenstein bound we find that

$$C_E=\frac{36}{71}\ .$$

We now prove Corollary 12 regarding Halmos's form Q.

Proof. Having calculated both C_E and C_f, we now employ the methods detailed in section 3 to compute and check eligible numbers. Note that

$$C_B=B(2)B(3)B(5),$$

since $B(p)>1$ for all $p>5$. With this, we compute that there are $5,634$ eligible primes and $343,203$ squarefree eligible numbers, the largest of which is 18047039010. Using the approximate boolean theta function of the split local cover $Q=x^2\oplus\left(2y^2+7z^2+13w^2\right)$, we compute the representability of each of these numbers. This approximation shows that all squarefree eligible numbers except 1 and 5 are represented. Computing the full theta series of Q we see that, while 1 is represented, 5 indeed is not. Therefore, we take $S_1=\{5\}$ to be the set of squarefree exceptions.

We hence compute that there are 28 eligible numbers of the form $5p^2$, and the set of exceptions of this form is $S_2 = \emptyset$. Thus, we have that $S = S_1 = \{5\}$ is the entire set of exceptions for this form, confirming Halmos's conjecture. Our implementation of this entire process takes approximately 2 minutes and 7 seconds. $\square$

Acknowledgments

This research was supported by the National Science Foundation (DMS-1461189). We thank Jeremy Rouse and Danny Krashen for helpful discussions and suggestions, and thank both Jonathan Hanke and Jeremy Rouse for providing access to code. Specifically, we give thanks to Jonathan Hanke for his Sage and C code implementing many useful quadratic form operations and to Jeremy Rouse for similar code in Magma. Lastly, we are grateful to Wake Forest University for providing access to servers and to the referee for helpful feedback.

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