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ISOMORPHIC CLASSIFICATION OF $L_{p,q}$-SPACES: THE CASE $p=2,\ 1\leq q<2$

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Abstract. Let $1\leq q<2$. We prove that the Banach space $l_{2,q}$ (respectively, $L_{2,q}(0,\infty)$) does not isomorphically embed into the space $L_{2,q}(0,1)$ (respectively, $L_{2,q}(0,1)\oplus l_{2,q}$).

1. Introduction

In this paper, we complete the isomorphic classification of $L_{p,q}$-spaces over resonant measure spaces by answering two open questions posed in [7, Section 5.1]. In that paper the authors proved that $l_{p,q}$ does not isomorphically embed into $L_{p,q}(0,1)$ and that the space $L_{p,q}(0,1)\oplus l_{p,q}$ does not contain a closed subspace which is isomorphic to $L_{p,q}(0,\infty)$ for all $1\leq p,q<\infty$, $p\neq q$, $p\neq 1$, $p\neq 2$, and $p=2$, $q>2$. The case $p=2$, $1\leq q<2$ was not amenable to the techniques employed there and was left open in [7]. We shall use here a rather different approach from that of [7], which is of interest in its own right.

2. Preliminaries

2.1. **$L_{p,q}$-spaces.** Let m be the Lebesgue measure on $\mathbb{R}^n$, $n=1,2$. Given a measurable real-valued function f defined on a measurable set $B\subset\mathbb{R}^n$, we define the *distribution* function d_f (of $|f|$) by setting

$$d_f(t)=m(\{|f|>t\})$$

and the *decreasing rearrangement* of $|f|$ by

$$f^*(t)=\inf\{s>0: d_f(s)\leq t\}.$$

For $0<p<\infty$ and $I=[0,1]$, or $I=[0,1]^2$, or $[0,\infty)$, the Lorentz function space $L_{p,q}(I)$ is the space (of equivalence classes) of all measurable functions f on I for which $\|f\|_{p,q}<\infty$, where

$$\|f\|_{p,q}=\left(\int_I f^*(t)^q d(t^{q/p})\right)^{1/q},\qquad q<\infty. \tag{1}$$

A symmetric sequence space analogue of $L_{p,q}$ is given by the space $l_{p,q}$ consisting of all scalar sequences $(x_i)_{i=1}^\infty$ for which $\|(x_i)\|_{p,q}<\infty$, where

$$\|(x_i)\|_{p,q}=\left\{\sum_{i=1}^\infty x_i^{*q}(i^{q/p}-(i-1)^{q/p})\right\}^{\frac{1}{q}},\qquad q<\infty, \tag{2}$$

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and where (x_i^*) is the decreasing rearrangement of $(|x_i|)$. Clearly, $l_{p,q}$ is isometric to a sublattice of $L_{p,q}[0,\infty)$. Also for any $p \geq 1$ we have $L_{p,p} = L_p$ and $l_{p,p} = l_p$; in this case we will simply write $\|\cdot\|_p$.

It is well known that for $1 \leq q \leq p < \infty$, (1) defines a norm under which $L_{p,q}$ is a separable, rearrangement invariant (r.i.) Banach function space; otherwise, (1) defines a quasi-norm on $L_{p,q}$ (which is known to be equivalent to a norm if $1 < p < q \leq \infty$).

Next recall that for any $0 < p < \infty$ and $0 < q \leq \infty$, $L_{p,q}$ is equal, up to an equivalent norm, to the space $[L_{p_1}, L_{p_2}]_{\theta,q}$ constructed using the *real interpolation method*, where $0 < p_1 < p_2 \leq \infty$, $0 < \theta < 1$, and $1/p = (1-\theta)/p_1 + \theta/p_2$. (See Theorem 5.2.4 in [2].)

Let $I = [0,1]$ or $I = [0,\infty)$ and let ψ be an increasing concave function on I with $\psi(0) = \psi(+0) = 0$, $1 \leq q < \infty$. The Lorentz space $\Lambda_{\psi,q}$ (when $q = 1$ we simply write Λ_ψ) consists of all measurable functions f on I for which

$$\|f\|_{\Lambda_{\psi,q}} := \left(\int_I f^*(t)^q d\psi(t) \right)^{\frac{1}{q}} < \infty.$$

If $\psi(t) = t$ for all $0 \leq t < \infty$, then $\Lambda_{\psi,q} = L_q$ with equality of norms. If $1 \leq q \leq p < \infty$, then $L_{p,q} = \Lambda_{\psi,q}$ with $\psi(t) = t^{q/p}$, $0 \leq t < \infty$.

2.2. **Main tool.** A key role in our proofs is played by the operator A_n defined below. It is a substantially modified operator A_n introduced in [10]. Let $r_n(t)$ denote the n^{th} Rademacher function, that is,

$$r_n(t) = sign \sin 2^n \pi t, \quad \text{for} \quad 0 \leq t \leq 1 \quad (n = 1, 2, \dots).$$

Define operators $A_n : L_{2,1}(0,1) \to L_{2,1}((0,1)^2)$, $n \geq 1$, by setting

$$A_n f = \sum_{k=1}^{n} k^{-\frac{1}{2}} (f \circ \gamma_k) \otimes r_k, \quad n \geq 1, \tag{3}$$

where $\gamma_k : (0,1) \to (0,1)$ is an arbitrary measure-preserving transformation, for every $k \geq 1$.

In the rest of this subsection we collect a number of (basically) known results. Firstly, we need to recall a well-known sufficient condition on a sequence $\{f_k\}_{k=1}^\infty \subseteq L_{p,q}$ guaranteeing that it has a subsequence whose closed linear span is isomorphic to the space l_q.

Lemma 1 ([3, Lemma 2.1], [4, Proposition 1])**.** *Let $I = [0,1]$ or $I = [0,1]^2$ or $[0,\infty)$, let $1 < p < \infty$, $1 \leq q < \infty$, and let $\{f_k\}_{k=1}^\infty$ be a sequence of semi-normalized elements in $L_{p,q}(I)$. If $f_k^* \xrightarrow{a.e.} 0$ as $k \to \infty$, then there is a subsequence of $\{f_k\}_{k=1}^\infty$ which is equivalent to the unit vector basis of l_q.*

The statement of the following subsequence splitting lemma is very similar to [5, Proposition 3.2] and [10, Theorem 3.2]. The only new component below is the assertion that the subsequence $\{f_k'\}_{k=1}^\infty$ and sequences $\{x_k\}_{k=1}^\infty, \{y_k\}_{k=1}^\infty \subseteq L_{p,q}(0,1)$ can be chosen to be unconditional. This assertion easily follows from a well-known fact that the space $L_{p,q}(0,1)$, $1 < p < \infty$, $1 \leq q < \infty$ admits an unconditional finite dimensional decomposition (see e.g. the proof of [1, Proposition 3.10]).

Lemma 2. *Let $1 < p < \infty$, $1 \le q < \infty$ and let $\{f_k\}_{k=1}^{\infty}$ be a weakly null sequence of elements in $L_{p,q}(0,1)$ with $\inf_k \|f_k\|_{p,q} > 0$. Then there exists an unconditional basic subsequence $\{f'_k\}_{k=1}^{\infty}$ of $\{f_k\}_{k=1}^{\infty}$ such that*

$$f'_k = x_k + y_k + d_k,\ k \ge 1, \tag{4}$$

where $\{x_k\}_{k=1}^{\infty}, \{y_k\}_{k=1}^{\infty}, \{d_k\}_{k=1}^{\infty} \subseteq L_{p,q}(0,1)$, $x_k^* = x_1^*$ for all k, $y_k y_j = 0$ for all $k \neq j$, $\|d_k\|_{p,q} \to 0, k \to \infty$, and both sequences $\{x_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ are unconditional basic sequences in $L_{p,q}(0,1)$.

3. Auxiliary results

Given a measurable real-valued function f on $B \subset \mathbb{R}^n$, $n = 1, 2$, we define the support of f by $supp(f) = \{f \neq 0\}$. In what follows χ_E denotes the indicator function of a Lebesgue measurable set E.

Lemma 3. *For every $k \ge 1$, let $S_k \subset (0,1)$ be a measurable set such that $m(S_k) = t \in (0,1]$. For all $n \in \mathbb{N}$, we have*

$$\Big\| \Big(\sum_{k=1}^{n} k^{-1} \chi_{S_k} \Big)^{\frac{1}{2}} \Big\|_{2,1} \le \sqrt{\frac{6}{\log(2)}} \cdot t^{\frac{1}{2}} \log(en).$$

Proof. Fix $n \ge 1$ and set $f = \sum_{k=1}^{n} \frac{1}{k} \chi_{S_k}$. For every $l \in \mathbb{Z}$, consider the set $\{2^l < f \le 2^{1+l}\}$. Obviously,

$$2^l \chi_{\{2^l < f \le 2^{1+l}\}} \le f \chi_{\{2^l < f \le 2^{1+l}\}} \le 2^{1+l} \chi_{\{2^l < f \le 2^{1+l}\}}, \quad l \in \mathbb{Z},$$

and therefore, setting

$$g = \sum_{l \in \mathbb{Z}} 2^l \chi_{\{2^l < f \le 2^{1+l}\}}$$

we obtain

$$g \le f \le 2g.$$

Next, observing that $\frac{1}{n} \chi_{supp(f)} \le f \le (1 + \log(n)) \chi_{supp(f)}$, we can rewrite the function g as a finite sum

$$g = \sum_{l=l_-(n)}^{l_+(n)} 2^l \chi_{\{2^l < f \le 2^{1+l}\}},$$

where $l_-(n)$ and $l_+(n)$ are integers depending on n such that $l_-(n) \le l_+(n)$.

It is immediate that

$$g^* = \sum_{l=l_-(n)}^{l_+(n)} 2^l \chi_{(d_f(2^{l+1}), d_f(2^l))}.$$

Using the inequality $\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}} \le (\alpha - \beta)^{\frac{1}{2}}$ for $0 < \beta < \alpha$ and the fact that $f \le 2g$, we obtain

$$\begin{aligned} \|f^{\frac{1}{2}}\|_{2,1} \le 2^{\frac{1}{2}} \|g^{\frac{1}{2}}\|_{2,1} &= 2^{\frac{1}{2}} \sum_{l=l_-(n)}^{l_+(n)} 2^{\frac{l}{2}} (d_f(2^l)^{\frac{1}{2}} - d_f(2^{1+l})^{\frac{1}{2}}) \\ &\le 2^{\frac{1}{2}} \sum_{l=l_-(n)}^{l_+(n)} 2^{\frac{l}{2}} (d_f(2^l) - d_f(2^{1+l}))^{\frac{1}{2}}. \end{aligned}$$

From Cauchy inequality, we have

$$\sum_{l=l_-(n)}^{l_+(n)} a_l \leq (l_+(n) - l_-(n) + 1)^{\frac{1}{2}} \cdot \Bigg(\sum_{l=l_-(n)}^{l_+(n)} a_l^2 \Bigg)^{\frac{1}{2}}.$$

Setting

$$a_l = 2^{\frac{l}{2}} (d_f(2^l) - d_f(2^{1+l}))^{\frac{1}{2}},$$

we obtain

$$\|f^{\frac{1}{2}}\|_{2,1} \leq 2^{\frac{1}{2}} (l_+(n) - l_-(n) + 1)^{\frac{1}{2}} \Bigg(\sum_{l=l_-(n)}^{l_+(n)} 2^l (d_f(2^l) - d_f(2^{1+l})) \Bigg)^{\frac{1}{2}}. \tag{5}$$

Also,

$$\sum_{l=l_-(n)}^{l_+(n)} 2^l (d_f(2^l) - d_f(2^{1+l})) = \|g\|_1 \leq \|f\|_1 = t \sum_{k=1}^{n} \frac{1}{k} \leq t \log(en). \tag{6}$$

From (5) and (6) we have

$$\|f^{\frac{1}{2}}\|_{2,1} \leq 2^{\frac{1}{2}} (l_+(n) - l_-(n) + 1)^{\frac{1}{2}} \cdot (t \log(en))^{\frac{1}{2}}. \tag{7}$$

Clearly,

$$\frac{1}{n} \leq \inf\{f(t) : \ f(t) \neq 0\} \in (2^{l_-(n)}, 2^{1+l_-(n)}],$$

$$\sum_{k=1}^{n} \frac{1}{k} \geq \sup\{f(t) : \ f(t) \neq 0\} \in (2^{l_+(n)}, 2^{1+l_+(n)}].$$

Thus,

$$\frac{-\log(n)}{\log(2)} \leq 1 + l_-(n), \quad \frac{\log(\sum_{k=1}^{n} \frac{1}{k})}{\log(2)} \geq l_+(n).$$

In either case, we have

$$l_+(n) \leq \frac{\log(\log(en))}{\log(2)} \leq \frac{\log(en)}{\log(2)},$$

$$l_-(n) \geq -\frac{\log(2n)}{\log(2)} \geq -\frac{\log(en)}{\log(2)}.$$

Hence,

$$1 + l_+(n) - l_-(n) \leq 1 + 2\frac{\log(en)}{\log(2)} \leq 3\frac{\log(en)}{\log(2)}.$$

Substituting this into (7), we arrive at

$$\|f^{\frac{1}{2}}\|_{2,1} \leq \sqrt{\frac{6}{\log(2)}} \cdot t^{\frac{1}{2}} \log(en).$$

This concludes the proof. □

The following fact should be compared with formula [6, (II.5.4)].

Fact 4. If $f \in \Lambda_\psi(0,1)$ is such that $f : (0,1) \to \mathbb{Z}_+$, then

$$\|f\|_{\Lambda_\psi} = \sum_{k=0}^{\infty} \psi(d_f(k)).$$

Proof. Indeed, we have that

$$f^*(t) = k+1, \quad t \in (d_f(k+1), d_f(k)).$$

Therefore,

$$\|f\|_{\Lambda_\psi} = \sum_{k=0}^{\infty}(k+1)\cdot(\psi(d_f(k)) - \psi(d_f(k+1))).$$

Summation by parts yields

$$\sum_{k\geq 0}\psi(d_f(k)) = \sum_{k\geq 0}\sum_{l\geq k}(\psi(d_f(l)) - \psi(d_f(l+1)))$$

$$= \sum_{l\geq 0}(\psi(d_f(l)) - \psi(d_f(l+1)))\sum_{k=0}^{l} 1$$

$$= \sum_{l\geq 0}(l+1)(\psi(d_f(l)) - \psi(d_f(l+1))).$$

Thus,

$$\|f\|_{\Lambda_\psi} = \sum_{k=0}^{\infty}\psi(d_f(k)).$$

□

In the lemma below we shall use the following simple decomposition. Suppose that $0 \leq f \in \Lambda_\psi(0,1)$. Fix $\epsilon > 0$ and consider the approximation

$$f_\epsilon = \sum_{k=0}^{\infty}(k+1)\epsilon\chi_{(k\epsilon,(k+1)\epsilon]}(f) = \sum_{k=0}^{\infty}(k+1)\epsilon(\chi_{(k\epsilon,\infty)}(f) - \chi_{((k+1)\epsilon,\infty)}(f))$$

$$= \sum_{k=0}^{\infty}\epsilon\chi_{(k\epsilon,\infty)}(f),$$

where $\chi_{(a,b)}(f) = \chi_{\{a<f<b\}}$.

Obviously, by construction, we have

$$\|f - f_\epsilon\|_{\Lambda_\psi} \leq \|f - f_\epsilon\|_\infty \leq \epsilon.$$

The following result could be inferred from [6, Lemma II.5.2]; however, we supply a short and self-contained proof for the convenience of the reader.

Lemma 5. *If $V : \Lambda_\psi(0,1) \to \Lambda_\psi(0,1)$ is a bounded operator, then*

$$\|V\|_{\Lambda_\psi\to\Lambda_\psi} = \sup_{\substack{A\subset(0,1)\\ m(A)>0}}\ \sup_{\substack{h\in\Lambda_\psi\\ |h|=\chi_A}} \frac{\|V(h)\|_{\Lambda_\psi}}{\|h\|_{\Lambda_\psi}}.$$

Proof. Assume for simplicity that the right hand side is 1. We aim to prove that $\|V\|_{\Lambda_\psi\to\Lambda_\psi} \leq 1$. Fix $f \in \Lambda_\psi$. We shall show below that

$$\|V(f)\|_{\Lambda_\psi} \leq \|f\|_{\Lambda_\psi}.$$

Let $f_+ = f\cdot\chi_{\{f>0\}}$ and $f_- = f\cdot\chi_{\{f<0\}}$.

Fix $\epsilon > 0$ and let[1]

$$g_1 = \epsilon\lceil\frac{f_+}{\epsilon}\rceil = \sum_{k=0}^{\infty}\epsilon\chi_{(k\epsilon,\infty)}(f_+),$$

[1]Here, $\lceil\cdot\rceil$ denotes the ceiling function.

$$g_2 = \epsilon\lceil\frac{f_-}{\epsilon}\rceil = \sum_{k=0}^{\infty}\epsilon\chi_{(k\epsilon,\infty)}(f_-).$$

These series converge in the norm of Λ_ψ and

$$\|f-(g_1-g_2)\|_{\Lambda_\psi} \le \|f-(g_1-g_2)\|_\infty \le \epsilon. \tag{8}$$

By triangle inequality, we have

$$\|V(g_1-g_2)\|_{\Lambda_\psi} \le \sum_{k=0}^{\infty}\epsilon\|V(\chi_{(k\epsilon,\infty)}(f_+)-\chi_{(k\epsilon,\infty)}(f_-))\|_{\Lambda_\psi}.$$

Observing that $\chi_{(k\epsilon,\infty)}(f_+)\cdot\chi_{(k\epsilon,\infty)}(f_-)=0$, $k\ge 0$, we see that by our assumption

$$\|V(\chi_{(k\epsilon,\infty)}(f_+)-\chi_{(k\epsilon,\infty)}(f_-))\|_{\Lambda_\psi} \le \|\chi_{(k\epsilon,\infty)}(f_+)-\chi_{(k\epsilon,\infty)}(f_-)\|_{\Lambda_\psi},\ \ k\ge 0,$$

and hence

(9)

$$\|V(g_1-g_2)\|_{\Lambda_\psi} \le \epsilon\sum_{k=0}^{\infty}\|\chi_{(k\epsilon,\infty)}(f_+)-\chi_{(k\epsilon,\infty]}(f_-)\|_{\Lambda_\psi} = \epsilon\sum_{k=0}^{\infty}\|\chi_{(k\epsilon,\infty)}(|f|)\|_{\Lambda_\psi}.$$

Denote for brevity

$$h = \lceil\frac{|f|}{\epsilon}\rceil.$$

We have

$$\chi_{(k\epsilon,\infty)}(|f|) = \chi_{(k,\infty)}(\frac{|f|}{\epsilon}) = \chi_{(k,\infty)}(h). \tag{10}$$

Clearly, h takes only values in $\mathbb{Z}_+$, and, therefore, one can apply Fact 4. We then have

$$\sum_{k=0}^{\infty}\|\chi_{(k\epsilon,\infty)}(|f|)\|_{\Lambda_\psi} = \sum_{k=0}^{\infty}\|\chi_{(k,\infty)}(h)\|_{\Lambda_\psi} = \sum_{k=0}^{\infty}\psi(d_h(k)) \overset{F.4}{=} \|h\|_{\Lambda_\psi}.$$

Therefore, we have

$$\|V(g_1-g_2)\|_{\Lambda_\psi} \le \epsilon\|h\|_{\Lambda_\psi} = \|g_1-g_2\|_{\Lambda_\psi} \le \|f\|_{\Lambda_\psi}+\epsilon. \tag{11}$$

Hence, from (10) and (11), we arrive at

$$\|V(f)\|_{\Lambda_\psi} \le \|V(g_1-g_2)\|_{\Lambda_\psi} + \|V(f-g_1+g_2)\|_{\Lambda_\psi} \le \|f\|_{\Lambda_\psi}+\epsilon\|V\|_{\Lambda_\psi\to\Lambda_\psi}.$$

Since $\epsilon>0$ is arbitrarily small, the assertion follows. ☐

The following three lemmas provide key estimates for the norm of the operator A_n, $n\ge 1$, and for the norms $\|A_nf\|_{2,q}$, $1\le q<2$, $n\ge 1$, for an arbitrary fixed element $f\in L_{2,q}(0,1)$. The notation c_{abs} stands for an absolute constant (whose value may change from line to line).

Lemma 6. *Let $A_n : L_{2,1}(0,1)\to L_{2,1}((0,1)^2)$, then*

$$\|A_n\|_{L_{2,1}\to L_{2,1}} \le c_{abs}\log(en), \quad n\ge 1. \tag{12}$$

Proof. Let $A\subset(0,1)$ be measurable. By Proposition 2.d.1 in [9] and Lemma 3, we have

$$\|A_n(\chi_A)\|_{2,1} = \Big\|\sum_{k=1}^{n}k^{-\frac{1}{2}}(\chi_A\circ\gamma_k)\otimes r_k\Big\|_{2,1} \le c_{abs}\Big\|\Big(\sum_{k=1}^{n}\frac{1}{k}(\chi_A\circ\gamma_k)^2\Big)^{\frac{1}{2}}\Big\|_{2,1}$$

$$= c_{abs}m(A)^{\frac{1}{2}}\log(en) = c_{abs}\log(en)\|\chi_A\|_{2,1}.$$

If A and B are Lebesgue measurable subsets of $(0,1)$ such that $A\cap B=\emptyset$, then

$$\|A_n(\chi_A-\chi_B)\|_{2,1}\le 2\|A_n(\chi_{A\cup B})\|_{2,1}\le 2c_{abs}\log(en)\|\chi_{A\cup B}\|_{2,1},$$

and we conclude that (12) holds, thanks to Lemma 5. □

Lemma 7. *Let* $A_n:L_{2,q}(0,1)\to L_{2,q}((0,1)^2)$, *for* $1\le q\le 2$. *Then*

$$\|A_n\|_{L_{2,q}\to L_{2,q}}\le c_{abs}\log^{\frac{1}{q}}(en),\quad n\ge 1. \tag{13}$$

Proof. It follows from Theorem 5.2.4 in [2] that $[L_{2,1},L_2]_{\frac{2q-2}{q},q}=L_{2,q}$. Using Proposition 2.g.15 in [9] we obtain

$$\|A_n\|_{L_{2,q}\to L_{2,q}}\le\|A_n\|^{\frac{2-q}{q}}_{L_{2,1}\to L_{2,1}}\|A_n\|^{\frac{2q-2}{q}}_{L_2\to L_2}. \tag{14}$$

Clearly,

$$\|A_n\|_{L_2\to L_2}=(\sum_{k=1}^n\frac{1}{k})^{\frac{1}{2}}\le\log^{\frac{1}{2}}(en). \tag{15}$$

From (14) and (15) and Lemma 6 we infer that

$$\|A_n\|_{L_{2,q}\to L_{2,q}}\le c_{abs}\Big(\log(en)\Big)^{\frac{2-q}{q}}\cdot\Big(\log^{\frac{1}{2}}(en)\Big)^{\frac{2q-2}{q}}=c_{abs}\log^{\frac{1}{q}}(en).$$

□

Lemma 8. *For every* $f\in L_{2,q}(0,1)$, $1\le q<2$, *we have*

$$\|A_nf\|_{2,q}=o(\log^{\frac{1}{q}}(en)),\quad n\to\infty.$$

Proof. Without loss of generality, we assume that $f\ge 0$ and fix $\epsilon>0$. Since $L_{2,q}$ is separable, it follows that there exists $t>0$ with $\|f^*(t)\chi_{(0,t)}\|_{2,q}\le\epsilon$. Set $f_1=(f-f^*(t))_+$ and $f_2=\min\{f,f^*(t)\}$. It is immediate that

$$\|A_nf\|_{2,q}\le\|A_n(f_1)\|_{2,q}+\|A_n(f_2)\|_{2,q}. \tag{16}$$

By Lemma 7, we have

$$\|A_n(f_1)\|_{2,q}\le c_{abs}\log(en)\|f_1\|_{2,q}\le c_{abs}\epsilon\log^{\frac{1}{q}}(en). \tag{17}$$

By Proposition 2.d.1 in [9], we have

(18)

$$\begin{aligned}\|A_n(f_2)\|_{2,q}\le{}&c_{abs}\Big\|\Big(\sum_{k=1}^n\frac{1}{k}(f_2\circ\gamma_k)^2\Big)^{\frac{1}{2}}\Big\|_{2,q}\le c_{abs}\Big\|\Big(\sum_{k=1}^n\frac{1}{k}(f_2\circ\gamma_k)^2\Big)^{\frac{1}{2}}\Big\|_{\infty}\\&=c_{abs}\Big\|\Big(\sum_{k=1}^n\frac{1}{k}(f_2\circ\gamma_k)^2\Big)\Big\|_{\infty}^{\frac{1}{2}}\le c_{abs}\Big(\sum_{k=1}^n\frac{1}{k}\|(f_2\circ\gamma_k)^2\|_{\infty}\Big)^{\frac{1}{2}}\\&=c_{abs}\Big(\sum_{k=1}^n\frac{1}{k}(f^*(t))^2\Big)^{\frac{1}{2}}=c_{abs}f^*(t)\Big(\sum_{k=1}^n\frac{1}{k}\Big)^{\frac{1}{2}}\le c_{abs}\log^{\frac{1}{2}}(en)f^*(t).\end{aligned}$$

From (16), (17), and (18) we have

$$\|A_n(f)\|_{2,q}\le c_{abs}\log^{\frac{1}{2}}(en)f^*(t)+c_{abs}\epsilon\log^{\frac{1}{q}}(en).$$

Taking into account that $q < 2$, we obtain

$$\limsup_{n\to\infty} \frac{1}{\log^{\frac{1}{q}}(en)} \|A_n f\|_{2,q} \le c_{abs}\epsilon.$$

Since $\epsilon > 0$ is arbitrarily small, the assertion follows. □

4. Main results

In the proof, $A \approx B$ means that $c^{-1}A \le B \le cA$ for some constant c which only depends on the isomorphic embedding.

Theorem 9. *If $1 \le q < 2$, then the Banach space $l_{2,q}$ does not isomorphically embed into $L_{2,q}(0,1)$.*

Proof. Suppose the contrary, that is, there exists T, the required embedding. Let $\{e_k\}_{k=1}^{\infty}$ be the standard unit basis in $l_{2,q}$. Set $f_k = T(e_k), k \ge 1$. It is obvious that $\{f_k\}_{k=1}^{\infty}$ is a basic sequence in $L_{2,q}(0,1)$, which is equivalent to the basis $\{e_k\}_{k=1}^{\infty}$ in $l_{2,q}$. Hence, for every $t \in (0,1)$ and for every $\alpha \in l_{2,q}$, we have

$$\Big\|\sum_{k\ge1} \alpha_k r_k(t) f_k\Big\|_{2,q} \approx \Big\|\sum_{k\ge1} \alpha_k r_k(t) e_k\Big\|_{2,q} \approx \|\alpha\|_{2,q}.$$

Thus,

$$\int_0^1 \Big\|\sum_{k\ge1} \alpha_k r_k(t) f_k\Big\|_{2,q} dt \approx \|\alpha\|_{2,q}.$$

By Theorem 1.d.6 and Proposition 2.d.1 in [9], we have

$$\int_0^1 \Big\|\sum_{k\ge1} \alpha_k r_k(t) f_k\Big\|_{2,q} dt \approx \Big\|\sum_{k\ge1} \alpha_k f_k \otimes r_k\Big\|_{2,q}$$

and therefore

$$\Big\|\sum_{k\ge1} \alpha_k f_k \otimes r_k\Big\|_{2,q} \approx \|\alpha\|_{2,q}. \tag{19}$$

Since the sequence $\{f_k\}_{k=1}^{\infty}$ is weakly null and $\inf_k \|f_k\|_{2,q} > 0$, it follows from Lemma 2 that we may further assume (passing to a subsequence, if necessary)[2] that $\{f_k\}_{k=1}^{\infty}$ is an unconditional basic sequence satisfying

$$f_k = u_k + v_k, \quad k \ge 1,$$

where $\{u_k\}_{k=1}^{\infty}, \{v_k\}_{k=1}^{\infty}$ are unconditional basic sequences in $L_{2,q}(0,1)$, such that $u_k^* = u_1^*$ for all $k \ge 1$ (hence, there exists $u \in L_{2,q}$ and measure-preserving transformations $\gamma_k : (0,1) \to (0,1)$ such that $u_k = u \circ \gamma_k$ for all $k \ge 1$) and $v_k v_i = 0$ for all $k \ne i \ge 1$, in particular, $v_k \to 0$ in measure as $k \to \infty$.

Suppose that $v_k \not\to 0$ in the norm of $L_{2,q}(0,1)$. Passing to a subsequence if needed, we may further assume (see Lemma 1) that the basic sequence $\{v_k\}_{k=1}^{\infty}$ in $L_{2,q}$ is equivalent to the basis $\{e_k\}_{k=1}^{\infty}$ in l_q. It follows from the triangle inequality that

$$\Big\|\sum_{k=1}^{n} k^{-\frac{1}{2}} v_k \otimes r_k\Big\|_{2,q} \le \Big\|\sum_{k=1}^{n} k^{-\frac{1}{2}} f_k \otimes r_k\Big\|_{2,q} + \Big\|\sum_{k=1}^{n} k^{-\frac{1}{2}} u_k \otimes r_k\Big\|_{2,q}. \tag{20}$$

[2]We get rid of the third sequence by applying Proposition 1.a.9 in [8].

By (2), we have

$$\left\|\left\{k^{-\frac{1}{2}}\right\}_{k=1}^{n}\right\|_{2,q} \approx \log^{\frac{1}{q}}(en), \quad n \geq 1, \tag{21}$$

and therefore we immediately infer from (19) that

$$\left\|\sum_{k=1}^{n} k^{-\frac{1}{2}} f_k \otimes r_k\right\|_{2,q} \approx \log^{\frac{1}{q}}(en), \quad n \geq 1.$$

Note that $u_k = u \circ \gamma_k$ and, hence, the second summand on the right hand side of (20) equals $\|A_n u\|_{2,q}$. Appealing to Lemma 6, we infer that this second summand is also $O(\log^{\frac{1}{q}}(en))$ when $n \to \infty$. So we obtain from (20)

$$\left\|\sum_{k=1}^{n} k^{-\frac{1}{2}} v_k \otimes r_k\right\|_{2,q} = O(\log^{\frac{1}{q}}(en)), \quad n \to \infty. \tag{22}$$

However,

$$\left\|\sum_{k=1}^{n} k^{-\frac{1}{2}} v_k \otimes r_k\right\|_{2,q} \approx \left\|\sum_{k=1}^{n} k^{-\frac{1}{2}} v_k\right\|_{2,q} \approx \left\|\left\{k^{-\frac{1}{2}}\right\}_{k=1}^{n}\right\|_{q} \approx n^{\frac{1}{q}-\frac{1}{2}}.$$

Combining the equivalences above with (22), we arrive at

$$n^{\frac{1}{q}-\frac{1}{2}} = O(\log^{\frac{1}{q}}(en)), \quad n \to \infty.$$

Taking into account the fact that $q < 2$, we obtain a contradiction, which shows that $\lim_{k\to\infty} \|v_k\|_{2,q} = 0$.

Again appealing to Proposition 1.a.9 in [8], we may further assume without loss of generality that $v_k = 0,\ k \geq 1$. Now, combining (21) with (19), we arrive at

$$\left\|\sum_{k=1}^{n} k^{-\frac{1}{2}} u_k \otimes r_k\right\|_{2,q} \approx \log^{\frac{1}{q}}(en), \quad n \to \infty. \tag{23}$$

Recalling that $u_k = u \circ \gamma_k$, the equivalence (23) can be rewritten as

$$\|A_n u\|_{2,q} \approx \log^{\frac{1}{q}}(en), \quad n \to \infty.$$

This contradiction with Lemma 8 completes the proof. □

For every $n, k \in \mathbb{N}$, $t \in (0, \infty)$ we define

$$r_{n,k}(t) := \begin{cases} r_n(t-k+1), & t \in (k-1, k], \\ 0, & \text{elsewhere}, \end{cases} \tag{24}$$

where $r_n(t),\ n \in \mathbb{N},\ t \in (0,1)$ are the Rademacher functions. Let $\mathbf{r} := \{r_{n,k}\}_{n,k\in\mathbb{N}}$.

Theorem 10. *Let $[r_{n,k}]_{k,n=1}^{\infty}$ be the closed linear subspace of $L_{2,q}(0,\infty)$ spanned by the system $\mathbf{r}$. If $1 \leq q < 2$, then $[r_{n,k}]_{k,n=1}^{\infty}$ does not isomorphically embed into $L_{2,q}(0,1) \oplus l_{2,q}$.*

Proof. The proof is essentially a verbatim repetition of the arguments given in the proof of [7, Theorem 11]. One needs only to consistently replace the references to [7, Theorem 10] and [7, Corollary 8] there with references to our Theorem 9. We omit further details. □

Corollary 11. *If $1 \leq q < 2$, then the space $L_{2,q}(0,1) \oplus l_{2,q}$ does not contain a closed subspace which is isomorphic to $L_{2,q}(0,\infty)$.*

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