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OFF-DIAGONAL ESTIMATES OF THE BERGMAN KERNEL ON HYPERBOLIC RIEMANN SURFACES OF FINITE VOLUME

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ABSTRACT. In this article, we derive off-diagonal estimates of the Bergman kernel associated to tensor-powers of the cotangent line bundle defined over a hyperbolic Riemann surface of finite volume.

1. INTRODUCTION

In this article, we derive off-diagonal quantitative estimates of the Bergman kernel associated to tensor-powers of the cotangent bundle defined on a hyperbolic Riemann surface of finite volume.

Estimates of Bergman kernels associated to high tensor-powers of holomorphic line bundles defined on complex manifolds has been an object of study for a long time. Tian, Zelditch, Demailly, Marinsecu, Ma et al. have done seminal work in this field.

In this article, we derive estimates of the Bergman kernel associated to tensor-powers of the cotangent bundle defined over a hyperbolic Riemann surface of finite volume, away from the diagonal, both in the compact and in the noncompact setting. Our estimates depend only on the injectivity radius of the hyperbolic Riemann surface, and tensor-powers of the cotangent bundle.

Results from literature. We now briefly discuss the history behind the problem, before we state our main theorem. In [Chr91], Christ has derived an estimate of the Bergman kernel associated to the trivial line bundle defined over $\mathbb{C}$, away from the diagonal. In [Del98], Delin has derived a similar estimate for the $\mathcal{C}^1$-seminorm of the Bergman kernel associated to the trivial line bundle defined over $\mathbb{C}^n$, away from the diagonal.

Let X be a compact Kähler manifold, and let $\mathcal{L}$ be a positive line bundle defined over X. Then, for any $k \in \mathbb{N}$, an off-diagonal estimate of the Bergman kernel associated to $\mathcal{L}^{\otimes k}$, is derived by Christ in [Chr13].

Let X be a compact sympletic manifold of real dimension-$2n$. Then, in [DLM06], Dai, Liu, and Ma have derived off-diagonal asymptotic expansion for the Bergman kernel of the spin^c Dirac operator associated to high tensor-powers of a positive line bundle. In particular, they derived estimates of the Bergman kernel along the

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diagonal. They also extended their estimates to compact sympletic orbifolds in the same article.

Let X be a complete, sympletic manifold of real dimension-$2n$ with bounded geometry, and let $\mathcal{L}$ and $\mathcal{F}$ be a positive line bundle and a vector bundle defined over X, respectively. In [MM15], for $n, k \in \mathbb{N}$, Ma and Marinescu have derived estimates of $\mathcal{C}^n$-norms of the Bergman kernel associated to the vector bundle $\mathcal{L}^{\otimes k} \otimes \mathcal{F}$, away from the diagonal. At any $z, w \in X$, the estimates derived in [MM15] are very general, and unlike the estimates derived in [Chr13], they do not impose any restriction on the geodesic distance between the points z and w.

When X is a compact hyperbolic Riemann surface, in [CF16], Chen and Fu have derived an estimate of the Bergman kernel associated to the cotangent bundle along the diagonal. In [ABMS16], the authors have derived an estimate of the Bergman kernel associated to tensor-powers of the line bundle of holomorphic cusp forms of weight-2, along the diagonal. As the line bundle of holomorphic cusp forms of weight-2 is isometric to the cotangent bundle, the estimate computed in [ABMS16] can also be viewed as an estimate of the Bergman kernel associated to tensor-powers of the cotangent bundle. The estimates derived in [CF16] and [ABMS16] are stable in covers of compact hyperbolic Riemann surfaces.

When X is a noncompact hyperbolic Riemann surface of finite volume, an estimate of the Bergman kernel associated to tensor-powers of the line bundle of holomorphic cusp forms of weight-2 is obtained in [FJK16], which is also stable in covers.

Let X be a noncompact Riemann surface, whose natural metric has singularities of Poincaré type at a finite set. Let $\mathcal{L}$ be a holomorphic line bundle whose curvature form is a scalar multiple of the hyperbolic metric outside a compact subset of X. In [AMM16a], Auvray, Ma, and Marinescu have derived optimal estimates of $\mathcal{C}^n$-norms of the Bergman kernel associated of tensor-powers of $\mathcal{L}$, along the diagonal. Their estimates in the setting of hyperbolic Riemann surfaces of finite volume coincide and are more explicit than the ones derived in [FJK16]. The estimates derived in [AMM16a] easily extend to the setting of Riemann orbifolds.

Furthermore, in [AMM16b], Auvray, Ma, and Marinescu have derived optimal estimates of $\mathcal{C}^n$-norms of the Bergman kernel associated to tensor-powers of $\mathcal{L}$, both along the diagonal, and away from the diagonal. The estimates derived in [AMM16b] easily extend to the setting of Riemann orbifolds.

The estimates derived in [AMM16a] and [AMM16b] also remain stable in covers of Riemann surfaces.

Statement of the main theorem. We now state the main theorem of the article.

Main theorem. *Let X be a hyperbolic Riemann surface of finite volume, and for any $k \in \mathbb{N}$, let $\mathcal{B}^k_{\Omega_X}$ denote the Bergman kernel associated to $\Omega_X^{\otimes k}$, where Ω_X denotes the cotangent bundle of X. Let $\|\cdot\|_{\mathrm{hyp}}$ denote the Hermitian metric on $\Omega_X^{\otimes k}$. For any $k \geq 3$, and $\delta \geq r_X$, let $z = x + iy$, $w = u + iv \in X$ (identifying X with its universal cover $\mathbb{H}$) with $d_{\mathrm{hyp}}(z, w) \geq \delta$, where $d_{\mathrm{hyp}}(z, w)$ denotes the geodesic distance between the points z and w on X. When X is compact, we have the following estimate:*

$$\| \mathcal{B}^k_{\Omega_X} \|_{\mathrm{hyp}}(z, w) \leq \mathcal{C}_X; \tag{1}$$

when X is noncompact, without loss of generality, we assume that $i\infty$ is the only puncture of X (identifying X with its universal cover $\mathbb{H}$). Then, we have the following estimate:

$$\| \mathcal{B}^k_{\Omega_X} \|_{\mathrm{hyp}}(z,w) \leq \mathcal{C}_X + \frac{2k-1}{4\pi\cosh^{2k}(\delta/2)} + \frac{(4yv)^k}{(y+v)^{2k-1}} \cdot \frac{(2k-1)\Gamma\big(k-1/2\big)}{2\sqrt{\pi}\Gamma(k)}, \tag{2}$$

where

$$\begin{aligned} \mathcal{C}_X := {} & \frac{(2k-1)\sinh(\delta+r_X)}{4\pi\cosh^{2k}\big((\delta-r_X)/2\big)\sinh(r_X)} + \frac{(2k-1)\sinh(\delta)}{2\pi\cosh^{2k}(\delta/2)} \cdot \frac{\cosh(r_X/4)}{\sinh(r_X/4)} \\ & + \frac{2k-1}{2\pi(2k-2)\cosh^{2k-2}(\delta/2)}\left(2+\frac{1}{\sinh^2(r_X/4)}\right) \\ & + \frac{2k-1}{\pi(2k-4)\cosh^{2k-4}(\delta/2)} \cdot \frac{1}{\sinh^2(r_X/4)}. \end{aligned} \tag{3}$$

Here r_X denotes the injectivity radius of X, which is as defined in equations (4) *and* (5)*, for compact and noncompact hyperbolic Riemann surfaces, respectively.*

Remark 1.1. Although, we assume that $i\infty$ is the only puncture of X, our estimate (2) easily extends to the case of multiple punctures.

Similar to the estimates of the Bergman kernel derived in [CF16], [ABMS16], [FJK16], [AMM16a], and [AMM16b], it is easy to show that our estimates (1) and (2) are stable in covers of Riemann surfaces, by following similar arguments as in [ABMS16] (see Remark 3.3).

It is also not difficult to extend estimates (1) and (2) to hyperbolic Riemann orbisurfaces of finite volume. Furthermore, in certain cases, our estimate (1) is slightly stronger than the more general estimate derived in [MM15] (see Remark 3.1).

For k sufficiently large, and for a fixed $w \in X$, as $z \in X$ approaches a puncture of X, estimate (2) gives a slightly stronger estimate than the one derived in [AMM16b] (see Remark 3.2). However, unlike our estimates, the estimates derived in [MM15], [AMM16a], and [AMM16b] have no restriction on the geodesic distance between z and w, and are also uniform in z and w.

Although we impose the condition that $k \geq 3$, one can easily derive similar estimates for the cases $k = 1$ and $k = 2$, by employing similar techniques.

2. Background material

In this section, we set up the notation and recall the background details needed for the proofs of the main theorem.

Let X denote a hyperbolic Riemann surface of finite volume. By uniformization theorem from complex analysis, X can be realized as the quotient space $\Gamma \backslash \mathbb{H}$, where $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is a cofinite Fuchsian subgroup, and $\mathbb{H}$ is the complex upper half-plane. Locally, we identify X with its universal cover $\mathbb{H}$, and hence, for brevity of notation, we denote the points on X by the same letter as the points on $\mathbb{H}$. Let X_Γ denote a fundamental domain for Γ.

When X is a noncompact hyperbolic Riemann surface of finite volume, without loss of generality, we assume that the point $i\infty$ is the only puncture of X, which we denote by ∞.

Let μ_{hyp} denote the natural hyperbolic metric on $\mathbb{H}$, which is of constant negative curvature -1. The natural metric on X is induced by the hyperbolic metric μ_{hyp}, which we again denote by μ_{hyp}. For any $z, w \in \mathbb{H}$, let $d_{\mathrm{hyp}}(z, w)$ denote the hyperbolic distance on $\mathbb{H}$, which is the natural distance function on $\mathbb{H}$, coming from the hyperbolic metric μ_{hyp}. Locally, for any $z, w \in X$, the geodesic distance between the points z and w on X is given by $d_{\mathrm{hyp}}(z, w)$.

The injectivity radius of a compact hyperbolic Riemann surface X is given by the following formula:

$$r_X := \inf\left\{\, d_{\mathrm{hyp}}(z, \gamma z) \,|\, z \in \mathbb{H},\ \gamma \in \Gamma \backslash \{\mathrm{Id}\}\right\}; \tag{4}$$

when X is a noncompact hyperbolic Riemann surface of finite volume, we define the injectivity radius of X by the following formula:

$$r_X := \inf\left\{\, d_{\mathrm{hyp}}(z, \gamma z) \,|\, z \in \mathbb{H},\ \gamma \in \Gamma \backslash \Gamma_\infty\right\}, \tag{5}$$

where Γ_∞ is the stabilizer of the cusp ∞.

Let Ω_X denote the cotangent bundle of holomorphic differential 1-forms on X. The global sections of this line bundle are of the form $f(z)dz$, where $f(z)$ is a holomorphic modular form of weight-2 with respect to Γ. For any $k \in \mathbb{N}$, let $\omega \in H^0(X, \Omega_X^{\otimes k})$ be a global differential k-form. Then, locally, at any $z \in X$, $\omega(z) = f(z)dz^{\otimes k}$, where f is a weight-$2k$ modular form with respect to Γ.

Furthermore, there exists a point-wise metric on $H^0(X, \Omega_X^{\otimes k})$, which is denoted by $\|\cdot\|_{\mathrm{hyp}}$, and locally, at the point $z = x + iy \in \mathbb{H}$, it is given by the following formula:

$$\|\omega\|_{\mathrm{hyp}}(z) = y^k |f(z)|.$$

Let

$$H^0_{(2)}\big(X, \Omega_X^{\otimes k}\big) := \left\{ \omega \in H^0\big(X, \Omega_X^{\otimes k}\big) \,\middle|\, \int_{X_\Gamma} \|\omega\|^2_{\mathrm{hyp}}(z)\, \mu_{\mathrm{hyp}}(z) = \int_{X_\Gamma} y^{2k} |f(z)|^2\, \mu_{\mathrm{hyp}}(z) < \infty \right\}$$

denote the space of L^2-global holomorphic sections of Ω_X.

For any $\omega(z) = f(z)dz^{\otimes k}, \eta(z) = g(z)dz^{\otimes k} \in H^0_{(2)}(X, \Omega_X^{\otimes k})$, the L^2-metric induced by $\|\cdot\|_{\mathrm{hyp}}$ on $H^0_{(2)}(X, \Omega_X^{\otimes k})$ is denoted by $\langle \cdot, \cdot \rangle_{\mathrm{hyp}}$, and is given by the following formula:

$$\langle \omega, \eta \rangle_{\mathrm{hyp}} = \int_{X_\Gamma} y^{2k} f(z) \overline{g(z)}\, \mu_{\mathrm{hyp}}(z).$$

The space $H^0_{(2)}(X, \Omega_X)$ can be identified with $\mathcal{S}^2(\Gamma)$, the complex vector-space of weight-2 cusp forms, and for any $k \in \mathbb{N}$, we can identify $H^0_{(2)}(X, \Omega_X^{\otimes k})$ with $\mathcal{S}^{2k}(\Gamma)$ as complex vector-spaces.

Let $\{\omega_1, \ldots, \omega_{j_k}\}$ denote a set of orthonormal basis of $H^0_{(2)}(X, \Omega_X^{\otimes k})$ with respect to the L^2-metric $\langle \cdot, \cdot \rangle_{\mathrm{hyp}}$. Then, locally, for any $z, w \in X$, the Bergman kernel $\mathcal{B}^k_{\Omega_X}(z, w)$ associated to the line bundle $\Omega_X^{\otimes k}$ is given by the following formula:

$$\mathcal{B}^k_{\Omega_X}(z, w) = \sum_{i=1}^{j_k} \omega_i(z) \overline{\omega_i(w)}.$$

It is easy to show that the Bergman kernel is independent of the choice of orthonormal bases for $H^0_{(2)}(X, \Omega_X^{\otimes k})$.

We now describe the Bergman kernel associated to $\mathcal{S}^{2k}(\Gamma)$, the vector-space of weight-$2k$ cusp forms. For $f \in \mathcal{S}^{2k}(\Gamma)$, we have the following point-wise metric at $z = x + iy \in \mathbb{H}$:

$$\|f\|_{\mathrm{pet}}(z) := y^k |f(z)|,$$

which induces an L^2-metric on $\mathcal{S}^{2k}(\Gamma)$, which is also known as the Petersson inner-product. For any $f, g \in \mathcal{S}^{2k}(\Gamma)$, the Petersson inner-product is given by the following formula:

$$\langle f, g\rangle_{\mathrm{pet}} := \int_{X_\Gamma} y^{2k} f(z)\overline{g(z)}\, \mu_{\mathrm{hyp}}(z).$$

Let $\{f_1, \ldots, f_{j_k}\}$ denote an orthonormal basis for $\mathcal{S}^{2k}(\Gamma)$ with respect to the Petersson inner-product. Then, for any $z = x + iy$, $w = u + iv \in \mathbb{H}$, the Bergman kernel associated to the complex vector-space $\mathcal{S}^{2k}(\Gamma)$ is given by the following formula:

$$\mathcal{B}_X^{2k}(z, w) := \sum_{i=1}^{j_k} f_i(z)\overline{f_i(w)}.$$

The Bergman kernel $\mathcal{B}_X^{2k}(z, w)$ is a holomorphic cusp form of weight-$2k$ in z, and an anti-holomorphic cusp form of weight-$2k$ in w. It can also be defined by the following infinite series (see Proposition 1.3 on p. 77 in [Fre90]):

$$\mathcal{B}_X^{2k}(z, w) = \frac{(2k-1)(2i)^{2k}}{4\pi} \sum_{\gamma\in\Gamma} \frac{1}{(\gamma z - \overline{w})^{2k}} \cdot \frac{1}{j(\gamma, z)^{2k}},$$

where for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $j(\gamma, z) = cz + d$.

Remark 2.1. The expression for the Bergman kernel $\mathcal{B}_X^{2k}(z, w)$ given in [Fre90] is missing a factor of $(2i)^{2k}$, which is taken into account in the above formula.

As $H^0_{(2)}(X, \Omega_X^{\otimes k}) \cong \mathcal{S}^{2k}(\Gamma)$ as complex vector-spaces, locally, we have the following relation of Bergman kernels:

$$\mathcal{B}^k_{\Omega_X}(z, w) = \mathcal{B}_X^{2k}(z, w)\big(dz^{\otimes k} \wedge d\overline{w}^{\otimes k}\big),$$

and the point-wise metric on $\Omega_X^{\otimes k}$ induces the following point-wise metric on $\mathcal{B}^k_{\Omega_X}(z, w)$:

$$\| \mathcal{B}^k_{\Omega_X} \|_{\mathrm{hyp}}(z, w) = \frac{(2k-1)(4yv)^k}{4\pi} \cdot \left| \sum_{\gamma\in\Gamma} \frac{1}{(\gamma z - \overline{w})^{2k}} \cdot \frac{1}{j(\gamma, z)^{2k}} \right|. \tag{6}$$

Recall that for any $z = x + iy$, $w = u + iv \in \mathbb{H}$, we have the following formula:

$$\cosh^2\big(\, d_{\mathrm{hyp}}(z, w)/2\big) = \frac{|z - \overline{w}|^2}{4yv}.$$

Using the above relation and equation (6), we derive the following inequality:

$$\| \mathcal{B}^k_{\Omega_X} \|_{\mathrm{hyp}}(z,w) \leq \frac{2k-1}{4\pi}\sum_{\gamma\in\Gamma}\left|\frac{(4yv)^k}{(\gamma z-\overline{w})^{2k}j(\gamma,z)^{2k}}\right|$$

$$= \frac{2k-1}{4\pi}\sum_{\gamma\in\Gamma}\frac{\big(4\,\mathrm{Im}(\gamma z)v\big)^k}{\big|\gamma z-\overline{w}\big|^{2k}} = \frac{2k-1}{4\pi}\sum_{\gamma\in\Gamma}\frac{1}{\cosh^{2k}\big(\, d_{\mathrm{hyp}}(\gamma z,w)/2\big)}. \tag{7}$$

For a hyperbolic Riemann surface of finite volume, we now state an inequality from [JR95], which is adapted to our setting. The inequality gives us an estimate for the number of elements in Γ or $\Gamma\setminus\Gamma_\infty$, depending on whether X is compact or noncompact, respectively.

For any positive, smooth, real-valued, and decreasing function f defined on $\mathbb{R}_{\geq 0}$, and for any $\delta > r_X/2$, and $z,w\in\mathbb{H}$, we have the following inequality:

$$\int_0^\infty f(\rho)dN_\Gamma(z,w;\rho) \leq \int_0^\delta f(\rho)dN_\Gamma(z,w;\rho) + f(\delta)\frac{2\cosh(r_X/4)\sinh(\delta)}{\sinh(r_X/4)}$$

$$+ \frac{1}{2\sinh^2(r_X/4)}\int_\delta^\infty f(\rho)\sinh(\rho+r_X/2)d\rho, \tag{8}$$

where $N_\Gamma(z,w;\rho) := \mathrm{card}\,\{\gamma|\,\gamma\in\Gamma\setminus\Gamma_\infty,\ d_{\mathrm{hyp}}(\gamma z,w)\leq\rho\}$.

From arguments similar to the ones used in deriving inequality (8) in [JR95], for any $\delta>0$, and $z,w\in\mathbb{H}$, we have the following inequality:

$$N_\Gamma(z,w;\delta) \leq \frac{\sinh(\delta+r_X)}{\sinh(r_X)}. \tag{9}$$

The above inequality has already been used in the above form in [JK06]. Notice that our definition for injectivity radius is two times the injectivity radius in [JR95], and both our inequalities, (8) and (9) take this fact into account.

Here it is understood that, when X is compact, $\Gamma_\infty=\emptyset$. So inequalities (8) and (9) also hold true in the compact setting.

3. Proof of the main theorem

Proof of estimate (1). We state that $X=\Gamma\setminus\mathbb{H}$ is a compact hyperbolic Riemann surface with injectivity radius r_X, and $\delta\geq r_X$. Furthermore, $\mathcal{B}^k_{\Omega_X}(z,w)$ is the Bergman kernel for the line bundle $\Omega_X^{\otimes k}$. Combining inequalities (7) and (8), for any $k\geq 3$, and $z,w\in X$ with $d_{\mathrm{hyp}}(z,w)\geq\delta$, we find

$$\| \mathcal{B}^k_{\Omega_X} \|_{\mathrm{hyp}}(z,w) \leq \frac{2k-1}{4\pi}\sum_{\gamma\in\Gamma}\frac{1}{\cosh^{2k}\big(\, d_{\mathrm{hyp}}(\gamma z,w)/2\big)}$$

$$= \frac{2k-1}{4\pi}\int_0^\delta \frac{dN_\Gamma(z,w;\rho)}{\cosh^{2k}\big(\, d_{\mathrm{hyp}}(\gamma z,w)/2\big)} + \frac{(2k-1)\sinh(\delta)}{2\pi\cosh^{2k}(\delta/2)}\cdot\frac{\cosh(r_X/4)}{\sinh(r_X/4)}$$

$$+ \frac{2k-1}{8\pi\sinh^2(r_X/4)}\int_\delta^\infty\frac{\sinh(\rho+r_X/2)d\rho}{\cosh^{2k}(\rho/2)}. \tag{10}$$

We now estimate the first term on the right-hand side of the equality in the above inequality. For $\gamma\in\Gamma$, and $z,w\in\mathbb{H}$ with $d_{\mathrm{hyp}}(z,w)\geq\delta$, using triangular inequality, we derive

$$d_{\mathrm{hyp}}(z,\gamma z)+d_{\mathrm{hyp}}(\gamma z,w)\geq d_{\mathrm{hyp}}(z,w)\geq\delta.$$

Using which, we compute

$$\inf_{\gamma\in\Gamma\setminus\{\mathrm{Id}\}}\big(d_{\mathrm{hyp}}(z,\gamma z)+d_{\mathrm{hyp}}(\gamma z,w)\big)=r_X+\inf_{\gamma\in\Gamma\setminus\{\mathrm{Id}\}}d_{\mathrm{hyp}}(\gamma z,w)\geq\delta,$$

which implies that for any $\gamma\in\Gamma$, we have

$$d_{\mathrm{hyp}}(\gamma z,w)\geq\delta-r_X\geq 0\implies\frac{1}{\cosh^{2k}\big((\delta-r_X)/2\big)}\geq\frac{1}{\cosh^{2k}\big(d_{\mathrm{hyp}}(\gamma z,w)/2\big)}.$$

So, combining the above inequality with inequality (9), we arrive at the following inequality:

$$\begin{aligned}&\int_0^{\delta}\frac{dN_{\Gamma}(z,w;\rho)}{\cosh^{2k}\big(d_{\mathrm{hyp}}(\gamma z,w)/2\big)}\\&\qquad\leq\big(N_{\Gamma}(z,w;\delta)\big)\cdot\sup_{\gamma\in S_{\Gamma}(z,w;\rho)}\left(\frac{1}{\cosh^{2k}\big(d_{\mathrm{hyp}}(\gamma z,w)/2\big)}\right)\\&\qquad\leq\frac{\sinh(\delta+r_X)}{\cosh^{2k}\big((\delta-r_X)/2\big)\sinh(r_X)},\end{aligned}$$

which implies that we have the following estimate for the first term on the right-hand side of the equality in (10):

$$\frac{2k-1}{4\pi}\int_0^{\delta}\frac{dN_{\Gamma}(z,w;\rho)}{\cosh^{2k}\big(d_{\mathrm{hyp}}(\gamma z,w)/2\big)}\leq\frac{(2k-1)\sinh(\delta+r_X)}{4\pi\cosh^{2k}\big((\delta-r_X)/2\big)\sinh(r_X)}.\tag{11}$$

We now estimate the third term on the right-hand side of the equality in (10). For any $\rho\geq\delta$, observe that

$$\begin{aligned}\sinh(\rho+r_X/2)&=\sinh(\rho)\cosh(r_X/2)+\cosh(\rho)\sinh(r_X/2)\\&\leq\sinh(\rho)\cosh(r_X/2)+\cosh(\rho)\sinh(\rho)\\&=2\sinh(\rho/2)\cosh(\rho/2)\big(\cosh(r_X/2)+\cosh(\rho)\big)\\&\leq2\sinh(\rho/2)\cosh(\rho/2)\big(\cosh(r_X/2)+2\cosh^2(\rho/2)\big).\end{aligned}$$

Using which, we derive that

$$\begin{aligned}&\int_{\delta}^{\infty}\frac{\sinh(\rho+r_X/2)d\rho}{\cosh^{2k}(\rho/2)}\leq\cosh(r_X/2)\int_{\delta}^{\infty}\frac{2\sinh(\rho/2)d\rho}{\cosh^{2k-1}(\rho/2)}+\int_{\delta}^{\infty}\frac{4\sinh(\rho/2)d\rho}{\cosh^{2k-3}(\rho/2)}\\&\quad=\frac{4\cosh(r_X/2)}{(2k-2)\cosh^{2k-2}(\delta/2)}+\frac{8}{(2k-4)\cosh^{2k-4}(\delta/2)}.\end{aligned}$$

Hence, we have the following estimate for the third term on the right-hand side of the equality in (10):

$$\begin{aligned}
&\frac{2k-1}{8\pi\sinh^2(r_X/4)}\int_\delta^\infty \frac{\sinh(\rho+r_X/2)d\rho}{\cosh^{2k}(\rho/2)}\\
&\quad\leq \frac{(2k-1)}{2\pi(2k-2)\cosh^{2k-2}(\delta/2)}\cdot\frac{\cosh(r_X/2)}{\sinh^2(r_X/4)}\\
&\qquad+\frac{2k-1}{\pi(2k-4)\cosh^{2k-4}(\delta/2)}\cdot\frac{1}{\sinh^2(r_X/4)}\\
&\quad=\frac{(2k-1)}{2\pi(2k-2)\cosh^{2k-2}(\delta/2)}\Big(2+\frac{1}{\sinh^2(r_X/4)}\Big)\\
&\qquad+\frac{2k-1}{\pi(2k-4)\cosh^{2k-4}(\delta/2)}\cdot\frac{1}{\sinh^2(r_X/4)}.
\end{aligned}\tag{12}$$

The proof of estimate (1) follows from combining estimates (10), (11), and (12). □

Proof of estimate (2). Now, let X be a noncompact hyperbolic Riemann surface of finite volume. For any $k\geq 3$, and $z,w\in X$ with $d_{\mathrm{hyp}}(z,w)\geq\delta$, from inequality (7), we have

$$\begin{aligned}
\|\mathcal{B}^k_{\Omega_X}\|_{\mathrm{hyp}}(z,w)\leq{}&\frac{2k-1}{4\pi}\sum_{\gamma\in\Gamma\setminus\Gamma_\infty}\frac{1}{\cosh^{2k}(d_{\mathrm{hyp}}(\gamma z,w)/2)}\\
&+\frac{2k-1}{4\pi}\sum_{\gamma\in\Gamma_\infty}\frac{1}{\cosh^{2k}\big(d_{\mathrm{hyp}}(\gamma z,w)/2\big)}.
\end{aligned}\tag{13}$$

From similar arguments as in the proof of estimate (1), we have the following estimate for the first term on the right-hand side of the above inequality:

$$\frac{2k-1}{4\pi}\sum_{\gamma\in\Gamma\setminus\Gamma_\infty}\frac{1}{\cosh^{2k}(d_{\mathrm{hyp}}(\gamma z,w)/2)}\leq\mathcal{C}_X,\tag{14}$$

where $\mathcal{C}_X$ is as in equation (3).

We now estimate the second term on the right-hand side of inequality (13). Without loss of generality, we assume that

$$\Gamma_\infty:=\Big\{\begin{pmatrix}1&n\\0&1\end{pmatrix}\,\Big|\,n\in\mathbb{Z}\Big\}.$$

This implies that for $z = x + iy$, $w = u + iv \in \mathbb{H}$, we have

$$\begin{aligned}
&\sum_{\gamma\in\Gamma_\infty} \frac{1}{\cosh^{2k}\big(\, d_{\mathrm{hyp}}(\gamma z, w)/2\big)} \\
&\quad = \frac{1}{\cosh^{2k}\big(\, d_{\mathrm{hyp}}(z, w)/2\big)} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(4yv)^k}{\big((x+n-u)^2 + (y+v)^2\big)^k} \\
&\quad \leq \frac{1}{\cosh^{2k}(\delta/2)} + \int_0^\infty \frac{(4yv)^k}{(y+v)^{2k}} \cdot \frac{d\alpha}{\Big(\big(\frac{\alpha+x-u}{y+v}\big)^2 + 1\Big)^k} \qquad (15)\\
&\qquad + \int_0^\infty \frac{(4yv)^k}{(y+v)^{2k}} \cdot \frac{d\alpha}{\Big(\big(\frac{\alpha+u-x}{y+v}\big)^2 + 1\Big)^k}.
\end{aligned}$$

Making the substitution $(\alpha + x - u)/(y + v) = \theta$, and using formula 3.251.2 from [GR15], we arrive at the following estimate for the second term on the right-hand side of inequality (15):

$$\begin{aligned}
\int_0^\infty \frac{(4yv)^k}{(y+v)^{2k}} \cdot \frac{d\alpha}{\Big(\big(\frac{\alpha+x-u}{y+v}\big)^2 + 1\Big)^k} &\leq \frac{(4yv)^k}{(y+v)^{2k-1}} \cdot \int_{-\infty}^\infty \frac{d\theta}{\big(\theta^2+1\big)^k} \\
&= \frac{(4yv)^k}{(y+v)^{2k-1}} \cdot \frac{\sqrt{\pi}\,\Gamma\big(k-1/2\big)}{\Gamma(k)}. \qquad (16)
\end{aligned}$$

Following similar arguments, we have the following estimate for the third term on the right-hand side of inequality (15):

$$\int_0^\infty \frac{(4yv)^k}{(y+v)^{2k}} \cdot \frac{d\alpha}{\Big(\big(\frac{\alpha+u-x}{y+v}\big)^2 + 1\Big)^k} \leq \frac{(4yv)^k}{(y+v)^{2k-1}} \cdot \frac{\sqrt{\pi}\,\Gamma\big(k-1/2\big)}{\Gamma(k)}. \qquad (17)$$

Combining inequalities (15), (16), and (17), we arrive at the following estimate for the second term on the right-hand side of inequality (13):

$$\begin{aligned}
&\frac{2k-1}{4\pi} \sum_{\gamma\in\Gamma_\infty} \frac{1}{\cosh^{2k}\big(\, d_{\mathrm{hyp}}(\gamma z, w)/2\big)} \qquad (18)\\
&\qquad \leq \frac{2k-1}{4\pi\cosh^{2k}(\delta/2)} + \frac{(4yv)^k}{(y+v)^{2k-1}} \cdot \frac{(2k-1)\Gamma\big(k-1/2\big)}{2\sqrt{\pi}\Gamma(k)}.
\end{aligned}$$

Combining estimates (13), (14), and (18) completes the proof of estimate (2), and also the proof of the main theorem. □

Remark 3.1. When X is compact, for any $k \geq 3$, $\delta \geq r_X$, and $z, w \in X$ with $d_{\mathrm{hyp}}(z, w) \geq \delta$, a careful analysis of each of the terms comprising the constant $\mathcal{C}_X$ given in equation (3), leads us to the conclusion that

$$\mathcal{C}_X = O_X\bigg(\frac{k}{\cosh^{2k-4}\big((\delta - r_X)/2\big)}\bigg).$$

Using the fact that $\cosh(u) \geq e^u/2$, for all $u \geq 0$, we observe that

$$\mathcal{C}_X = O_X\left(\frac{k \cdot 2^{2k-4}}{e^{(k-2)(\delta - r_X)}}\right) = O_X\Big(ke^{-(k-2)\left(\delta - r_X - 2\ln 2\right)}\Big).$$

For k sufficiently large, and for $z, w \in X$ with $d_{\mathrm{hyp}}(z,w) > (r_X + 2\ln 2)$, estimate (1) is a slightly stronger estimate than the more general estimates derived in [MM15] and [Chr13].

Remark 3.2. When X is noncompact, for any $k \geq 3$, and $z, w \in X$ with $d_{\mathrm{hyp}}(z,w) \geq r_X$, substituting $\delta = d_{\mathrm{hyp}}(z,w)$ in estimate (14), and from Remark 3.1, we have

$$\frac{2k-1}{4\pi} \sum_{\gamma \in \Gamma \backslash \Gamma_\infty} \frac{1}{\cosh^{2k}\big(d_{\mathrm{hyp}}(\gamma z, w)/2\big)} \leq \mathcal{C}_X = O_X\left(\frac{k}{\cosh^{2k-4}\big((d_{\mathrm{hyp}}(z,w) - r_X)/2\big)}\right).$$

Furthermore, for any $z = x + iy$, $w = u + iv \in \mathbb{H}$, we find

$$\frac{1}{\cosh^{2k-4}\big(d_{\mathrm{hyp}}(z,w)/2\big)} = \left(\frac{4yv}{(x-u)^2 + (y+v)^2}\right)^{k-2} \leq \left(\frac{(4yv)}{(y+v)^2}\right)^{k-2} \leq \left(\frac{4v}{y}\right)^{k-2}. \tag{19}$$

Hence, for any $k \geq 3$, and for a fixed $w \in \mathbb{H}$, as $z \in \mathbb{H}$ approaches $i\infty$, we can conclude that

$$\frac{2k-1}{4\pi} \sum_{\gamma \in \Gamma \backslash \Gamma_\infty} \frac{1}{\cosh^{2k}\big(d_{\mathrm{hyp}}(\gamma z, w)/2\big)} = O_X\left(\frac{ke^{(k\, r_X)/2}}{\cosh^{2k-4}\big(d_{\mathrm{hyp}}(z,w)/2\big)}\right) = O_{X,w}\left(\frac{ke^{(k\, r_X)/2}}{y^{k-2}}\right).$$

The coordinate function in the neighborhood of the puncture ∞ is given by $q(z) := e^{2\pi i z}$, which implies that in local coordinates, we have the following estimate:

$$\frac{2k-1}{4\pi} \sum_{\gamma \in \Gamma \backslash \Gamma_\infty} \frac{1}{\cosh^{2k}\big(d_{\mathrm{hyp}}(\gamma z, w)/2\big)} = O_{X,w}\left(\frac{ke^{(k\, r_X)/2}}{\big|\log|q(z)|\big|^{k-2}}\right). \tag{20}$$

Similarly, for any $k \geq 3$, and for a fixed $w \in \mathbb{H}$, as $z \in \mathbb{H}$ approaches $i\infty$, substituting $\delta = d_{\mathrm{hyp}}(z,w)$ in estimate (18) and combining it with estimate (19), we have

$$\frac{2k-1}{4\pi} \sum_{\gamma \in \Gamma_\infty} \frac{1}{\cosh^{2k}\big(d_{\mathrm{hyp}}(\gamma z, w)/2\big)} \leq \frac{2k-1}{4\pi}\left(\frac{4v}{y}\right)^k + \frac{(4v)^k}{y^{k-1}} \cdot \frac{(2k-1)\,\Gamma\left(k - 1/2\right)}{2\sqrt{\pi}\,\Gamma(k)}.$$

Furthermore, for any $k > 0$, from the asymptotics of the Gamma function, we have the following estimate:

$$\begin{aligned}
&\frac{(2k-1)\Gamma\big(k-1/2\big)}{2\sqrt{\pi}\Gamma(k)} = O\big(\sqrt{k}\big)\\
(21)\qquad &\Longrightarrow\ \frac{2k-1}{4\pi}\sum_{\gamma\in\Gamma_\infty}\frac{1}{\cosh^{2k}\big(\,d_{\mathrm{hyp}}(\gamma z,w)/2\big)} = O_w\bigg(\frac{k}{y^{k-1}}\bigg)\\
&\qquad\qquad = O_w\bigg(\frac{k}{\big|\log|q(z)|\big|^{k-1}}\bigg).
\end{aligned}$$

So, for any $k\geq 3$, and a fixed $w\in X$, as $z\in X$ approaches the puncture ∞, combining estimates (20) and (21), we deduce that

$$\|\,\mathcal{B}^k_{\Omega_X}\,\|_{\mathrm{hyp}}(z,w) = O_{X,w}\bigg(\frac{ke^{(k\,r_X)/2}}{\big|\log|q(z)|\big|^{k-2}}\bigg).$$

For k sufficiently large, in the case of w being fixed, and z approaching the puncture, the above estimate is a slightly stronger estimate than the one derived in Theorem 6.1 in [AMM16b]. However, the estimate derived in Theroem 6.1 in [AMM16b] is a more general estimate, and has no restriction on the distance between the two points z and w, and is also uniform in z and w.

Remark 3.3. Let $X_1=\Gamma_1\backslash\mathbb{H}$, $X_0=\Gamma_0\backslash\mathbb{H}$ be two compact hyperbolic Riemann surfaces. Let X_1 be a finite cover of X_0, which implies that Γ_1 is a finite index subgroup of Γ_0. So, for any $k\geq 3$, and $z,w\in X_1$ with $d_{\mathrm{hyp}}(z,w)\geq\delta$, from estimates (7) and (1), we find that

$$\begin{aligned}
\|\mathcal{B}^k_{\Omega_{X_1}}\|_{\mathrm{hyp}}(z,w) &\leq \frac{2k-1}{4\pi}\sum_{\gamma\in\Gamma_1}\frac{1}{\cosh^{2k}\big(\,d_{\mathrm{hyp}}(\gamma z,w)/2\big)}\\
&\leq \frac{2k-1}{4\pi}\sum_{\gamma\in\Gamma_0}\frac{1}{\cosh^{2k}\big(\,d_{\mathrm{hyp}}(\gamma z,w)/2\big)}\leq \mathcal{C}_{X_0}.
\end{aligned}$$

This implies that our estimate (1) is stable in covers of compact hyperbolic Riemann surfaces. Following the same argument, we can conclude that our estimate (2) is also stable in covers of noncompact hyperbolic Riemann surfaces of finite volume.

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