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# INNER RADIUS OF NODAL DOMAINS OF QUANTUM ERGODIC EIGENFUNCTIONS

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ABSTRACT. In this short note we show that the lower bounds of Mangoubi on the inner radius of nodal domains can be improved for quantum ergodic sequences of eigenfunctions, according to a certain power of the radius of shrinking balls on which the eigenfunctions equidistribute. We prove such improvements using a quick application of our recent results [Anal. PDE 11 (2018), 855–871], which give modified growth estimates for eigenfunctions that equidistribute on small balls. Since by Nonlinearity 28 (2015), 3263–3288, Adv. Math. 290 (2016), 938–966 small scale QE holds for negatively curved manifolds on logarithmically shrinking balls, we get logarithmic improvements on the inner radius of eigenfunctions on such manifolds. We also get improvements for manifolds with ergodic geodesic flows. In addition using the small scale equidistribution results of Comm. Math. Phys. 350 (2017), 279–300, one gets polynomial betterments of Comm. Partial Differential Equations 33 (2008), 1611–1621 for toral eigenfunctions in dimensions $n \geq 3$. The results work only for a full density subsequence of eigenfunctions.

## 1. Introduction

Let $(X, g)$ be a smooth compact connected boundaryless Riemannian manifold of dimension $n$. Suppose $\Delta_g$ is the positive Laplace-Beltrami operator on $(X, g)$ and $\psi_\lambda$ is a real-valued $L^2$-normalized eigenfunction of $\Delta_g$ with eigenvalue $\lambda > 0$, i.e., $\Delta_g \psi_\lambda = \lambda \psi_\lambda$. Let $\Omega_\lambda$ be a nodal domain of $\psi_\lambda$ and let $\mathrm{in}(\Omega_\lambda)$ be its inner radius.[1] Mangoubi [Ma08a, Ma08b] has shown that[2]

$$a_1 \lambda^{-\frac{1}{2} - \frac{(n-1)(n-2)}{4n}} \leq \mathrm{in}(\Omega_\lambda) \leq a_2 \lambda^{-\frac{1}{2}}, \tag{1.1}$$

where $a_1$ and $a_2$ depend only on $(X, g)$. In this note we show that

**Theorem 1.** *There exists $r_0(g) > 0$ such that if $\lambda^{-1/2} < r_0(g)$ and if for some $r \in [\lambda^{-1/2}, r_0(g)]$ and for all geodesic balls $\{B_r(x)\}_{x \in X}$ we have*

$$K_1 r^n \leq \int_{B_r(x)} |\psi_\lambda|^2 dv_g \leq K_2 r^n, \tag{1.2}$$

*for some positive constants $K_1$ and $K_2$ independent of $x$, then for $n \geq 3$,*

$$a_1 \lambda^{-\frac{1}{2}} (r^2 \lambda)^{-\frac{(n-1)(n-2)}{4n}} \leq \mathrm{in}(\Omega_\lambda)\,. \tag{1.3}$$

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[1] $\mathrm{in}(\Omega_\lambda)$ is the supremum of the radii of geodesic balls contained in $\Omega_\lambda$.

[2] In particular in dimension two, one has the optimal lower bound $a_1 \lambda^{-1/2}$.

A result of [HeRi16][3] shows that on negatively curved manifolds, (1.2) holds for a full density subsequence with $r = (\log \lambda)^{-\kappa}$ for any $\kappa \in (0, \frac{1}{2n})$. Hence the following unconditional result on such manifolds is quickly obtained.

**Corollary 1.1.** *Let $(X, g)$ be a boundaryless compact connected smooth Riemannian manifold of dimension $n \geq 3$, with negative sectional curvatures. Let $\{\psi_{\lambda_j}\}_{j\in\mathbb{N}}$ be any ONB of $L^2(X)$ consisting of real-valued eigenfunctions of $\Delta_g$ with eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$. Let $\varepsilon > 0$ be arbitrary. Then there exists $S_\varepsilon \subset \mathbb{N}$ of full density [4] such that for $j \in S_\varepsilon$,*

$$a_1 (\log \lambda_j)^{\frac{(n-1)(n-2)}{4n^2} - \varepsilon} \lambda_j^{-\frac{1}{2} - \frac{(n-1)(n-2)}{4n}} \leq \mathrm{in}(\Omega_\lambda),$$

*for some $a_1$ that only depends on $(X, g)$ and $\varepsilon$.*

One can also get the following improvements for quantum ergodic sequences of eigenfunctions. In fact it is enough to assume equidistribution on the configuration space $X$.

**Corollary 1.2.** *Let $(X, g)$ be a boundaryless compact connected smooth Riemannian manifold of dimension $n \geq 3$. Let $\{\psi_{\lambda_j}\}_{j\in S}$ be a sequence of real-valued $L^2$-normalized eigenfunctions of $\Delta_g$ with eigenvalues $\{\lambda_j\}_{j\in S}$ such that for all $r \in (0, R_0)$, for some fixed $R_0 > 0$, and for all $x \in X$,*

$$\int_{B_r(x)} |\psi_{\lambda_j}|^2 \to \frac{\mathrm{Vol}_g(B_r(x))}{\mathrm{Vol}_g(X)}, \qquad \lambda_j \xrightarrow{j\in S} \infty. \tag{1.4}$$

*Then along this sequence*

$$\lim_{j\to\infty} \lambda_j^{\frac{1}{2} + \frac{(n-1)(n-2)}{4n}} \mathrm{in}(\Omega_{\lambda_j}) = \infty.$$

In particular the above corollary holds for manifolds with ergodic geodesic flows by the quantum ergodicity theorem of [Sh74, CdV85, Ze87]. This means that given any ONB of eigenfunctions on such a manifold one can find a full density subsequence where (1.4), hence Corollary 1.2, holds.

We must also mention the work of Lester-Rudnick [LeRu16], where they proved that for a full density subsequence of toral eigenfunctions one has equidistribution at the shrinking rate $r = \lambda^{-\frac{1}{2n-2} + \epsilon}$. Of course, one can also use this and Theorem 1 to get improved lower bounds for such toral eigenfunctions.

**Corollary 1.3.** *For any ONB $\{\psi_{\lambda_j}\}_{j\in\mathbb{N}}$ of real-valued eigenfunctions of $\Delta$ on the flat torus $\mathbb{T}^{n\geq 3}$ and any $\varepsilon > 0$, there exists a full density subset $S_\varepsilon \subset \mathbb{N}$ such that for $j \in S_\varepsilon$,*

$$a_1 \lambda_j^{-\frac{1}{2} - \frac{(n-2)^2}{4n} - \varepsilon} \leq \mathrm{in}(\Omega_{\lambda_j}),$$

*where $a_1$ is a positive constant that depends only on $n$ and $\varepsilon$.*

*Remark* 1.4. In the real analytic case, a very recent result of Georgiev [Ge16] gives the lower bound

$$a\lambda^{-1} \leq \mathrm{in}(\Omega_\lambda),$$

which is better than Mangoubi's lower bounds for dimensions $n \geq 5$. The paper [Ge16] also uses the idea of this paper, namely, modified doubling estimates,

[3] In [Han15], this is proved for $\kappa \in (0, \frac{1}{3n})$.

[4] It means that $\lim_{N\to\infty} \frac{1}{N} \mathrm{card}(S \cap [1, N]) = 1$.

and gives logarithmic improvements on the above lower bound for a full density subsequence of eigenfunctions on negatively curved manifolds.

1.1. **Proofs of Theorem 1.** The main idea is to use the modified growth estimates of our recent preprint [He16]. We recall that Donnelly and Fefferman (see Theorem 4.2.ii of [DoFe88]) showed that an eigenfunction $\psi_\lambda$ of $\Delta_g$ with eigenvalue $\lambda$ satisfies

$$N(B_s(x)) := \log\left(\frac{\sup_{B_{2s}(x)}|\psi_\lambda|^2}{\sup_{B_s(x)}|\psi_\lambda|^2}\right) \leq c\sqrt{\lambda},$$

for all $s < s_0$ where $s_0$ and $c$ depend only on $(X, g)$. In [He16], we have shown that (see Lemma 2.1 of [He16]):

**Lemma 1.5.** *Under the assumption*

$$K_1 r^n \leq \int_{B_r(x)} |\psi_j|^2 \leq K_2 r^n,$$

*we have*

$$N(B_s(x)) \leq c\, r\sqrt{\lambda}, \qquad \text{for all } s \leq r, \tag{1.5}$$

*where $c$ is positive and is uniform in $x$, $r$, $s$, and $\lambda$, but depends on $K_1$, $K_2$, and $(X, g)$.*

*Remark* 1.6. For the sake of completeness and independence, we present a short proof of this lemma in the next section. We also give a proof without the upper bound assumption $\int_{B_r(x)} |\psi_j|^2 \leq K_2 r^n$ however under the additional mild condition $r \geq (\log\lambda)\lambda^{-\frac{1}{2}}$, $\lambda \geq 2$.

We apply the growth estimates in the above lemma to the proof of [Ma08b]. In [Ma08b], it is first proved (see Theorem 4.4 and inequality 5.1 of [Ma08b]) that there exists $\varepsilon_0 \in (0, 1)$, sufficiently small and only dependent on $(X, g)$, such that for all embedded geodesic balls $B_R(p)$ with $\{\psi_\lambda = 0\} \cap B_{R/2}(p) \neq \emptyset$ we have

$$\frac{\mathrm{Vol}\big(\{\psi_\lambda \geq 0\} \cap B_R(p)\big)}{\mathrm{Vol}(B_R(p))} \geq \frac{a_3}{\beta(\lambda)^{n-1}}, \tag{1.6}$$

where

$$\beta(\lambda) = \sup_{x\in X, s\leq \varepsilon_0 \lambda^{-\frac{1}{2}}} N(B_s(x))$$

and $a_3 > 0$ depends only on $(X, g)$. One can think of (1.6) as a local asymmetry property of the nodal domain $\Omega_\lambda$. Then it is shown that (see Theorem 6.2 of [Ma08b]) for $n \geq 3$ one has[5]

$$\mathrm{in}(\Omega_\lambda) \geq a_4 \lambda^{-\frac{1}{2}} \inf_{B\in\mathcal{B}_\lambda} \left(\frac{\mathrm{Vol}\big(\{\psi_\lambda \geq 0\} \cap B_R(p)\big)}{\mathrm{Vol}(B_R(p))}\right)^{\frac{n-2}{2n}},$$

where $a_4 > 0$ depends only on $(X, g)$ and the infimum in taken over the set $\mathcal{B}_\lambda$ of balls $B_R(p)$ such that $\{\psi_\lambda = 0\} \cap B_{R/2}(p) \neq \emptyset$. Combining the last two inequalities, one obtains

$$\mathrm{in}(\Omega_\lambda) \geq a_5 \lambda^{-\frac{1}{2}} \beta(\lambda)^{-\frac{(n-1)(n-2)}{2n}}.$$

Since $s \leq \varepsilon_0 \lambda^{-1/2} \leq r$, we can use our improved doubling estimates (1.5) to get $\beta(\lambda) \leq cr\sqrt{\lambda}$, which implies the theorem immediately.

[5] The condition $n \geq 3$ seems to be essential in this part of the argument of [Ma08b].

1.2. **Proof of Corollary 1.2.** This follows quickly from the following lemma (see Lemma 2.9 of [He16]) together with Theorem 1.

**Lemma 1.7.** *Let $\{\psi_j\}_{j\in S}$ be a sequence of eigenfunctions of $\Delta_g$ with eigenvalues $\{\lambda_j\}_{j\in S}$ such that for some $R_0 > 0$, all $r \in (0, R_0)$, and all $x \in X$,*

$$\int_{B_r(x)} |\psi_j|^2 \to \frac{\mathrm{Vol}_g(B_r(x))}{\mathrm{Vol}_g(X)}, \qquad \lambda_j \xrightarrow{j\in S} \infty. \tag{1.7}$$

*Then there exists $r_0(g)$ such that for each $r \in (0, r_0(g))$ there exists $\Lambda_r$ such that for $\lambda_j \geq \Lambda_r$ we have*

$$K_1 r^n \leq \int_{B_r(x)} |\psi_j|^2 \leq K_2 r^n,$$

*uniformly for all $x \in X$. Here, $K_1$ and $K_2$ are independent of $r$, $j$, and $x$.*

We note that this lemma is obvious when $x$ is fixed, but to show that it holds uniformly in $x$ one can use a covering argument as presented in the proof of Lemma 2.9 of [He16].

*Remark* 1.8. It is clear from Theorem 1 that if (1.2) holds for $r = \lambda^{-\frac{1}{2}+\varepsilon}$ for arbitrary $\varepsilon > 0$, then we have

$$a_1(\varepsilon)\lambda^{-\frac{1}{2}-\varepsilon} \leq \mathrm{in}(\Omega_\lambda) \leq a_2\lambda^{-\frac{1}{2}},$$

for any $\varepsilon > 0$. It is a natural but seemingly very difficult to prove conjecture that $r = \lambda^{-\frac{1}{2}+\varepsilon}$ is the optimal rate for the eigenfunctions of negatively curved manifolds. A result of [LeRu16] shows that this optimal rate of shrinking is satisfied for a full density subsequence of any ONB of eigenfunctions of the flat 2-torus $\mathbb{T}^2$.

*Remark* 1.9. We have used both local and global harmonic analysis of eigenfunctions. The local analysis is contained in the work of Mangoubi, where lower bounds for the inner radius in terms of the doubling index are given. The global analysis is contained in the small scale QE results of [Han15, HeRi16] and also the QE results of [Sh74, CdV85, Ze87], where the global behavior of the geodesic flow and the wave operator are considered to obtain equidistribution on small balls. Hence, any non-optimal results that give improvements on the results of [Ma08b] and are only based on the local analysis of eigenfunctions (for example via doubling methods) can be improved using our hybrid method.

*Remark* 1.10. The upper bound in (1.1) is an immediate consequence of a result of Brüning [Br78], which says that there exists $a_2 > 0$ dependent only on $(X, g)$ such that every geodesic ball of radius $a_2\lambda^{-1/2}$ contains a zero of $\psi_\lambda$. See [Ze08] for a simple proof using the domain monotonicity property of Dirichlet eigenvalues.

*Remark* 1.11. The paper [GeMa16] gives a refinement of the result of Mangoubi using Brownian motion techniques. To be precise, the authors show that the inscribed ball that gives the lower bound of [Ma08b] can be centered at the maximum of the eigenfunctions $\psi_\lambda$ on its nodal domain $\Omega_\lambda$. It would be interesting to see whether our improved lower bounds also hold when the ball is centered at a maximum.

## 2. Proof of Lemma 1.5

Here we will prove Lemma 1.5 and we will also prove a new version of it without assuming the upper bound $\int_{B_r(x)} |\psi_j|^2 \leq K_2 r^n$, but under the additional mild assumption that $r \geq (\log \lambda)\lambda^{-\frac{1}{2}}$. Our proof relies on a refined doubling estimate of Mangoubi [Ma12] (see Theorem 3.2), which asserts that

**Theorem 2.** *Let $(X, g)$ and $\psi_\lambda$ be as in the introduction. Denote $S = \sup_X |\mathrm{Sec}(g)|$. Then for all $s \leq t \leq CS^{-1/2}$ and all $x \in X$ we have*

$$\sup_{B_{3s}(x)} |\psi_\lambda|^2 \leq c_0 e^{c_1 t\sqrt{\lambda}} \left( \frac{\sup_{B_{3t}(x)} |\psi_\lambda|^2}{\sup_{B_t(x)} |\psi_\lambda|^2} \right)^{1+c_2 s^2 S} \sup_{B_{2s}(x)} |\psi_\lambda|^2,$$

*where $C$, $c_1$, and $c_2$ are positive constants which depend only on the injectivity radius of $(X, g)$, and $c_0$ depends on bounds on $(g^{-1})_{ij}$, its derivatives, and its ellipticity constant on $(X, g)$.*

Using this theorem twice, we get for $s \leq t \leq CS^{-1/2}$:

$$\frac{\sup_{B_{2s}(x)} |\psi_\lambda|^2}{\sup_{B_s(x)} |\psi_\lambda|^2} \leq \frac{\sup_{B_{\frac{9}{4}s}(x)} |\psi_\lambda|^2}{\sup_{B_{\frac{3}{2}s}(x)} |\psi_\lambda|^2} \frac{\sup_{B_{\frac{3}{2}s}(x)} |\psi_\lambda|^2}{\sup_{B_s(x)} |\psi_\lambda|^2} \tag{2.1}$$

$$\leq c_0^2 e^{2c_1 t\sqrt{\lambda}} \left( \frac{\sup_{B_{3t}(x)} |\psi_\lambda|^2}{\sup_{B_t(x)} |\psi_\lambda|^2} \right)^{2+c_2' s^2 S}, \tag{2.2}$$

for a new constant $c_2'$. Now we put $t = r$ into this inequality. To estimate

$$\frac{\sup_{B_{3r}(x)} |\psi_\lambda|^2}{\sup_{B_r(x)} |\psi_\lambda|^2},$$

we observe that since by assumption $\int_{B_r(x)} |\psi_\lambda|^2 \geq K_1 r^n$, we must have

$$\sup_{B_r(x)} |\psi_\lambda|^2 \geq \frac{r^n}{\mathrm{Vol}(B_r(x))} K_1.$$

By making $r$ sufficiently small (only dependent on $(X, g)$), we obtain that

$$\sup_{B_r(x)} |\psi_\lambda|^2 \geq a K_1,$$

for some constant $a$ which is uniform in $x$, $r$, and $\lambda$. We also note that by the standard supnorm estimates of $L^2$ normalized eigenfunctions,

$$\sup_{B_{3r}(x)} |\psi_\lambda|^2 \leq \sup_X |\psi_\lambda|^2 \leq c_3 \lambda^{\frac{n-1}{2}}. \tag{2.3}$$

Applying these estimates to (2.2) gives us

$$\frac{\sup_{B_{2s}(x)} |\psi_\lambda|^2}{\sup_{B_s(x)} |\psi_\lambda|^2} \leq b_0 e^{2c_1 r\sqrt{\lambda}} \lambda^{b_1},$$

for some new constants $b_0$ and $b_1$ independent of $r$, $x$, and $\lambda$. Note that if we assume $r \geq (\log \lambda)\lambda^{-\frac{1}{2}}$ and $\lambda \geq 2$, by choosing $b_2$ sufficiently large we have

$$e^{2c_1 r\sqrt{\lambda}} \lambda^{b_1} \leq e^{b_2 r\sqrt{\lambda}}.$$

In the above proof we did not use the upper bound assumption $\int_{B_r(x)} |\psi_j|^2 \leq K_2 r^n$ at the price of assuming $r \geq (\log \lambda)\lambda^{-\frac{1}{2}}$. To prove Lemma 1.5 in the full generality $r \geq \lambda^{-\frac{1}{2}}$, we will need the upper bound on the local $L^2$ norms. All we have to change in the above proof is to replace the estimate (2.3), using a result of Sogge [So16] (see estimate (3.3) on page 391), with

$$\sup_X |\psi_\lambda|^2 \leq c_4 r^{-1} \lambda^{\frac{n-1}{2}} \sup_{x \in X} \int_{B_r(x)} |\psi_\lambda|^2 \leq c_4 K_2 r^{n-1} \lambda^{\frac{n-1}{2}} = c_4 K_2 (r\sqrt{\lambda})^{n-1}.$$

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