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UNIVERSAL MEASURE FOR PONCELET-TYPE THEOREMS

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Abstract. We give a simple proof of the Emch closing theorem by introducing a new invariant measure on the circle. Special cases of that measure are well known and have been used in the literature to prove Poncelet's and the zigzag theorems. Some further generalizations are also obtained by applying the new measure.

1. Introduction

Invariant measures on circles and conics provide powerful tools in the study of closing theorems such as Poncelet's and Steiner's porisms, the zigzag theorem, etc. We present an elementary formula for a universal measure that generalizes several well-known invariant measures. We show that every pair of circles generates a function on the plane, whose restriction to an arbitrary circle defines an invariant measure on it. Remarkable properties of that measure give new results as well as new proofs of known facts.

The *Poncelet closing theorem* discovered in 1813 and published in 1822 [16] states that if for two circles (or quadrics) α and δ, there is an n-sided polygon $\boldsymbol{x}_1 \dots \boldsymbol{x}_n$ inscribed in δ and circumscribed around α (i.e., the straight lines containing its sides are tangent to α), then there exist infinitely many such polygons whose vertices $\boldsymbol{x}_1$ can be chosen on δ arbitrarily, provided $\boldsymbol{x}_1 \notin \alpha$.

There are several methods to prove Poncelet's theorem. They are all nontrivial and based on various ideas [3, 4, 8, 10, 11]. The invariant measure approach originated with Jacobi in 1828, then was improved by Bertrand, and was developed further in [1, 2, 13, 19], etc., giving an elegant and natural proof. Consider, for example, the case when the circle α lies inside δ. Suppose there is a measure $m(\cdot)$ on δ such that all oriented arcs $\breve{\boldsymbol{xy}} \subset \delta$ whose chords touch the circle α have the same value $m(\breve{\boldsymbol{xy}}) = \tilde{m}$. Then the Poncelet n-gon exists if and only if the number $n\,\tilde{m}$ is an integer multiple of $m(\delta)$. Since this property does not depend on the location of the first vertex of the polygon, the Poncelet theorem for two circles follows.

For arbitrary circles α and δ, a measure on δ is called invariant if its density $\rho = m'$ satisfies the equality $\rho(\boldsymbol{x})|d\boldsymbol{x}| = \rho(\boldsymbol{y})|d\boldsymbol{y}|$, where $d\boldsymbol{x}, d\boldsymbol{y}$ are oriented lengths of small arcs after perturbation of an arbitrary chord $\boldsymbol{xy}$ touching α. If a function $\rho : \delta \to \mathbb{R}_+$ possesses this property, then $m(\breve{\boldsymbol{xy}}) = \int_{\boldsymbol{x}}^{\boldsymbol{y}} \rho(\boldsymbol{s})d\boldsymbol{s}$ is an invariant measure

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(the integration is over the arc $\breve{\boldsymbol{xy}}$). For arbitrary circles α and δ, such a measure is readily available by the formula $\rho(\boldsymbol{x}) = 1/\sqrt{|f(\boldsymbol{x})|}$, where $f(\boldsymbol{x}) = |\boldsymbol{x} - \boldsymbol{c}|^2 - r^2$ is the power with respect to the circle α of radius r centered at $\boldsymbol{c} \in \mathbb{R}^2$. This is the *Jacobi-Bertrand measure.* Moreover, as was observed by Khovansky (see [1] for an overview), if we consider an arbitrary quadratic polynomial $f(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^2$, then the same formula also defines an invariant measure, which proves Poncelet's theorem for the circle δ and the quadric $\alpha = \{\boldsymbol{x} \in \mathbb{R}^2 \mid f(\boldsymbol{x}) = 0\}$. By a suitable projective transform, this leads to the general case of two quadrics.

In 1974 Black, Howland, and Howland [5] found an invariant measure for another well-known closing theorem:

Zigzag Theorem. *If for given circles α, δ and for a number $l > 0$, there is a polygon with $2n$ sides all of length l, with odd vertices (i.e., vertices $\boldsymbol{x}_k$ with odd k) on δ and even vertices on α, then there exist infinitely many such polygons, whose vertex $\boldsymbol{x}_1$ can be chosen on δ arbitrarily, provided the distance from $\boldsymbol{x}_1$ to α is smaller than l.*

Thus, if a grasshopper jumps from one circle to the other making a closed walk after $2n$ jumps, then his walk from any point of the first circle closes after $2n$ steps, provided he can make the first jump. This theorem was established by Emch in 1901 [4], then rediscovered by Bottema in 1965 [4] and in 1974 in [5]. It holds for two circles in the space as well, but we consider only the plane version.

We mention also the third popular closing theorem, the *Steiner theorem.* Given two circles α_0, α_1, one inside the other. Circles $\{\omega_k\}_{k\in\mathbb{N}}$ inscribed in the annulus between α_0 and α_1 touch each other in succession (ω_k and ω_{k+2} are different and both tangent to ω_{k+1} , $k \in \mathbb{N}$). If this series closes after n steps, i.e., $\omega_{n+1} = \omega_1$, then it does for an arbitrary initial circle ω_1.

Those three closing theorems are actually special cases of the Emch theorem on circular series [9]. To formulate it we need to introduce some notation. The tangency of two circles is called interior if one of the circles lies inside the other. Suppose α_0, α_1 are circles on the plane. Then for an arbitrary circle ω touching both α_0 and α_1 the *index of tangency* is 0 if there is an even number of interior tangencies among the two ones: ω with α_0 and ω with α_1. If this number is odd, then the index is 1. For $i = 0, 1$, let $\mathcal{M}_i$ denote the family of circles touching α_0, α_1 with index i.

Let α_0, α_1, and δ be an arbitrary triple of circles on the plane. Choose some $i \in \{0, 1\}$ and take the family $\mathcal{M}_i$ of circles touching α_0, α_1 with index i. We assume $\delta \notin \mathcal{M}_i$. Take arbitrary circle $\omega_1 \in \mathcal{M}_i$ that intersects δ at two points $\boldsymbol{x}_1, \boldsymbol{x}_2$. The family $\mathcal{M}_i$ contains two circles passing through $\boldsymbol{x}_2$: one of them is ω_1; take the other and denote it ω_2. The circles ω_2 and δ have two points of intersection: one of them is $\boldsymbol{x}_2$; take the other and denote it $\boldsymbol{x}_3$, etc. This way we obtain a *circular series* $\{\omega_k\}_{k\in\mathbb{N}}$. Each ω_k touches α_0 and α_1 with index i and meets the circle δ at points $\boldsymbol{x}_k$ and $\boldsymbol{x}_{k+1}$. This series closes after n steps if $\omega_{n+1} = \omega_1$.

Theorem of Emch [9]**.** *Let α_0, α_1, and δ be arbitrary circles, $i \in \{0, 1\}$, and $\delta \notin \mathcal{M}_i$. If for some initial circle $\omega_1 \in \mathcal{M}_i$, the circular series closes after n steps, then it does for arbitrary $\omega_1 \in \mathcal{M}_i$.*

This theorem was formulated by Emch in 1901 [9], but he gave a proof only for nonintersecting circles α_0 and α_1. See Figure 1. Probably, the author missed this aspect. It was not before 1996 that the proof for the general position of circles

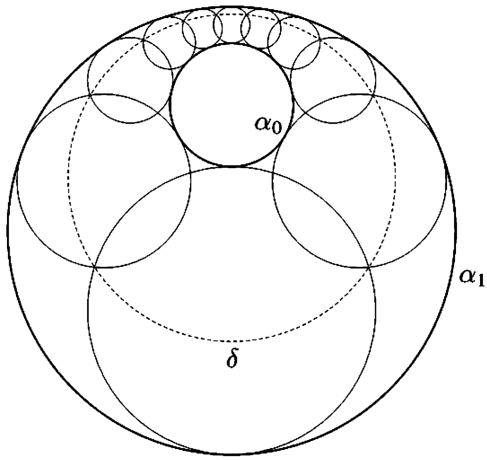

FIGURE 1. Theorem of Emch.

$\alpha_0, \alpha_1, \delta$ was given in [3]. In [17] the Emch theorem was deduced from Poncelet's theorem for quadrics; in [18] an elementary geometrical proof was found.

All three famous closing theorems follow directly from the Emch theorem. If the radius of α_1 tends to infinity, then we obtain in the limit the Poncelet theorem for circles α_0 and δ. If α_0 and α_1 are concentric circles, then we obtain the zigzag theorem. Finally, if δ is a locus of points of tangency for pairs of circles from $\mathcal{M}_i$, then we come to the Steiner theorem. See [17] for more details.

A natural question arises if the Emch theorem admits a proof by an invariant measure. We show that such a measure exists and, moreover, is explicitly given by a simple formula. For an arbitrary pair of circles α_0 and α_1, we consider the function $\rho(\boldsymbol{x}) = \frac{1}{\sqrt{|f_0(\boldsymbol{x}) f_1(\boldsymbol{x})|}}$ on the plane $\mathbb{R}^2$, where f_j is a power w.r.t. the circle α_j, $j = 0, 1$. In Theorem 1 we show that this function defines an invariant measure on any circle $\delta \subset \mathbb{R}^2$. This gives a geometric proof for the Emch theorem. Both the Jacobi-Bertrand measure and the Black–Howland measure are special cases of this measure $\rho(\cdot)$. Therefore, it can be considered as a universal measure for Poncelet-type theorems. Simple algebraic manipulations with the formula for $\rho(\boldsymbol{x})$ give generalizations of Emch's theorem to pencils of circles (Section 4), to a cyclic instead of two circles (Section 5), and prove the equivalence of Emch's theorem with Poncelet's theorem for quadrics (Section 6).

In the next section we formulate Theorem 1 and observe its special cases for the Poncelet, zigzag, and Steiner theorems. In Section 3 we give a geometrical proof of Theorem 1. For the sake of simplicity, in Sections 4-6 we deal with the case of *nested circles* $\alpha_0, \delta, \alpha_1$. This means that α_0 is inside δ, which is inside α_1, and all the circles ω_k are inscribed in the annulus between α_0 and α_1; i.e., they are from the family $\mathcal{M}_1$. Then, in Section 7, we prove Emch's theorem for the general position of circles. The proof remains short but becomes less obvious than for the nested circles.

In what follows, we denote points and vectors from $\mathbb{R}^2$ by bold letters, all distances are Euclidean, the distance between points $\boldsymbol{x}$ and $\boldsymbol{y}$ is denoted either as $\boldsymbol{xy}$ or as $|\boldsymbol{x} - \boldsymbol{y}|$. By $\boldsymbol{c}_i, r_i$, and $f_i(\boldsymbol{x}) = |\boldsymbol{x} - \boldsymbol{c}_i| - r_i^2$ we denote the center of the circle α_i, its radius, and the power w.r.t. α_i, respectively, $i = 0, 1$. For two different quadrics $\gamma_j = \{\boldsymbol{x} \in \mathbb{R}^2 \mid q_j(\boldsymbol{x}) = 0\}$, $j = 0, 1$, we denote by $\{\gamma_0, \gamma_1\}$ the pencil passing through them, which is the one-parametric family of quadrics $\gamma_t = \{\boldsymbol{x} \in \mathbb{R}^2 \mid (q_0 + tq_1)(\boldsymbol{x}) = 0\}$, $t \in \bar{\mathbb{R}}$, where $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. For a circular series $\{\omega_k\}_{k\in\mathbb{N}}$, we denote by $\boldsymbol{x}_k, \boldsymbol{x}_{k+1}$ the points of intersection of the circle ω_k with δ and by $\boldsymbol{t}_0^k, \boldsymbol{t}_1^k$ the points of its tangency with α_0 and α_1, respectively.

2. The main result

Let α_0, α_1, and δ be arbitrary circles. Consider a circle ω tangent to both α_0 and α_1 and intersecting δ at some points $\boldsymbol{x}, \boldsymbol{y}$. Let ω' be a circle close to ω and also touching α_0, α_1; let $\boldsymbol{x}', \boldsymbol{y}'$ be the corresponding points of intersection ($\boldsymbol{x}'$ is close to $\boldsymbol{x}$). The oriented lengths of small arcs $\breve{\boldsymbol{x}'\boldsymbol{x}}$ and $\breve{\boldsymbol{y}'\boldsymbol{y}}$ of the circle δ as $\omega' \to \omega$ are $d\boldsymbol{x}$ and $d\boldsymbol{y}$. Thus, if one slightly perturbs the circle ω, its points of intersection with the circle δ move to $d\boldsymbol{x}$ and $d\boldsymbol{y}$.

Definition 1. Given three circles $\alpha_0, \alpha_1, \delta$ and an index $i \in \{0, 1\}$. A Lebesgue measurable function $\rho : \delta \to \mathbb{R}_+$ defines an invariant measure if for almost all circles ω touching α_0, α_1 with index i we have

$$\rho(\boldsymbol{x})\,|d\boldsymbol{x}| \;=\; \rho(\boldsymbol{y})\,|d\boldsymbol{y}|\,, \tag{1}$$

where $\boldsymbol{x}, \boldsymbol{y}$ are points of intersection of the circles ω and δ.

For an arbitrary arc $\breve{\boldsymbol{x}\boldsymbol{y}} \subset \delta$ we denote by $m(\breve{\boldsymbol{x}\boldsymbol{y}}) = \int_{\boldsymbol{x}}^{\boldsymbol{y}} \rho(\boldsymbol{s})d\boldsymbol{s}$ its measure, or *mass*. In case of nested circles $\alpha_0, \delta, \alpha_1$, any slight perturbation of a circle ω moves the points $\boldsymbol{x}$ and $\boldsymbol{y}$ in the same direction. Hence, $d\boldsymbol{x}$ and $d\boldsymbol{y}$ always have the same sign, and equality (1) becomes $\rho(\boldsymbol{x})d\boldsymbol{x} = \rho(\boldsymbol{y})d\boldsymbol{y}$. Integrating, we obtain $m(\breve{\boldsymbol{x}\boldsymbol{y}}) \equiv \text{const}$. Thus, all circles $\omega \in \mathcal{M}_i$ cut arcs of the same mass $\tilde{m}$ from the circle δ. In particular, in Emch's theorem, $m(\boldsymbol{x}_k\breve{\boldsymbol{x}}_{k+1}) = \tilde{m}$ for all k. Hence, the circular series closes after n steps if and only if $n\,\tilde{m}$ is an integer multiple of $m(\delta)$. This proves Emch's theorem in case of nested circles. The general case is more delicate, and we consider it in Section 7.

Theorem 1. *The function* $\rho(\boldsymbol{x}) = \frac{1}{\sqrt{|f_0(\boldsymbol{x})f_1(\boldsymbol{x})|}}$ *defines an invariant measure on any circle* δ.

Note that the function $\rho(\boldsymbol{x})$ is defined on the whole plane (including the circles α_0, α_1, where it equals $+\infty$) and does not depend on the circle δ. The restriction of this function to any circle defines an invariant measure on it. Before we prove Theorem 1 we observe some of its special cases.

1. **The circle α_1 is infinitely big: Poncelet's theorem.** If we increase the radius of α_1 leaving its center and all other circles unmoved, then $f_1(\boldsymbol{x})/r_1^2 \to -1$ uniformly on any compact subset of $\mathbb{R}^2$ as $r_1 \to \infty$. Hence, on the circle δ, the function f_1 becomes equivalent to an identical constant. Consequently, the function $\rho(\cdot)$ becomes proportional to $1/\sqrt{|f_0(\cdot)|}$, which is the Jacobi-Bertrand measure. On the other hand, all the circles ω_k also enlarge as $r_1 \to \infty$, and their arcs touching α_0 become close to line segments. Therefore, in the limit as $r_1 \to \infty$, the Emch theorem becomes the Poncelet theorem (for circles) and the measure ρ becomes the Jacobi-Bertrand measure. Hence, the invariance property of the Jacobi-Bertrand measure follows from Theorem 1.

2. **The circles α_0, α_1, and δ belong to one pencil: Steiner's theorem.** If the circle δ belongs to the pencil $\{\alpha_0, \alpha_1\}$, then the functions f_0 and f_1 are proportional on δ: $f_1(\boldsymbol{x}) = -cf_0(\boldsymbol{x}),\ \boldsymbol{x} \in \delta$. Hence $\rho(\boldsymbol{x}) = \frac{1}{c\, f_0(\boldsymbol{x})}$. So, in this case the reciprocal of the power w.r.t. the circle α_0 is an invariant measure on the circle δ. If δ is the locus of points of tangency of two circles both touching α_0 and α_1, we obtain the Steiner theorem.

3. **The circles α_0 and α_1 are concentric: The zigzag theorem.** If α_0 and α_1 are concentric, then the Emch theorem becomes the zigzag theorem for the circles δ and α (the circle α is of radius $r = \frac{r_0+r_1}{2}$ and is concentric to α_0, α_1) and for the jump length $l = \frac{|r_1-r_0|}{2}$. The measure $\rho(\cdot)$ on the circle δ becomes the Black–Howland measure $b(\cdot)$ for the zigzag theorem [5]. It is defined as $b(\boldsymbol{x}) = 1/|(\boldsymbol{x} - \boldsymbol{c}_0) \times (\boldsymbol{x} - \boldsymbol{z})|$, where $\times$ denotes the operation of cross (vector) product, and $\boldsymbol{x} \in \delta$ and $\boldsymbol{z} \in \alpha$ is such that $|\boldsymbol{x} - \boldsymbol{z}| = l$. In other terms, $1/b(\boldsymbol{x})$ is the double area of a triangle with the sidelengths $x = |\boldsymbol{x} - \boldsymbol{c}_0|$, $\frac{r_1+r_0}{2}$, and $\frac{|r_1-r_0|}{2}$ (Figure 2).

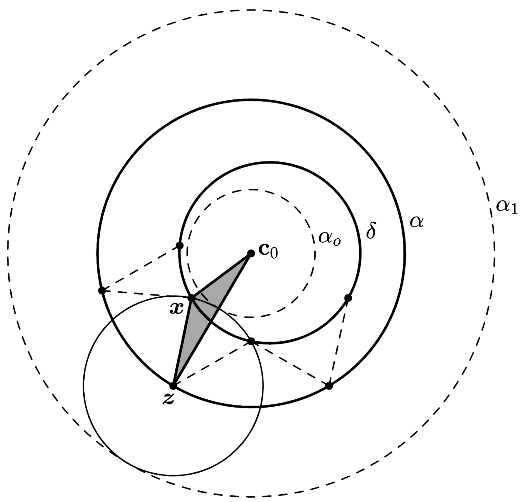

FIGURE 2. The invariant measure for the zigzag theorem.

The Heron formula yields

$$\frac{1}{b(\boldsymbol{x})} = \frac{1}{2}\sqrt{(r_1+x)(r_1-x)(x+r_0)(x-r_0)} = \frac{1}{2}\sqrt{(r_1^2-x^2)(x^2-r_0^2)}$$
$$= \frac{1}{2}\sqrt{-f_1(\boldsymbol{x})\cdot f_0(\boldsymbol{x})}\,.$$

Hence $b(\boldsymbol{x}) = 2\rho(\boldsymbol{x})$ for all $\boldsymbol{x} \in \delta$. Thus, the Black–Howland measure is the special case of $\rho(\cdot)$, when the circles α_0 and α_1 are concentric.

3. Proof of the main theorem

Let a circle ω touch both α_0 and α_1. We consider the line connecting the two tangent points and denote by $h(\boldsymbol{x})$ the distance from a point $\boldsymbol{x}$ to that line. We are going to show that the function ρ on the circle ω is proportional to $1/h$.

Proposition 1. *Suppose ω is an arbitrary circle touching α_0 and α_1 at points $\boldsymbol{t}_0$ and $\boldsymbol{t}_1$, respectively. Then the restriction of the function $\rho(\boldsymbol{x}) = 1/\sqrt{|f_0(\boldsymbol{x})f_1(\boldsymbol{x})|}$ to ω is proportional to the reciprocal of the distance to the line $\boldsymbol{t}_0\boldsymbol{t}_1$. Thus, $\rho(\boldsymbol{x}) \sim 1/h(\boldsymbol{x}),\ \boldsymbol{x} \in \omega$.*

Proof. Let the line $\boldsymbol{x}\boldsymbol{t}_0$ meet the circle α_0 for the second time at point $\boldsymbol{z}_0$. Note that $|f_0(\boldsymbol{x})| = \boldsymbol{x}\boldsymbol{t}_0 \cdot \boldsymbol{x}\boldsymbol{z}_0 = c \cdot \boldsymbol{x}\boldsymbol{t}_0^2$, where c is a constant. Indeed, since the circles α_0 and ω are homothetic with respect to the point of tangency $\boldsymbol{t}_0$, the ratio $\boldsymbol{z}_0\boldsymbol{t}_0/\boldsymbol{x}\boldsymbol{t}_0$ is constant, and hence so is the ratio $\boldsymbol{x}\boldsymbol{z}_0/\boldsymbol{x}\boldsymbol{t}_0$. Similarly, $|f_1(\boldsymbol{x})|$ is proportional to $(\boldsymbol{x}\boldsymbol{t}_1)^2$. Thus, $\sqrt{|f_0(\boldsymbol{x}) \cdot f_1(\boldsymbol{x})|} \sim \boldsymbol{x}\boldsymbol{t}_0 \cdot \boldsymbol{x}\boldsymbol{t}_1$, which is proportional to the area of the triangle $\triangle\, \boldsymbol{t}_0\boldsymbol{x}\boldsymbol{t}_1$ (because $\sin(\angle\, \boldsymbol{t}_0\boldsymbol{x}\boldsymbol{t}_1)$ is constant), which is, in turn, proportional to its altitude $h(\boldsymbol{x})$, since this triangle has a constant base $\boldsymbol{t}_0\boldsymbol{t}_1$. □

A different proof of Proposition 1 based on properties of pencils of quadrics is given in Section 5, where we prove a generalization of Emch's theorem.

Proposition 2. *Suppose a circle ω passes through points $\boldsymbol{k}$ and $\boldsymbol{l}$ and meets a circle δ at points $\boldsymbol{x}$ and $\boldsymbol{y}$. Then a small perturbation of ω that passes through $\boldsymbol{k}$ and $\boldsymbol{l}$ satisfies $\frac{|d\boldsymbol{y}|}{|d\boldsymbol{x}|} = \frac{q(\boldsymbol{y})}{q(\boldsymbol{x})}$, where $q(\cdot)$ is the distance to the line $\boldsymbol{k}\boldsymbol{l}$.*

Proof. Since three pairwise chords of three circles concur, the lines $\boldsymbol{x}\boldsymbol{y}$ and $\boldsymbol{x}'\boldsymbol{y}'$ meet on the line $\boldsymbol{k}\boldsymbol{l}$ at some point $\boldsymbol{n}$. Equalities $\boldsymbol{x}'\boldsymbol{n} \cdot \boldsymbol{y}'\boldsymbol{n} = \boldsymbol{k}\boldsymbol{n} \cdot \boldsymbol{l}\boldsymbol{n} = \boldsymbol{x}\boldsymbol{n} \cdot \boldsymbol{y}\boldsymbol{n}$ imply similarity of triangles $\triangle\, \boldsymbol{x}\boldsymbol{n}\boldsymbol{x}' \sim \triangle\, \boldsymbol{y}'\boldsymbol{n}\boldsymbol{y}$, which yields $\frac{\boldsymbol{y}\boldsymbol{y}'}{\boldsymbol{x}\boldsymbol{x}'} = \frac{\boldsymbol{n}\boldsymbol{y}'}{\boldsymbol{n}\boldsymbol{x}}$. Replacing $\boldsymbol{n}\boldsymbol{y}'$ by a close value $\boldsymbol{n}\boldsymbol{y}$ and $\frac{\boldsymbol{n}\boldsymbol{y}}{\boldsymbol{n}\boldsymbol{x}}$ by $\frac{q(\boldsymbol{y})}{q(\boldsymbol{x})}$, we conclude the proof. See Figure 3. □

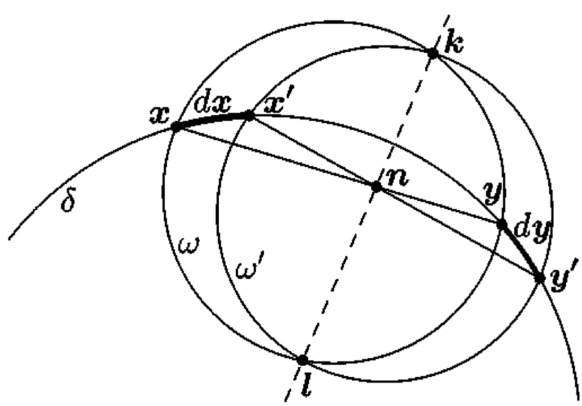

FIGURE 3. Proof of Proposition 2.

Proof of Theorem 1. Let ω be a circle touching α_0, α_1 and intersecting δ at points $\boldsymbol{x}, \boldsymbol{y}$; let ω' be a small perturbation of ω. Replacing in the equality (1) the function $\rho(\cdot)$ by $1/h(\cdot)$ (Proposition 1) and $h(\cdot)$ by a close value $q(\cdot)$, which is the distance to the common chord of the circles ω and ω' (this chord tends to the line $\boldsymbol{t}_0\boldsymbol{t}_1$, hence $q/h \to 1$ as $\omega' \to \omega$), we come to an equivalent assertion $\frac{|d\boldsymbol{x}|}{q(\boldsymbol{x})} = \frac{|d\boldsymbol{y}|}{q(\boldsymbol{y})}$, which follows from Proposition 2. □

4. Generalizations to pencils of circles

For the sake of simplicity, in Sections 4-6 we consider the case of nested circles $\alpha_0, \delta, \alpha_1$. The general case is analyzed in Section 7.

The measure ρ provides a simple way to generalize Emch's theorem from one pair of circles (α_0, α_1) to an arbitrary sequence of pairs $(\alpha_0^{(k)}, \alpha_1^{(k)})_{k\in\mathbb{N}}$, where each $\alpha_i^{(k)}$ is taken from a given pencil of circles $\mathcal{A}_i$. Such an extension for Poncelet's theorem is known; it was proved by Poncelet himself [16], then developed by Lebesgue [14]; see also [4]. A similar extension for Emch's theorem originated in [18]. Let $\mathcal{A}_0, \mathcal{A}_1$ be arbitrary pencils of circles both containing the circle δ. Take arbitrary sequences $\{\alpha_0^{(k)}\}_{k\in\mathbb{N}} \subset \mathcal{A}_0$ and $\{\alpha_1^{(k)}\}_{k\in\mathbb{N}} \subset \mathcal{A}_1$.

Proposition 3. *All the pairs $(\alpha_0^{(k)}, \alpha_1^{(k)})$, $k \in \mathbb{N}$, generate invariant measures on the circle δ that are proportional to one measure ρ.*

Proof. Let $f_i^{(k)}$ denote the power w.r.t. the circle $\alpha_i^{(k)}$, $i = 0, 1$. Since this circle belongs to the pencil $\{\delta, \alpha_i^{(1)}\}$, it follows that $f_i^{(k)} = (1 - t_{i,k})f_\delta + t_{i,k} f_i^{(1)}$, for some $t_{i,k} \in \bar{\mathbb{R}}$. For all $\boldsymbol{x} \in \delta$, we have $f_\delta(\boldsymbol{x}) = 0$, and hence $f_0^{(k)}(\boldsymbol{x}) f_1^{(k)}(\boldsymbol{x}) = t_{0,k}t_{1,k} f_0^{(1)}(\boldsymbol{x}) f_1^{(1)}(\boldsymbol{x})$; i.e., the measures generated by the kth pair and by the first pair are proportional on δ. □

Thus, for given pencils $\mathcal{A}_0, \mathcal{A}_1$ containing a circle δ, every pair $(\alpha_0, \alpha_1) \in \mathcal{A}_0 \times \mathcal{A}_1$ generates an invariant measure, and all those measures are proportional on δ. Hence, the following generalized Emch's theorem holds. Let $(\alpha_0^{(k)}, \alpha_1^{(k)}) \in \mathcal{A}_0 \times \mathcal{A}_1$, $k \in \mathbb{N}$, be an arbitrary sequence of pairs. Consider a circular series $\{\omega_k\}_{k\in\mathbb{N}}$, where ω_k touches the kth pair (Figure 4).

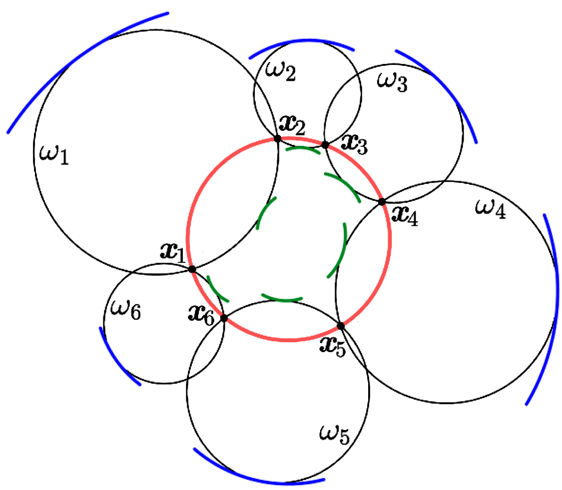

Figure 4. Emch theorem for pencils of circles.

If for some initial circle ω_1, we have $\omega_{n+1} = \omega_1$, then it holds for arbitrary ω_1 touching the first pair. Moreover, after an arbitrary change of order of the pairs $(\alpha_0^{(1)}, \alpha_1^{(1)}), \ldots, (\alpha_0^{(n)}, \alpha_1^{(n)})$, this series still closes after n steps. See [18] for the precise formulation. The proof is literally the same as for the Emch theorem.

The closing after n steps takes place if and only if the sum of masses of n arcs $\overset{\smile}{\boldsymbol{x}_1\boldsymbol{x}_2}, \dots, \overset{\smile}{\boldsymbol{x}_n\boldsymbol{x}_{n+1}}$ of the circle δ cut by the circles ω_k is equal to a multiple of the total mass of δ. This equality depends neither on the location of the initial circle ω_1 (due to the invariance of the measure) nor on the ordering of the circles (due to commutativity of summation).

Several corollaries can be drawn from Proposition 3 even if the circular series does not close. They are based on the following simple observation.

Proposition 4. *Under the assumptions of Emch's theorem, for every $\tilde{m} > 0$, the following holds: all circles ω that cut δ arcs of the same mass $\tilde{m}$ (generated by the measure $\rho(\boldsymbol{x}) = 1/\sqrt{|f_0(\boldsymbol{x})f_1(\boldsymbol{x})|}$) and touch α_0 with a given index touch a fixed circle from the pencil $\mathcal{A}_1 = \{\delta, \alpha_1\}$.*

Proof. For an arbitrary circle ω, the pencil $\mathcal{A}_1$ contains a unique circle α_1' touching ω with a given index. By Proposition 3, the measure ρ is invariant for the pair (α_0, α_1'); hence all circles ω' touching this pair with a given index cut the same mass m on δ. □

Corollary 1. *Let us have two circular series $\{\omega_k\}_{k\in\mathbb{N}}$ and $\{\omega_k'\}_{k\in\mathbb{N}}$ touching circles α_0, α_1 with the same index and having the same direction. Let ω_k and ω_k' intersect the circle δ at points $\boldsymbol{x}_k, \boldsymbol{x}_{k+1}$ and $\boldsymbol{x}_k', \boldsymbol{x}_{k+1}'$, respectively. Denote by γ_k the circle passing through the points $\boldsymbol{x}_k$ and $\boldsymbol{x}_k'$ and touching α_0 with the same index. Then all γ_k touch a fixed circle from the pencil $\mathcal{A}_1 = \{\delta, \alpha_1\}$.*

Proof. All the arcs $\overset{\smile}{\boldsymbol{x}_k\boldsymbol{x}_k'}$ of the circle δ have the same masses. Invoking Proposition 4 completes the proof. □

Corollary 2. *Let $\{\omega_k\}_{k\in\mathbb{N}}$ be a circular series touching α_0, α_1, and let ω_k intersect the circle δ at points $\boldsymbol{x}_k, \boldsymbol{x}_{k+1}$. Fix $r \in \mathbb{N}$ and for every k, consider a circle passing through $\boldsymbol{x}_k$ and $\boldsymbol{x}_{k+r}$ and touching α_0 with the same index. Then all those circles touch a fixed circle $\alpha_r \in \mathcal{A}_1$.*

Proof. We apply Corollary 1 with $\omega_k' = \omega_{k+r}$. □

Thus, the situation is the same as for the diagonals of Poncelet's polygons [4]. Here, if a curvilinear broken line is inscribed in a circle δ and its sides touch a pair of circles α_0, α_1, then all its diagonals of rth order touching α_0 also touch a fixed circle from the pencil $\mathcal{A}_1 = \{\delta, \alpha_1\}$.

5. Emch's theorem for cyclics

A *cyclic* is a plane algebraic curve of order four defined by the equation

$$F(x_1, x_2) = \lambda(x_1^2 + x_2^2)^2 + (x_1^2 + x_2^2)\,\ell(x_1, x_2) + Q(x_1, x_2) = 0, \tag{2}$$

where ℓ is a linear form and Q is a polynomial of degree at most two. In what follows we assume $\lambda = 1$. The general case follows either by normalization (if $\lambda \neq 0$) or by a limit passage (if $\lambda = 0$). A pair of circles on the plane is always a cyclic, but not vice versa. An arbitrary quadric is a cyclic as well. Some properties of cyclics can be found in [7, Chapter 4, Section 2]. Nilov in [15] proved that the Emch theorem remains true after replacing the pair of circles α_0, α_1 by an arbitrary cyclic Γ. In this case, all circles ω_k have *double tangency* (i.e., two points of tangency) with Γ. See Figure 5.

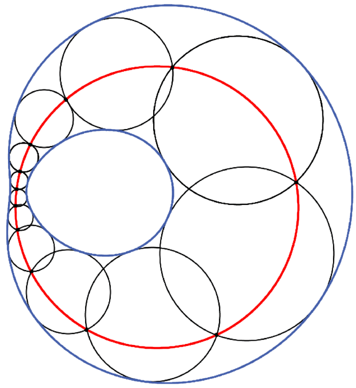

FIGURE 5. The Emch theorem for a cyclic.

If we take into account the complex tangency, then there exist four families of circles with double tangency with Γ; all ω_k belong to one of them [7]. The definition of invariant measure remains the same. The proof in [15] is geometrical and relies on the Poncelet theorem for quadrics. The universal measure ρ enables us to give a self-contained proof using the following generalization of Theorem 1.

Theorem 1′. *The function* $\rho(\boldsymbol{x}) = 1/\sqrt{|F(\boldsymbol{x})|}$ *generated by a cyclic* $\Gamma = \{\boldsymbol{x} \in \mathbb{R}^2 \mid F(\boldsymbol{x}) = 0\}$ *defines an invariant measure on any circle.*

The proof is based on the following generalization of Proposition 1 (Section 3) to cyclics.

Proposition 1′. *If a circle* ω *touches a given cyclic* $\Gamma = \{\boldsymbol{x} \in \mathbb{R}^2 \mid F(\boldsymbol{x}) = 0\}$ *at points* $\boldsymbol{t}_0, \boldsymbol{t}_1$*, then the function* $F|_\omega$ *is proportional to the square of the distance to the line* $\boldsymbol{t}_0\boldsymbol{t}_1$*.*

Combining this with Proposition 2 we prove Theorem 1′ in the same way as Theorem 1. The proof of Proposition 1′ uses an algebraic argument and one auxiliary result.

Definition 2. For an arbitrary circle δ, two algebraic curves are called δ-equivalent if the polynomials defining those curves are proportional on δ.

The δ-equivalence of curves $g_1(\boldsymbol{x}) = 0$ and $g_2(\boldsymbol{x}) = 0$ means that for some $\mu \neq 0$, the polynomial $g_1 - \mu g_2$ is divisible by f_δ (the power w.r.t. δ).

Lemma 1. *For an arbitrary cyclic* Γ *and a circle* δ*, the closure of the set of quadrics* δ*-equivalent to* Γ *is a pencil of quadrics containing* δ*.*

Proof. Let $f_\delta(\boldsymbol{x}) = x_1^2 + x_2^2 + \ell_\delta(x_1, x_2) + A_\delta = 0$, where ℓ_δ is a linear form and A_δ is a constant. A polynomial $p(\boldsymbol{x})$ possesses the property $\deg(F - p\, f_\delta) \leq 2$ if and only if

$$p(\boldsymbol{x}) = x_1^2 + x_2^2 + \ell_p(x_1, x_2) + A_p\,, \qquad \text{with} \qquad \ell_p + \ell_\delta = \ell\,, \quad A_p \in \mathbb{R}\,, \tag{3}$$

where ℓ is from the equation of cyclic (2). Denote by p_0 the polynomial (3) with $A_p = 0$. For arbitrary $A_p \in \mathbb{R}$, we have a quadratic polynomial $F - p\, f_\delta = F - (p_0 + A_p) f_\delta = \big(F - p_0\, f_\delta\big) - A_p f_\delta$. When A_p runs over $\bar{\mathbb{R}}$, these polynomials define a pencil of quadrics which contains δ (for $A_p = \infty$). □

Proof of Proposition 1′. Any quadric ω-equivalent to Γ touches the circle ω at points $\boldsymbol{t}_0$ and $\boldsymbol{t}_1$. By Lemma 1, those quadrics form a pencil. On the other hand, all quadrics touching a circle at two points form a pencil that contains a double line connecting those points [4, Section 16.4.10]. Hence, these two pencils coincide. In particular, the double line $\boldsymbol{t}_0\boldsymbol{t}_1$ is ω-equivalent to Γ. So, the function $F|_\omega$ is proportional to the square of the distance to the line $\boldsymbol{t}_0\boldsymbol{t}_1$. □

6. The Emch theorem and Poncelet's theorem for quadrics

By Lemma 1, a cyclic is equivalent to a quadric on every circle. Moreover, if a cyclic Γ and a circle δ are fixed, then all such quadrics form a pencil $\mathcal{Q}$. This implies that the invariant measure $\rho = 1/\sqrt{|F|}$ generated by Γ on the circle δ is proportional to the Jacobi-Bertrand measure $1/\sqrt{|q|}$ generated by any quadric from $\mathcal{Q}$. Therefore, $\mathcal{Q}$ contains a quadric γ tangent to all lines $\boldsymbol{x}_k\boldsymbol{x}_{k+1}$, $k \in \mathbb{N}$, corresponding to a circular series $\{\omega_k\}$. Hence the Emch theorem follows from Poncelet's theorem for quadrics δ and γ.

It was first noted by Hraskó [12] that the zigzag theorem can be derived from Poncelet's theorem for quadrics. Then in [17] this result was extended to Emch's theorem and in [15] to cyclics. The proofs in those works are different and nontrivial. Now we see that this is actually a consequence of equivalence of a cyclic to a certain quadric on a circle. Moreover, it is possible to find the desired quadric γ explicitly. We have $F(\boldsymbol{x}) = f_0(\boldsymbol{x})f_1(\boldsymbol{x})$, where $f_i(\boldsymbol{x}) = x_1^2 + x_2^2 + \ell_i(x_1, x_2) + B_i = 0$ is the power w.r.t. the circle α_i, $i = 0, 1$. Applying (3) we see that the polynomial $p(\boldsymbol{x}) = x_1^2 + x_2^2 + \ell_p(x_1, x_2) + A_p$ satisfies the equalities $\ell_p + \ell_\delta = \ell_0 + \ell_1$. The quadric γ is thus given by the equation $q(\boldsymbol{x}) = (f_0 f_1 - p f_\delta)(\boldsymbol{x}) = 0$. Simplifying, we get

$$
\begin{aligned}
q(\boldsymbol{x}) = {} & \big(\ell_0(\boldsymbol{x}) + B_0\big)\big(\ell_1(\boldsymbol{x}) + B_1\big) - \big(\ell_\delta(\boldsymbol{x}) + A_\delta\big)\big(\ell_p(\boldsymbol{x}) + A_p\big) \\
& + \big(x_1^2 + x_2^2\big)\big(B_0 + B_1 - A_\delta - A_p\big),
\end{aligned}
\tag{4}
$$

where $\ell_p = \ell_0 + \ell_1 - \ell_\delta$, and the parameter A_p is found by the tangency condition.

The inverse implication can also be easily realized. If we have a circle δ and a quadric γ, then one can find functionals ℓ_0, ℓ_1, ℓ_p and constants B_0, B_1, A_p such that $\ell_0 + \ell_1 = \ell_p + \ell_\delta$ and (4) holds. This way we find circles α_0, α_1 such that all chords $\boldsymbol{x}_k\boldsymbol{x}_{k+1}$, $k \in \mathbb{N}$, in the Emch theorem are tangent to γ. Hence, Emch's theorem implies Poncelet's theorem for a circle and a quadric, which is equivalent to the case of two quadrics (by means of a suitable stereographic projection).

Thus, *the Poncelet theorem for quadrics follows from Emch's theorem.*

7. The Emch theorem for general position of circles

As we noted in Section 2, the very existence of an invariant measure immediately implies the Emch theorem for nested circles. In this case, the differentials $d\boldsymbol{x}$ and $d\boldsymbol{y}$ in (1) always have the same sign. In particular, for a small perturbation of the circular series $\{\omega_k\}$, we have $\rho(\boldsymbol{x}_k)d\boldsymbol{x}_k \equiv \text{const}$, $k \in \mathbb{N}$. Integrating, we obtain that if the circle ω_1 moves to a circle ω_1', then for the series $\{\omega_k\}$ and $\{\omega_k'\}$, we have $m(\overset{\smile}{\boldsymbol{x}_k\boldsymbol{x}_k'}) \equiv \text{const}$, $k \in \mathbb{N}$. In particular, $m(\overset{\smile}{\boldsymbol{x}_1\boldsymbol{x}_1'}) = m(\overset{\smile}{\boldsymbol{x}_{n+1}\boldsymbol{x}_{n+1}'})$; hence if $\boldsymbol{x}_{n+1} = \boldsymbol{x}_1$, then $\boldsymbol{x}_{n+1}' = \boldsymbol{x}_1'$, which completes the proof. In the general case, however, a small perturbation of ω_1 can move the points $\{\boldsymbol{x}_k\}_{k\in\mathbb{N}}$ in different directions. That is why, to prove Emch's theorem in the general case, we need to modify the invariant $\rho(\boldsymbol{x})|d\boldsymbol{x}|$ to respect the sign of the differential $d\boldsymbol{x}$.

For an arbitrary triangle $\triangle\, \boldsymbol{abc}$ we denote by $\tau(\boldsymbol{abc})$ its orientation: $\tau(\boldsymbol{abc}) = 1$ if its vertices follow in the positive direction or, equivalently, the pair of vectors $\boldsymbol{b}-\boldsymbol{a}$ and $\boldsymbol{c}-\boldsymbol{a}$ is positively oriented. Otherwise, $\tau(\boldsymbol{abc}) = -1$. To avoid considering two cases, we make the following assumption:

Assumption 1. The circle ω_1 lies inside α_1.

This assumption is not restrictive; it can always be achieved by a suitable inversion. Note also that if ω_1 lies inside α_1, then so does ω_2 (since it intersects ω_1), and ω_3, etc. Thus, Assumption 1 means that the whole series $\{\omega_k\}$ is inside α_1.

Theorem 2. *For any circles $\alpha_0, \alpha_1, \delta$ and for an arbitrary circular series $\{\omega_k\}_{k\in\mathbb{N}}$ touching α_0, α_1 and satisfying Assumption 1, we have $\tau(\boldsymbol{x}_k \boldsymbol{t}_0^k \boldsymbol{t}_1^k)\, \rho(\boldsymbol{x}_k)\, d\boldsymbol{x}_k \equiv$ const, $k \in \mathbb{N}$.*

The proof of Theorem 2 requires two auxiliary facts. The first one is a generalization of Proposition 2.

Proposition 5. *Under the assumptions of Proposition 2, we have*

$$\frac{d\boldsymbol{y}}{d\boldsymbol{x}} = -\frac{\tau(\boldsymbol{ykl})}{\tau(\boldsymbol{xkl})}\,\frac{q(\boldsymbol{y})}{q(\boldsymbol{x})}.$$

Proof. If the chords $\boldsymbol{kl}$ and $\boldsymbol{xy}$ intersect, then the arcs $\breve{\boldsymbol{xx}'}$ and $\breve{\boldsymbol{yy}'}$ have the same sign and $\frac{\tau(\boldsymbol{xkl})}{\tau(\boldsymbol{ykl})} = -1$. Otherwise those arcs have opposite signs and $\frac{\tau(\boldsymbol{xkl})}{\tau(\boldsymbol{ykl})} = -1$. Applying Proposition 2, we conclude the proof. □

The proof of the following fact is elementary and we omit it.

Lemma 2. *Let circles ω and ν pass through a point $\boldsymbol{m}$, and let circles α_0 and α_1 touch them with index 0 at points $\boldsymbol{t}_0, \boldsymbol{t}_1$ and $\boldsymbol{s}_0, \boldsymbol{s}_1$, respectively. Then $\tau(\boldsymbol{m}\boldsymbol{t}_0\boldsymbol{t}_1) = -\tau(\boldsymbol{m}\boldsymbol{s}_0\boldsymbol{s}_1)$.*

Proof of Theorem 2. Arguing as in the proof of Theorem 1 and using Proposition 5 for $\boldsymbol{x} = \boldsymbol{x}_k, \boldsymbol{y} = \boldsymbol{x}_{k+1}$, we obtain $\tau(\boldsymbol{x}_k \boldsymbol{t}_0^k \boldsymbol{t}_1^k)\rho(\boldsymbol{x}_k)d\boldsymbol{x}_k = -\tau(\boldsymbol{x}_{k+1} \boldsymbol{t}_0^k \boldsymbol{t}_1^k)\rho(\boldsymbol{x}_{k+1})d\boldsymbol{x}_{k+1}$. Applying now Lemma 2 to the circles $\omega = \omega_k$, $\nu = \omega_{k+1}$ and taking into account that α_0 and α_1 touch them with index 0, because ω_k lies inside α_1, we conclude that $\tau(\boldsymbol{x}_{k+1} \boldsymbol{t}_0^k \boldsymbol{t}_1^k) = -\tau(\boldsymbol{x}_{k+1} \boldsymbol{t}_0^{k+1} \boldsymbol{t}_1^{k+1})$. □

Now we are ready to prove Emch's theorem in the general case.

Proof of the Emch theorem. Consider a perturbation of the circular series $\{\omega_k\}$ that moves it to a series $\{\omega_k'\}$. The orientation of all triangles $\triangle \boldsymbol{x}_k \boldsymbol{t}_0^k \boldsymbol{t}_1^k$, $k = 1, \ldots, n+1$, is not changed whenever the perturbation is small enough. If $\omega_{n+1} = \omega_1$, then the points $\boldsymbol{x}_{n+1}, \boldsymbol{t}_0^{n+1}, \boldsymbol{t}_1^{n+1}$ coincide with $\boldsymbol{x}_1, \boldsymbol{t}_0^1, \boldsymbol{t}_1^1$, respectively. Hence $\tau(\boldsymbol{x}_{n+1} \boldsymbol{t}_0^{n+1} \boldsymbol{t}_1^{n+1}) = \tau(\boldsymbol{x}_1 \boldsymbol{t}_0^1 \boldsymbol{t}_1^1)$, and therefore $\rho(\boldsymbol{x}_{n+1})d\boldsymbol{x}_{n+1} = \rho(\boldsymbol{x}_1)d\boldsymbol{x}_1$. Integrating, we obtain $m(\boldsymbol{x}_{n+1}\boldsymbol{x}_{n+1}') = m(\boldsymbol{x}_1\boldsymbol{x}_1')$; hence $\boldsymbol{x}_{n+1}' = \boldsymbol{x}_1'$. We see that the assertion $\boldsymbol{x}_{n+1}' = \boldsymbol{x}_1'$ is locally stable (under small perturbations). The continuity implies that it holds identically. □

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