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EIGENVALUE ESTIMATES ON A CONNECTED FINITE GRAPH

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ABSTRACT. Based on gradient estimates of the eigenfunction, we prove lower bound estimates for the first nonzero eigenvalue of the μ-Laplacian on a connected finite graph through the curvature-dimension conditions. These estimates are parallel to the results on compact Riemannian manifolds with the Ricci curvature bounded from below.

1. INTRODUCTION

Lower bound estimates for the first nonzero eigenvalue of the Laplacian on compact manifolds with the Ricci curvature bounded from below are the fundamental results in the geometric analysis. The cases that the Ricci curvature is bounded from below by a positive number, zero or a negative number, were considered by Lichnerowicz [12] and Obata [13], Li-Yau [10] and Zhong-Yang [14] or Li-Yau [11], respectively. A general lower bound estimate was established in [1, 3, 6] independently, which can also be generalized to the compact metric measure space satisfying the Remannian curvature-dimension condition $RCD^*(K, N)$ [9].

A similar question exists on a connected finite graph through the curvature-dimension conditions, and some estimates have been established. In [4] the authors established an estimate $\lambda_{deg} \geq \frac{m}{m-1}K$ on a connected finite graph through the $CD(m, K)$ condition (see Definition 2.2) for some $m > 1$ and $K > 0$, where λ_{deg} is the first nonzero eigenvalue of the normalized graph Laplacian $\triangle_{deg}$ (see Remark 2.1 for its definition). When on a connected finite graph which satisfies the $CD(m, K)$ condition for some $K \leq 0$, the lower bound estimate for λ_{deg} was established in [7].

The chain rule always fails on graphs, even on the lattice $\mathbf{Z}^n$. In a recent paper [5] the authors showed a way to bypass the chain rule in the graphic setting; they also introduced the $CDE(m, K)$ condition (see Definition 2.3). By using the maximum principle they could establish the parabolic type gradient estimates for positive solutions to the linear heat equation

$$(\partial_t - \triangle_\mu)u = 0,$$

where $\triangle_\mu$ is the $\mu-$Laplacian and is defined in (2.1). The Harnack inequalities and the estimate of the heat kernel can also be derived.

In this paper we derive a gradient estimate for the eigenfunction of the μ-Laplacian $\triangle_\mu$ on a connected finite graph through the $CDE(m, K)$ condition for

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some constant $K \leq 0$. Based on this estimate we can get the following lower bound estimate for the first nonzero eigenvalue of the μ-Laplacian.

Theorem 1.1. *Let $G(V, E)$ be a connected finite graph satisfying the $CDE(m, -K)$ condition for some $m > 0, K \geq 0$. Then there exist constants $c_1, c_2 > 0$ depending on m, μ_{max}, w_{min} alone, such that the first nonzero eigenvalue of the μ-Laplacian satisfies*

$$\lambda \geq \frac{c_1}{d^2} \exp(-c_2\sqrt{K}d), \tag{1.1}$$

where μ_{max}, w_{min} are defined in Section 2 and $d = \max\{d(x, y), x, y \in V\}$ denotes the diameter of the graph.

When $G(V, E)$ satisfies $CDE(m, 0)$ condition for some $m > 0$, we can get the following corollary from the proof of Theorem 1.1.

Corollary 1.2. *Let $G(V, E)$ be a connected finite graph satisfying the $CDE(m, 0)$ condition for some $m > 0$. Then the first nonzero eigenvalue of the $\mu-$Laplacian satisfies*

$$\lambda \geq \frac{w_{min}}{2(m+1)e^2\mu_{max}d^2}. \tag{1.2}$$

When on a connected finite graph satisfying the $CD(m, 0)$ condition, the estimate of the first nonzero eigenvalue of the normalized graph Laplacian $\triangle_{deg}$ was considered in [7, 8]. For example, the authors of [7] showed that the first nonzero eigenvalue of $\triangle_{deg}$ on a connected finite unweighted graph (see Remark 2.1 for its definition) with the $CD(m, 0)$ condition satisfies

$$\lambda_{deg} \geq \frac{1}{\mu_{max}(4 - \frac{1}{m})d^2}. \tag{1.3}$$

The estimates in (1.2) and (1.3) can be improved, by which, we let λ be the first nonzero eigenvalue of the μ-Laplacian on a connected finite graph and let u be the corresponding eigenfunction. Due to Lemma 2.6, it is possible to arrange that

$$a - 1 = \inf_{x\in V} u(x), a + 1 = \sup_{x\in V} u(x)$$

by multiplying with a constant, where $0 \leq a(u) < 1$ is the median of u.

Theorem 1.3. *Let $G(V, E)$ be a connected finite graph satisfying the $CD(1, 0)$ condition. Then the first nonzero eigenvalue of the μ-Laplacian satisfies*

$$\lambda \geq \frac{2w_{min}(\arcsin\sqrt{\frac{1+a}{2+a}})^2}{\mu_{max}(1 + a)d^2}. \tag{1.4}$$

When on a connected finite graph satisfying the $CD(m, K)$ condition for some $m > 1$ and $K > 0$, the estimate of the first nonzero eigenvalue of $\triangle_{deg}$ was considered in [4]. In Section 6 we will prove the following similar estimate for the μ-Laplacian $\triangle_\mu$ by using an important identity observed in [5].

Theorem 1.4. *Let $G(V,E)$ be a connected finite graph satisfying the $CD(m,K)$ condition for some $m>1$ and $K>0$. Then the first nonzero eigenvalue of the μ-Laplacian satisfies*

$$\lambda \geq \frac{m}{m-1}K. \tag{1.5}$$

2. Notation and lemmas

Let $G(V,E)$ be a connected finite graph whose edge $xy\in E$ from x to y has weight $w_{xy}>0$. We also assume that

$$w_{min}:=\inf_{e\in E} w_e>0,$$

and for all $x\in V$,

$$\deg(x):=\sum_{y\sim x} w_{xy}.$$

Given a measure $\mu: V\to \mathbf{R}$ on V, the μ-Laplacian on G is the operator $\triangle_\mu$ defined by

$$\triangle_\mu f(x)=\frac{1}{\mu(x)}\sum_{y\sim x} w_{xy}(f(y)-f(x)). \tag{2.1}$$

Remark 2.1. We should point out that some special cases of the μ-Laplacian have been studied in literature. For example if $\mu=1$, then the μ-Laplacian is the standard graph Laplacian $\triangle$, and the case where

$$\mu(x)=\sum_{y\sim x} w_{xy}=deg(x),$$

which yields the normalized graph Laplacian $\triangle_{deg}$. We say that G is unweighted if $w_{xy}=1$ for all $xy\in E$. In this paper we always consider the μ-Laplacian $\triangle_\mu$, except when it is important to emphasize the effect of the measure.

Moreover we define

$$\mu_{max}=\max_{x\in V}\mu(x),$$

$$\begin{aligned}\Gamma(f,g)(x) &= \frac{1}{2}[\triangle_\mu(fg)-f\triangle_\mu g-g\triangle_\mu f](x)\\ &= \frac{1}{2}\frac{1}{\mu(x)}\sum_{y\sim x} w_{xy}(f(y)-f(x))(g(y)-g(x)),\end{aligned}$$

and

$$\Gamma_2(f,g)=\frac{1}{2}[\triangle_\mu\Gamma(f,g)-\Gamma(f,\triangle_\mu g)-\Gamma(\triangle_\mu f,g)].$$

For convenience, we write $\Gamma(f)=\Gamma(f,f), \Gamma_2(f)=\Gamma_2(f,f)$.

Definition 2.2. For $m>0, K\in\mathbf{R}$, we say that a graph $G(V,E)$ satisfies the $CD(m,K)$ condition if for all $x\in V$ and any function $f:V\to\mathbf{R}$,

$$\Gamma_2(f)(x)\geq \frac{1}{m}(\triangle_\mu f)^2(x)+K\Gamma(f)(x). \tag{2.2}$$

When studying gradient estimates on manifolds we always need the identity

$$\triangle_\mu \ln u = \frac{\triangle_\mu u}{u} - |\nabla \ln u|^2,$$

which comes from the chain rule formula. However the chain rule formula is always false in the graphic setting. In [5] the authors observed that in the graphic setting the identity

$$2\sqrt{u}\triangle_\mu\sqrt{u} = \triangle_\mu u - 2\Gamma(\sqrt{u}) \tag{2.3}$$

is always right for any positive function u. In order to establish Li-Yau's gradient estimate, they also introduced the following $CDE(m, K)$ condition.

Definition 2.3. We say that a graph $G(V, E)$ satisfies the $CDE(m, K)$ condition if for all $x \in V$, for any positive function $f : V \to \mathbf{R}$ such that $(\triangle_\mu f)(x) < 0$ we have

$$\tilde{\Gamma}_2(f)(x) \geq \frac{1}{m}(\triangle_\mu f)^2(x) + K\Gamma(f)(x), \tag{2.4}$$

where

$$\begin{aligned} \tilde{\Gamma}_2(f) &= \Gamma_2(f) - \Gamma(f, \frac{\Gamma(f)}{f}) \\ &= \frac{1}{2}\triangle_\mu\Gamma(f) - \frac{1}{2}\Gamma(f, \frac{\triangle_\mu f^2}{f}). \end{aligned} \tag{2.5}$$

Remark 2.4. As pointed out in [2], if $\triangle_\mu$ satisfies the $CD(m, 0)$ condition and generates a diffusion semigroup, then Li-Yau's famous gradient estimate, established in [11], holds. But $\triangle_\mu$ does not generate a diffusion semigroup on the graphic setting. The $CDE(m, K)$ condition was introduced in [5] to derive Li-Yau's gradient estimate. As pointed out in [5], both the $CD(m, K)$ condition and the $CDE(m, K)$ condition are local properties, and in some sense the $CDE(m, K)$ condition is weaker than the $CD(m, K)$ condition, i.e., if the semigroup generated by $\triangle_\mu$ is a diffusion semigroup, then the $CD(m, K)$ condition implies the $CDE(m, K)$ condition.

The following maximum principle, which can be viewed as the elliptic version of the maximum principle established in [5] for the parabolic operator $\triangle_\mu - \partial_t$, is useful.

Lemma 2.5. *Let $G(V, E)$ be a graph, and let $g, F : V \to \mathbf{R}$ be two functions. Suppose that $g \geq 0$ and that F has a local maximum at x_0. Then*

$$\triangle_\mu(gF)(x_0) \leq (\triangle_\mu g)(x_0)F(x_0). \tag{2.6}$$

Assume that λ is a nonzero constant and u is a nonconstant function satisfying

$$\triangle_\mu u = -\lambda u \tag{2.7}$$

on a connected finite graph.

Lemma 2.6. *u must change sign and $\lambda > 0$.*

Proof. If $\lambda < 0$, we assume that u achieves its maximum at $x_0 \in V$; then

$$u(x_0) = -\frac{1}{\lambda\mu(x_0)} \sum_{y\sim x_0} w_{x_0 y}(u(y) - u(x_0)) \leq 0.$$

Hence for all $x \in V$, $u(x) \leq 0$, which implies that $u \equiv 0$ and contradicts the fact that u is nonconstant. If $\lambda = 0$, we assume that u achieves its maximum at $x_0 \in V$; then

$$0 = \frac{1}{\mu(x_0)} \sum_{y\sim x_0} w_{x_0 y}(u(y) - u(x_0)) \leq 0.$$

Hence for all $y \sim x_0$, $u(y) = u(x_0)$, which implies that $u \equiv u(x_0)$ and also contradicts the fact that u is nonconstant. So we conclude that $\lambda > 0$. A similar discussion shows that the maximum and minimum of u are positive and negative, respectively. □

3. Gradient estimate

Assume that λ is a nonzero constant and u is a nonconstant function satisfying (2.7). Due to Lemma 2.6, we may normalize u to satisfy $\min u = -1$ and $\max u \leq 1$. The main result in this section is the following gradient estimate for u.

Theorem 3.1. *Let $G(V, E)$ be a connected finite graph satisfying the $CDE(m, -K)$ condition for some $m > 0$, $K \geq 0$, and let $u : V \to \mathbf{R}$ be a solution to* (2.7) *satisfying* $\min u = -1$ *and* $\max u \leq 1$. *Then for all $a > 1$ and all $x \in V$,*

$$\begin{aligned} \frac{\Gamma(\sqrt{a+u})}{a+u}(x) \quad &\leq \quad \frac{mK}{2} + \frac{(ma+1)\lambda}{2(a-1)} \\ &\quad + \frac{1}{2}\sqrt{m^2(K + \frac{\lambda}{a-1})^2 + \frac{m}{a-1}(2K\lambda + \frac{a}{a-1}\lambda^2)}. \end{aligned} \tag{3.1}$$

Proof. For $a > 1$ let $v = a + u$. Then v satisfies

$$\triangle_\mu v = -\lambda(v - a). \tag{3.2}$$

Let

$$F(x) = \frac{2\Gamma(\sqrt{v}) - \triangle_\mu v}{v} = \frac{-2\triangle_\mu \sqrt{v}}{\sqrt{v}}. \tag{3.3}$$

We assume that F achieves its maximum at x_0. If $F(x_0) \leq 0$, then for all $x \in V$, $F(x) \leq 0$, due to the fact that $v(x) \geq a - 1$. Thus we have

$$\frac{\Gamma(\sqrt{a+u})}{a+u}(x) \leq \frac{-\lambda(v-a)}{2v} \leq \frac{\lambda}{2(a-1)},$$

and (3.1) follows. Hence we can assume that $F(x_0) > 0$, which implies that $\triangle_\mu\sqrt{v}(x_0) < 0$. By the maximum principle we have that at x_0,

$$\begin{aligned}
(\triangle_\mu v)F &\geq \triangle_\mu(vF) = \triangle_\mu(2\Gamma(\sqrt{v}) - \triangle_\mu v) \\
&= 4\tilde{\Gamma}_2(\sqrt{v}) + 2\Gamma(\sqrt{v}, \frac{\triangle_\mu v}{\sqrt{v}}) - (\triangle_\mu\triangle_\mu v) \\
&\geq \frac{4}{m}(\triangle_\mu\sqrt{v})^2 - 4K\Gamma(\sqrt{v}) - 2\lambda\Gamma(\sqrt{v}, \frac{v-a}{\sqrt{v}}) - \lambda^2(v-a) \\
&= \frac{4}{m}(\triangle_\mu\sqrt{v})^2 - 4K\Gamma(\sqrt{v}) - 2\lambda\Gamma(\sqrt{v}) + 2\lambda a\Gamma(\sqrt{v}, \frac{1}{\sqrt{v}}) - \lambda^2(v-a). \qquad (3.4)
\end{aligned}$$

By (2.3), (3.2), and (3.3) we have

$$\triangle_\mu\sqrt{v} = -\frac{1}{2}\sqrt{v}F, \qquad (3.5)$$

$$\Gamma(\sqrt{v}) = \frac{1}{2}vF - \frac{1}{2}\lambda(v-a). \qquad (3.6)$$

Due to the fact that $v(x) \geq a-1$ for all $x \in V$, we have that for all $x \in V$,

$$\begin{aligned}
\Gamma(\sqrt{v}, \frac{1}{\sqrt{v}})(x) &= \frac{1}{\mu(x)}\sum_{y\sim x} w_{xy}(\sqrt{v}(y) - \sqrt{v}(x))(\frac{1}{\sqrt{v}(y)} - \frac{1}{\sqrt{v}(x)}) \\
&= -\frac{1}{\mu(x)}\sum_{y\sim x} w_{xy}\frac{(\sqrt{v}(y) - \sqrt{v}(x))^2}{\sqrt{v}(y)\sqrt{v}(x)} \\
&\geq -\frac{1}{(a-1)\mu(x)}\sum_{y\sim x} w_{xy}(\sqrt{v}(y) - \sqrt{v}(x))^2 \\
&\geq -\frac{1}{a-1}\Gamma(\sqrt{v})(x). \qquad (3.7)
\end{aligned}$$

Plugging (3.5), (3.6), and (3.7) into (3.4), we have that at x_0,

$$\frac{v}{m}F^2 - (2Kv + \frac{a\lambda}{a-1}v + a\lambda)F + (2K\lambda + \frac{a}{a-1}\lambda^2)(v-a) \leq 0.$$

Due to the fact that $a-1 < v \leq a+1$, we have that at x_0,

$$\frac{1}{m}F^2 - (2K + \frac{2a\lambda}{a-1})F - \frac{1}{a-1}(2K\lambda + \frac{a}{a-1}\lambda^2) \leq 0,$$

which implies that

$$F(x_0) \leq mK + \frac{ma\lambda}{a-1} + \sqrt{m^2(K + \frac{\lambda}{a-1})^2 + \frac{m}{a-1}(2K\lambda + \frac{a}{a-1}\lambda^2)}.$$

Hence for all $x \in V$,

$$F(x) \leq mK + \frac{ma\lambda}{a-1} + \sqrt{m^2(K + \frac{\lambda}{a-1})^2 + \frac{m}{a-1}(2K\lambda + \frac{a}{a-1}\lambda^2)},$$

which implies that

$$\begin{aligned}\frac{\Gamma(\sqrt{a+u})}{a+u}(x) &\leq \frac{-\lambda(v-a)}{2v}+\frac{mK}{2}+\frac{ma\lambda}{2(a-1)}\\ &\quad+\frac{1}{2}\sqrt{m^2(K+\frac{\lambda}{a-1})^2+\frac{m}{a-1}(2K\lambda+\frac{a}{a-1}\lambda^2)}\\ &\leq \frac{mK}{2}+\frac{(ma+1)\lambda}{2(a-1)}\\ &\quad+\frac{1}{2}\sqrt{m^2(K+\frac{\lambda}{a-1})^2+\frac{m}{a-1}(2K\lambda+\frac{a}{a-1}\lambda^2)}.\end{aligned}$$

Here in the last inequality we have used the fact that $v(x) \geq a-1$. □

4. Proof of Theorem 1.1

Based on the gradient estimate established in Theorem 3.1, we can derive the lower bound estimate of the first nonzero eigenvalue of the μ-Laplacian stated in Theorem 1.1.

Proof. We firstly assume that $x \sim y$. Note that for any $a, b > 0$, $\ln\frac{b}{a} \leq \frac{b-a}{a}$. By (3.1) we have

$$\begin{aligned}\ln\frac{\sqrt{v}(y)}{\sqrt{v}(x)} &\leq \frac{\sqrt{v}(y)-\sqrt{v}(x)}{\sqrt{v}(x)}\\ &\leq [\frac{2\mu_{max}\Gamma(\sqrt{v})(x)}{w_{min}}]^{\frac{1}{2}}\frac{1}{\sqrt{v}(x)}\\ &\leq \sqrt{\frac{2C(m,K,\lambda)\mu_{max}}{w_{min}}},\end{aligned}$$

where

$$C(m,K,\lambda)=\frac{mK}{2}+\frac{(ma+1)\lambda}{2(a-1)}+\frac{1}{2}\sqrt{m^2(K+\frac{\lambda}{a-1})^2+\frac{m}{a-1}(2K\lambda+\frac{a}{a-1}\lambda^2)}.$$

When x and y are not adjacent, we simply let $x = x_0; x_1; \cdots; x_k = y$ denote a path P between x and y. Then

$$\begin{aligned}\ln\frac{\sqrt{v}(y)}{\sqrt{v}(x)} &= \sum_{i=1}^{k}\ln\frac{\sqrt{v}(x_i)}{\sqrt{v}(x_{i-1})}\\ &\leq \sum_{i=1}^{k}\sqrt{\frac{2C(m,K,\lambda)\mu_{max}}{w_{min}}}\\ &= k\sqrt{\frac{2C(m,K,\lambda)\mu_{max}}{w_{min}}}.\end{aligned}$$

Choosing a suitable P so that $k = \mathrm{dist}(x, y)$, we will get that

$$\ln\frac{\sqrt{v}(y)}{\sqrt{v}(x)} \leq \mathrm{dist}(x,y)\sqrt{\frac{2C(m,K,\lambda)\mu_{max}}{w_{min}}} \leq d\sqrt{\frac{2C(m,K,\lambda)\mu_{max}}{w_{min}}}.$$

We assume that $u(x) = -1, u(y) = \max u$; then

$$\ln \frac{\sqrt{a}}{\sqrt{a-1}} \leq \ln \frac{\sqrt{a+\max u}}{\sqrt{a-1}} = \ln \frac{\sqrt{v}(y)}{\sqrt{v}(x)} \leq d\sqrt{\frac{2C(m,K,\lambda)\mu_{max}}{w_{min}}}. \tag{4.1}$$

A direct calculation shows that

$$\frac{1}{2}\sqrt{m^2(K+\frac{\lambda}{a-1})^2 + \frac{m}{a-1}(2K\lambda + \frac{a}{a-1}\lambda^2)} \leq \frac{mK}{2} + \frac{(ma+1)\lambda}{2(a-1)}.$$

Hence

$$C(m,K,\lambda) \leq mK + \frac{(ma+1)\lambda}{a-1}.$$

Plugging into (4.1) leads to

$$\begin{aligned}(\ln \frac{\sqrt{a}}{\sqrt{a-1}})^2 &\leq d^2\frac{2\mu_{max}}{w_{min}}[mK + \frac{(ma+1)\lambda}{a-1}] \\ &\leq d^2\frac{2\mu_{max}}{w_{min}}[mK + \frac{(m+1)a\lambda}{a-1}].\end{aligned}$$

Hence

$$\lambda \geq \frac{t}{m+1}[\frac{w_{min}}{8d^2\mu_{max}}(\ln \frac{1}{t})^2 - mK],$$

where

$$t = \frac{a-1}{a} \in (0,1).$$

Maximizing the right-hand side as a function of t by setting

$$t = \exp(-1 - \sqrt{1 + \frac{8\mu_{max}mKd^2}{w_{min}}}),$$

we obtain the estimate

$$\begin{aligned}\lambda &\geq \frac{w_{min}}{4(m+1)\mu_{max}d^2}(1 + \sqrt{1 + \frac{8\mu_{max}mKd^2}{w_{min}}}) \\ &\times \exp(-1 - \sqrt{1 + \frac{8\mu_{max}mKd^2}{w_{min}}}),\end{aligned} \tag{4.2}$$

which implies (1.1). □

The proof of Theorem 1.1 implies Corollary 1.2. In fact, letting $K = 0$ in (4.2) leads to (1.2).

5. Proof of Theorem 1.3

In this section we will prove Theorem 1.3.

Proof. Let $\phi = u - (a+\epsilon)$, where $0 < \epsilon < 1$ small enough so that

$$\sqrt{\frac{1+a+\epsilon}{2+a+\epsilon}}(-1-\epsilon) > -1. \tag{5.1}$$

Then ϕ satisfies

$$\triangle_\mu \phi = -\lambda(\phi + a + \epsilon). \tag{5.2}$$

Let

$$P = \Gamma(\phi) + \lambda(1+a+\epsilon)\phi^2. \tag{5.3}$$

Then

$$\triangle_\mu P = \triangle_\mu \Gamma(\phi) + \lambda(1+a+\epsilon)(2\phi\triangle_\mu\phi + 2\Gamma(\phi)). \tag{5.4}$$

Due to (2.2) we have

$$\triangle_\mu \Gamma(\phi) \quad \geq \quad 2(\triangle_\mu\phi)^2 + 2\Gamma(\phi, \triangle_\mu\phi). \tag{5.5}$$

By (5.2), (5.3), (5.4), (5.5), we have

$$\begin{aligned} \triangle_\mu P \quad &\geq \quad 2\lambda(a+\epsilon)\Gamma(\phi) - 2(1+a+\epsilon)(a+\epsilon+\phi)\lambda^2\phi + 2\lambda^2(\phi+a+\epsilon)^2 \\ &= \quad 2\lambda(a+\epsilon)P + 2\lambda^2[(a+\epsilon)\phi - 2(a+\epsilon)\phi^2 \\ &\quad\quad +(a+\epsilon)^2 - (a+\epsilon)^2\phi - (a+\epsilon)^2\phi^2] \\ &\geq \quad 2\lambda(a+\epsilon)P - 2\lambda^2[2(a+\epsilon) + (a+\epsilon)^2]. \end{aligned}$$

We assume that P achieves its maximum at x_0; then $\triangle_\mu P(x_0) \leq 0$, which implies that

$$P(x_0) \leq (2+a+\epsilon)\lambda.$$

Hence for all $x \in V$,

$$P(x) \leq (2+a+\epsilon)\lambda,$$

which implies that

$$\Gamma(\phi)(x) \leq (2+a+\epsilon)\lambda - (1+a+\epsilon)\lambda\phi^2(x)$$

or

$$\Gamma(v)(x) \leq (1+a+\epsilon)\lambda(1-v^2(x)), \tag{5.6}$$

where

$$v = \sqrt{\frac{1+a+\epsilon}{2+a+\epsilon}}\phi = \sqrt{\frac{1+a+\epsilon}{2+a+\epsilon}}(u-a-\epsilon). \tag{5.7}$$

We firstly assume that $x \sim y$. Note that for any a, b satisfying $0 \leq a \leq b < 1$,

$$\arcsin b - \arcsin a = \frac{b-a}{\sqrt{1-\xi^2}} \leq \frac{b-a}{\sqrt{1-b^2}}, \tag{5.8}$$

where $\xi \in (a,b)$. We claim that

$$\arcsin v(y) - \arcsin v(x) \leq \sqrt{\frac{2\mu_{max}(1+a+\epsilon)\lambda}{w_{min}}}. \tag{5.9}$$

In fact, by (5.8) we have

$$\arcsin v(y) - \arcsin v(x) \quad \leq \quad \frac{v(y)-v(x)}{\sqrt{1-v^2(x)}}.$$

Due to the definition of Γ, we conclude that

$$\begin{aligned} \arcsin v(y) - \arcsin v(x) \quad &\leq \quad \sqrt{\frac{2\mu_{max}\Gamma(v)(x)}{w_{min}}}\frac{1}{\sqrt{1-v^2(x)}} \\ &\leq \quad \sqrt{\frac{2\mu_{max}(1+a+\epsilon)\lambda}{w_{min}}}. \end{aligned}$$

Here we have used (5.6).

When x and y are not adjacent, we simply let $x = x_0; x_1; \cdots; x_k = y$ denote a path P between x and y. By (5.9) we have

$$\begin{aligned}\arcsin v(y) - \arcsin v(x) &= \sum_{i=1}^{k} (\arcsin v(x_i) - \arcsin v(x_{i-1})) \\ &\le \sum_{i=1}^{k} \sqrt{\frac{2\mu_{max}(1+a+\epsilon)\lambda}{w_{min}}} \\ &= k\sqrt{\frac{2\mu_{max}(1+a+\epsilon)\lambda}{w_{min}}}.\end{aligned}$$

Choosing a suitable P so that $k = \text{dist}(x, y)$, we will get that

$$\begin{aligned}\arcsin v(y) - \arcsin v(x) &\le \text{dist}(x,y)\sqrt{\frac{2\mu_{max}(1+a+\epsilon)\lambda}{w_{min}}} \\ &\le d\sqrt{\frac{2\mu_{max}(1+a+\epsilon)\lambda}{w_{min}}}.\end{aligned}$$

We assume that $u(x) = a - 1, u(y) = a + 1$; then,

$$\arcsin\sqrt{\frac{1+a+\epsilon}{2+a+\epsilon}}(1-\epsilon) + \arcsin\sqrt{\frac{1+a+\epsilon}{2+a+\epsilon}}(1+\epsilon) \le d\sqrt{\frac{2\mu_{max}(1+a)\lambda}{w_{min}}}.$$

Letting $\epsilon \searrow 0$ we conclude that

$$2\arcsin\sqrt{\frac{1+a}{2+a}} \le d\sqrt{\frac{2\mu_{max}(1+a)\lambda}{w_{min}}},$$

which implies (1.4). □

Remark 5.1. Since $0 \le a \le 1$, we easily deduce from (1.4) that

$$\lambda \ge \frac{\pi^2 w_{min}}{16\mu_{max} d^2},$$

which implies that (1.4) is better than (1.3) for the case that $m = 1$.

6. Proof of Theorem 1.4

In this section we shall prove Theorem 1.4, based on the important identity (2.3) observed in [5].

Proof. Let λ be the first nonzero eigenvalue of the μ-Laplacian on a connected finite graph and let u be the corresponding eigenfunction. We assume that

$$Q = \Gamma(u) + \frac{\lambda}{m}u^2.$$

By (2.3) we have

$$2u\triangle_\mu u = \triangle_\mu u^2 - 2\Gamma(u).$$

Hence

$$\begin{aligned} \triangle_\mu Q &= \triangle_\mu \Gamma(u) + \frac{\lambda}{m}\triangle_\mu u^2 \\ &= \triangle_\mu \Gamma(u) + \frac{\lambda}{m}(2u\triangle_\mu u + 2\Gamma(u)). \end{aligned}$$

Note that

$$\begin{aligned} \Gamma_2(u) &= \frac{1}{2}(\triangle_\mu \Gamma(u) - 2\Gamma(u, \triangle_\mu u)) \\ &\geq \frac{1}{m}(\triangle_\mu u)^2 + K\Gamma(u). \end{aligned}$$

Hence

$$\begin{aligned} \triangle_\mu Q &\geq \frac{2}{m}(\triangle_\mu u)^2 + 2K\Gamma(u) + 2\Gamma(u, \triangle_\mu u) + \frac{\lambda}{m}(2u\triangle_\mu u + 2\Gamma(u)) \\ &= -\frac{2(m-1)}{m}(\lambda - \frac{m}{m-1}K)\Gamma(u). \end{aligned} \tag{6.1}$$

If $\lambda \leq \frac{m}{m-1}K$, then Q is a subharmonic function. By the boundedness of the graph and the maximum principle, Q must be identically constant and all the inequalities in (6.1) are equalities. In particular, the right-hand side of (6.1) must be identically 0. Hence $\lambda = \frac{m}{m-1}K$ since u is nonconstant. □

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