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THE BERGMAN KERNEL ON FORMS: GENERAL THEORY

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Abstract. The goal of this paper is to explore the Bergman projection on forms. In particular, we show that some of most basic facts used to construct the Bergman kernel on functions, such as pointwise evaluation in $L^2_{0,q}(\Omega) \cap \ker \bar\partial_q$, fail for (p,q)-forms, $q \geq 1$, $p \geq 0$. We do, however, provide a careful construction of the Bergman kernel and explicitly compute the Bergman kernel on $(0,n-1)$-forms. For the ball in $\mathbb{C}^2$, we also show that the size of the Bergman kernel on $(0,1)$-forms is not governed by the control metric, in stark contrast to the Bergman kernel on functions.

1. Introduction

On a domain $\Omega \subset \mathbb{C}^n$, the Bergman projection B_q is the orthogonal projection $B_q : L^2_{0,q}(\Omega) \to \ker \bar\partial_q \cap L^2_{0,q}(\Omega)$. The basic theory of the classical Bergman projection B_0 is, well, classical and can be found in several complex variables textbook, e.g., [Kra01]. The Bergman projection B_0 is one of the most basic objects in the analysis of both one and several variables, and its mapping properties have been exhaustively (though not conclusively) researched, as have formulas for its kernel. See, for example, [Cat83, Cat87, KN65, FK72, PS77, McN89, NRSW89, CD06, NS06, McN94, MS94, KR, Fef74, D'A78, D'A94] for just a small sampling of the results in the literature. Surprisingly, when $q \geq 1$, only mapping properties have been investigated — regularity properties for the Bergman projection often follow from estimates of the $\bar\partial$-Neumann operator and Kohn's formula (see, for example, [HR15, BS90, HM06]). There is essentially no literature about an explicit construction of the kernels, pointwise size estimates, or geometry.

A standard discussion of B_0 includes a formal construction of the integral kernel, its transformation law under biholomorphic mappings, and a computation of the Bergman kernel on the ball (and perhaps the polydisk). One of the goals of this paper is to show that several of the main features of B_0 and its construction fail for B_q, $q \geq 1$. In particular, we show that:

(1) Pointwise evaluation is not a bounded linear functional on $L^2_{0,q}(\Omega) \cap \ker(\bar\partial_q)$.
(2) It is unrealistic for a transformation formula to hold for $B_{p,q}(z,w)$ unless $p, q \in \{0, n\}$.
(3) In $\mathbb{C}^2$, the Bergman kernel $B_1(z,w)$ on the ball does not behave according to control geometry (in stark contrast to $B_0(z,w)$).

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There is no additional information to be gained by looking at the Bergman projection on $L^2_{p,q}(\Omega)$, so we focus on the $p = 0$ case, except when we investigate the existence of transformation formulas because the $B_{p,0}$ behaves poorly.

We start by carefully constructing B_q, which, while using the well-known Hilbert space and distribution theory, does not seem to appear in the literature. We then exploit Kohn's formula and the knowledge of the $\bar{\partial}$-Neumann problem in the top degree to give a general formula for the Bergman projection B_{n-1} and its associated integral kernel $B_{n-1}(z, w)$. We conclude the paper with a discussion on the ball. We compute B_{n-1} explicitly and then restrict ourselves to the $\mathbb{C}^2$ case. There, we observe that the control geometry, which governs the size of $B_0(z, w)$, does not reflect the scaling present in the kernel $B_1(z, w)$. We conclude with a remark about future directions.

Fix $q \geq 1$. The kernel, $\ker \bar{\partial}_q$, is a closed subspace of $L^2_{0,q}(\Omega)$, so the projection B_q onto $\ker \bar{\partial}_q \cap L^2_{0,q}(\Omega)$ can be given as a Fourier series in terms of a basis. The construction of B_q can proceed as follows: suppose that $\{\phi_j\}_{j=1}^\infty$ is an orthonormal basis of $\ker \bar{\partial}_q \cap L^2_{0,q}(\Omega)$. The vector projection of $f \in L^2_{0,q}(\Omega)$ onto $\operatorname{span} \phi_j$ is $(f, \phi_j)\phi_j$, where the inner product

$$(f, \phi_j) = \int_\Omega \langle f, \phi_j \rangle \, dV = \int_\Omega f \wedge \star\phi_j,$$

where $\star$ is the Hodge-$\star$ operator (see, e.g., [CS01, p. 208]) and dV is Lebesgue measure. The orthogonal projection of f on $\ker \bar{\partial}_q \cap L^2_{0,q}(\Omega)$ is therefore given by the Fourier series

$$B_q f(z) = \sum_{j=1}^\infty (f, \phi_j)\phi_j(z),$$

where the sum converges in $L^2_{0,q}(\Omega)$.

Working formally, we see that

$$B_q f(z) = \sum_{j=1}^\infty \Big(\int_\Omega f(w) \wedge \star\phi_j(w) \Big) \phi_j(z) = \int_\Omega f(w) \wedge \Big(\sum_{j=1}^\infty \star\phi_j(w) \wedge \phi_j(z) \Big).$$

This suggests that the Bergman kernel ought to be

$$B_q(z, w) = \sum_{j=1}^\infty \star\phi_j(w) \wedge \phi_j(z)$$

for *any* orthonormal basis $\{\phi_j\}$ of $\ker \bar{\partial}_q \cap L^2_{0,q}(\Omega)$. For this formula to be rigorous, of course, the sum defining $B_q(\cdot, w)$ must converge in $L^2_{0,q}(\Omega)$, be independent of the orthonormal system $\{\phi_j\}$, and be the orthogonal projection onto $\ker \bar{\partial}_q \cap L^2_{0,q}(\Omega)$. This is contained in Theorem 1.1, our structure theorem for the Bergman projection. To state our results, we need the following notation. Let $\mathcal{I}_q = \{J = (j_1, \dots, j_q) \in \mathbb{N}^q : 1 \leq j_1 < \cdots < j_q \leq n\}$ be the set of increasing q-tuples and let

$$\widetilde{d\bar{z}_j} = d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_j} \wedge \cdots \wedge d\bar{z}_n,$$

where $\widehat{d\bar{z}_j}$ represents the omission of $d\bar{z}_j$ from the wedge product. We will also use $[\hat{I}]$ to denote the $(n - |I|)$-tuple $\{1, \dots, n\} \setminus I$. Also, let $d\bar{z}^J = d\bar{z}_{j_1} \wedge \cdots d\bar{z}_{j_q}$.

Theorem 1.1. *Let $\Omega \subset \mathbb{C}^n$ be a domain, and let $1 \le q \le n-1$. Then:*

(1) *There exists an integral kernel $B_q(z,w)$ so that the Bergman projection $B_q : L^2_{0,q}(\Omega) \to L^2_{0,q}(\Omega) \cap \ker \bar\partial_q$ is given by*
$$B_q f(z) = \int_\Omega f(w) \wedge B_q(z,w)$$
for any $f \in L^2_{0,q}(\Omega)$.

(2) *Moreover, there exist bounded operators $B_{J'J} : L^2(\Omega) \to L^2(\Omega)$ so that if $f = \sum_{J\in\mathcal{I}_q} f_J \, d\bar z^J$, then*
$$B_q f(z) = \sum_{J,J'\in\mathcal{I}_q} B_{J'J} f_J(z) \, d\bar z^{J'}.$$

(3) *Given any orthonormal basis $\{\phi_j\} \subset L^2_{0,q}(\Omega) \cap \ker \bar\partial_q$,*
$$B_q(z,w) = \sum_{j=1}^\infty \star\phi_j(w) \wedge \phi_j(z),$$
where the sum converges in $L^2_{(0,q),(n,n-q)}(\Omega \times \Omega)$.

We have additional information about the operators $B_{J'J}$ in the case that $q = n-1$, though to state the theorem, we need to introduce two pieces of notation. The operator N_n is a $\bar\partial$-Neumann operator of degree n and has integral kernel $N_n(z,w)$. Also, the operator ϑ_q is the formal (or integration by parts) adjoint of $\bar\partial_q$. It differs from the L^2-adjoint of $\bar\partial_q$ because forms in $\mathrm{Dom}(\vartheta_q)$ only have an integrability requirement. Forms in $\mathrm{Dom}(\bar\partial_q^*)$ have both an integrability requirement and a boundary condition.

Theorem 1.2. *Let $\Omega \subset \mathbb{C}^n$ be a domain, and let $G(z,w)$ be the Green's function for the Laplacian $\triangle$. Then*

(1)
$$B_{n-1} f(z) = f(z) - \int_\Omega f(w) \wedge \vartheta_{n-1,z} \partial^*_{n-1,w} N_n(z,w).$$

(2)
$$B_{[\hat k][\hat j]}(z,w) = \delta_{jk}\delta_z(w) + (-1)^{n+j+k-1} 4 \frac{\partial^2 G(z,w)}{\partial z_k \partial \bar w_j}, \tag{1.1}$$
where δ_{jk} is the Kronecker δ and $\delta_z(w)$ is the Dirac δ.

(3) *In the case that $\Omega = B(0,1)$ is the unit ball, then*
$$\begin{aligned} B_{[\hat k][\hat j]}(z,w) &= \delta_{jk}\delta_z(w) \\ &\quad + (-1)^{n+j+k-1}\frac{(n-1)!}{\pi^n}\Bigg[\frac{\delta_{jk}}{|z-w|^{2n}} - n\frac{(z_k-w_k)(\bar z_j - \bar w_j)}{|z-w|^{2n+2}} \\ &\quad - \frac{\delta_{jk} - \bar z_j w_k}{(|z-w|^2 + (1-|w|^2)(1-|z|^2))^n} \\ &\quad + n\frac{((z_k-w_k) + w_k(1-|z|^2))((\bar z_j - \bar w_j) - \bar z_j(1-|w|^2))}{(|z-w|^2 + (1-|w|^2)(1-|z|^2))^{n+1}}\Bigg]. \end{aligned}$$

Our final result is the failure of the boundedness of pointwise evaluation in $L^2_{0,q}(\Omega) \cap \ker \bar\partial_q$, $q \ge 1$. This result stands in stark contrast to the result for B_0. In fact, boundedness of pointwise evaluation in $L^2(\Omega)$ is a critical fact for B_0

and (more generally) one of the defining assumptions in the expansive theory of reproducing kernel Hilbert spaces; see, e.g., [BTA04]. To observe the first instance of the boundedness of pointwise evaluation in the theory of the Bergman projection, we simply need to recall the standard construction for B_0. This construction works equally well for reproducing kernels in reproducing kernel Hilbert spaces. Suppose that the evaluation functional $e_z(\varphi) = \varphi(z)$ was a bounded, linear functional, i.e., $|e_z(\varphi)| \leq C\|\varphi\|_{L^2_{0,q}(\Omega)}$ for some constant C that may depend on z but not on φ. This would mean that for any $f \in \ker\bar\partial_q \cap L^2_{0,q}(\Omega)$, $|f(z)| \leq C\|f\|_{L^2_{0,q}(\Omega)}$, where $C = C(z)$ does not depend on f. This is critical for the following reason: for any $\{a_j\} \in \ell^2$, $f(z) = \sum_{j=1}^\infty a_j\varphi_j(z) \in \ker\bar\partial_q \cap L^2_{0,q}(\Omega)$, with the consequence that

$$\begin{aligned}
|K(z,z)| &= \sum_{j=1}^\infty |\varphi_j(z)|^2\,dV(z) \\
&= \Big(\sup_{\substack{\{a\}\in\ell^2 \\ \|a\|_{\ell^2}=1}} \Big|\sum_{j=1}^\infty a_j\varphi_j(z)\Big|\Big)^2\,dV(z) \\
&= \sup_{\substack{f\in\ker\bar\partial \\ \|f\|_{L^2}=1}} |f(z)|^2\,dV(z).
\end{aligned}$$

Consequently, boundedness on the diagonal implies finiteness of $\sup_{\substack{f\in\ker\bar\partial \\ \|f\|_{L^2}=1}} |f(z)|$. From Theorem 1.2, it is immediate that $B_{n-1}(z,w)$ blows up as $w \to z$.

Theorem 1.3. *Let $\Omega \subset \mathbb{C}^n$ be a domain. If $1 \leq q \leq n$, then pointwise evaluation is not a bounded, linear functional on $L^2_{0,q}(\Omega) \cap \ker\bar\partial_q$.*

Proof. Since forms are not functions, we consider pointwise evaluation to be the pointwise evaluation functionals $\varphi \mapsto \varphi_J$ for each $J \in \mathcal{I}_q$. Without loss of generality, we may suppose that $0 \in \Omega$. Let $q \geq 1$, $J \in \mathcal{I}_q$, and $\psi \in (C^\infty_c)_{0,q-1}(\Omega)$ so that $(\bar\partial\psi(0))_J \neq 0$. Set $\varphi(z) = \frac{\bar\partial\psi(z)}{\|(\bar\partial\psi)_J\|_{L^2(\Omega)}}$. Then $\varphi_\epsilon(z) = \epsilon^{-\frac{n}{2}}\varphi(z/\epsilon) \in (C^\infty_c)_{0,q}(\Omega) \cap \ker\bar\partial_q$ since $\bar\partial^2 = 0$. Moreover, our normalization ensure $\|(\varphi_\epsilon)_J\|_{L^2(\Omega)} = 1$ for all $\epsilon > 0$, but $|(\varphi_\epsilon)_J(z)| \to \infty$ as $\epsilon \to 0$. $\square$

Remark 1.4. It is very unlikely that the Bergman kernel $B_{p,q}(z,w)$ satisfies a nice transformation formula under biholomorphisms unless $p, q \in \{0, n\}$. The transformation law for B_0 essentially follows from the pullback relationship $F^*\bar\partial = \bar\partial F^*$ and the fact that $J_\mathbb{R}F = |J_\mathbb{C}F|^2$, where $J_\mathbb{R}F$ is the determinant of the real Jacobian and $J_\mathbb{C}F$ is the determinant of the complex Jacobian. In general, while the pullback interacts nicely with $\bar\partial$, it behaves poorly with respect to L^2-inner products. In particular, if $F : \Omega_1 \to \Omega_2$ is a biholomorphism and $\phi, \psi \in L^2_{p,q}(\Omega_2)$, then

$$\begin{aligned}
&\big(F^*\phi, F^*\psi\big) = \int_{\Omega_1} F^*\phi(w) \wedge \star\big(F^*\psi(w)\big) \\
&= \sum_{\substack{I,I',K\in\mathcal{I}_p \\ J,J',L'\in\mathcal{I}_q}} \int_{\Omega_1} \Big(\phi_{IJ}\circ F(w)\Big)\Big(\overline{\psi_{I',J'}\circ F(w)}\Big)\left|\frac{\partial F^I}{\partial w^K}\right|\left|\overline{\frac{\partial F^J}{\partial w^L}}\right|\left|\overline{\frac{\partial F^{[\hat I]}}{\partial w^{[\hat K]}}}\right|\left|\frac{\partial F^{[\hat J]}}{\partial w^{[\hat L]}}\right| dV(w),
\end{aligned}$$

where $\frac{\partial F^I}{\partial w^K}$ is the $p \times p$ minor of the complex Jacobian of the mapping $F = (F_1, \dots, F_n)$ given by

$$\frac{\partial F^I}{\partial w^K} = \Big(\frac{\partial F_{I_j}}{\partial w_{K_k}}\Big)_{j,k=1}^p,$$

where $I = (I_1, \dots, I_p)$ and $K = (K_1, \dots, K_p)$, and similarly for the other terms. The complicated product of determinants only simplifies dramatically in the cases $p, q \in \{0, n\}$ to $J_{\mathbb{R}}F$ and a change of variables may proceed as in the B_0 case.

1.1. **Existence of the Bergman kernel and the proof of Theorem 1.1.** The proof of Theorem 1.1 requires no complex analysis and follows in a straightforward manner from functional analysis. Consequently, we sketch the details. Suppose $f = \sum_{J \in \mathcal{I}_q} f_J \, d\bar z^J$. For each $J, J' \in \mathcal{I}_q$, let $B_{J'J}$ be the component piece of B_q that takes the $d\bar z^J$ coefficient of f and maps it to the $d\bar z^{J'}$ coefficient of $B_q\{f_J \, d\bar z^J\}$. It follows easily from the boundedness and linearity of B_q that the operator norm $\|B_{J'J}\|_{L^2 \to L^2} \le 1$ and

$$B_q f = \sum_{J, J' \in \mathcal{I}_q} B_{J'J}\{f_J\} \, d\bar z^{J'}.$$

Since the maps $B_{J'J} : L^2(\Omega) \to L^2(\Omega)$ boundedly, they certainly map from $C_c^\infty(\Omega) \to \mathcal{D}'(\Omega)$. Consequently, the Schwartz Kernel Theorem ([Hör90, Theorem 5.2.1]) applies to each $B_{J'J}$. As a result, the Bergman kernel on $(0, q)$-forms exists as a distributional kernel, and we can write (for $f, g \in \mathcal{D}_{0,q}(\Omega)$)

$$(B_q f, g) = \int_\Omega \int_\Omega f(w) \wedge B_q(z, w) \wedge *g(z) \, dV(w) \, dV(z) = K_q(f \otimes g),$$

where the integral is understood in the distributional sense. This establishes parts (1) and (2) of Theorem 1.1

We now turn to establishing greater regularity for $B_q(z, w)$. Let $\{\phi_j\}$ be an orthonormal basis of $L^2_{0,q}(\Omega) \cap \ker \bar\partial_q$,

$$K_N(z, w) = \sum_{j=1}^N *\phi_j(w) \wedge \phi_j(z),$$

and let $\mathcal{K}_N$ be the operator with kernel K_N. We will show that

$$K_N(z, w) \to B_q(z, w) \text{ in } L^2_{(0,q),(n,n-q)}(\Omega).$$

Since $\mathcal{K}_N f \to Bf$ in $L^2_{0,q}(\Omega)$ and $\{\phi_j\}$ are orthogonal, given $\epsilon > 0$, there exists $N' > 0$ so that if $M \ge N \ge N'$, then

$$\begin{aligned}\Big|K_M(f \otimes g) - K_N(f \otimes g)\Big| &= \Big| \int_{\Omega \times \Omega} \sum_{j=N+1}^M f(w) \wedge *\phi_j(w) \wedge \phi_j(z) \wedge *g(z)\Big| \\ &< \epsilon \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.\end{aligned}$$

Consequently, the sequence of operators $\{\mathcal{K}_N\}$ with distributional kernels $\{K_N\}$ forms a Cauchy sequence acting on $L^2_{0,q}(\Omega) \times L^2_{0,q}(\Omega)$ and therefore converges to an operator B' acting on $L^2_{0,q}(\Omega) \times L^2_{0,q}(\Omega)$ and with distributional kernel $B_q(z, w)$. Moreover, since $K_N(z, w)$ forms a Cauchy sequence in $L^2_{0,q}(\Omega) \otimes L^2_{n,n-q}(\Omega)$, it follows that $B_q(z, w) \in L^2_{0,q}(\Omega) \otimes L^2_{n,n-q}(\Omega) \subset L^2_{(0,q),(n,n-q)}(\Omega \times \Omega)$. That this sum is independent of the basis is a standard Hilbert space fact. This concludes the proof of Theorem 1.1.

2. The Bergman projection B_{n-1} and the proof of Theorem 1.2, parts (1) and (2)

Recall that the boundary condition for a form $u = \sum_{J\in\mathcal{I}_q} u_J \, d\bar{z}^J \in L^2_{0,q}(\Omega)$ to be an element of $\mathrm{Dom}(\bar\partial^*)$ is that

$$\sum_{j=1}^n u_{jK} \frac{\partial\rho}{\partial z_j} = 0 \text{ in } \mathrm{b}\Omega \text{ for all } K \in \mathcal{I}_{q-1},$$

where

$$u_{jK} = \sum_{J\in\mathcal{I}_q} \epsilon^{jK}_J u_J.$$

If $q = n$ the boundary requirement is exactly that $u_{\{1,\dots,n\}} \frac{\partial\rho}{\partial z_j} = 0$ for all $j = 1,\dots,n$, i.e., $u = 0$ on $\mathrm{b}\Omega$. This is the Dirichlet boundary condition, and the $\bar\partial$-Neumann problem reduces to the standard Dirichlet problem for the Laplacian. We normalize the Laplacian $\triangle$ so that $\triangle = -4\sum_{j=1}^n \frac{\partial^2}{\partial z_j \partial \bar z_j}$. Consequently, if $G(z,w)$ is the Green's function for the Laplacian on Ω, then the $\bar\partial$-Neumann operator on the top degree is

$$N_n(z,w) = 4G(z,w)\, dw \wedge d\bar z$$

with the notation $dw = dw_1 \wedge \cdots \wedge dw_n$ and $d\bar z = d\bar z_1 \wedge \cdots \wedge d\bar z_n$. The integral operator N_n applied to a $(0,n)$-form $F = f\, d\bar z$ is then

$$N_n F(z) = \int_\Omega F(w) \wedge N_n(z,w) \wedge d\bar z = 4\Big[\int_\Omega f(w) G(z,w)\, dV(w)\Big] d\bar z.$$

Thus we have an explicit integral kernel for N_n for every case for which there is an explicit formula for $G(z,w)$.

Recall Kohn's formula for the Bergman projection:

$$B_q = I - \bar\partial_q^* N_{q+1} \bar\partial_q = I - \vartheta_q N_{q+1} \bar\partial_q.$$

We now compute B_{n-1} and recall that $G(x,y) = 0$ whenever either $x \in \mathrm{b}\Omega$ or $y \in \mathrm{b}\Omega$. Suppose $f \in L^2_{0,n-1}(\Omega)$. Then

$$\begin{aligned} B_{n-1} f(z) &= f(z) - \vartheta_{n-1,z} \int_\Omega \bar\partial_{n-1,w} f(w) \wedge N_n(z,w) \\ &= f(z) - \vartheta_{n-1,z} \int_\Omega f(w) \wedge \partial^*_{n-1,w} N_n(z,w). \end{aligned}$$

We would like to bring the operator $\vartheta_{n-1,z}$ inside the integral, but this requires care because the Newtonian potential on $\mathbb{C}^n$ is

$$\Phi(z) = \frac{(n-2)!}{4\pi^n} \frac{1}{|z|^{2n-2}}$$

and two derivatives mean that the kernel would blow up like a singular integral. In point of fact, this will not cause a problem because derivatives of two derivatives of $\Phi(z)$ generate a Calderón–Zygmund singular integral. But care certainly must be taken! In particular, the Green's function $G(z,w)$ is built from the Newtonian potential and a harmonic function. Therefore, the singularity of $\frac{\partial^2 G(z,w)}{\partial z_j \partial \bar w_k}$ can only come from the $\frac{\partial^2}{\partial z_j \partial \bar{w}_k} \frac{1}{|w-z|^{2n-2}}$, which we now compute:

$$\frac{\partial^2}{\partial z_j \partial \bar{w}_k} \frac{1}{|w-z|^{2n-2}} = (n-1)\frac{\delta_{jk}}{|w-z|^{2n}} - n(n-1)\frac{(w_k - z_k)\overline{(w_j - z_j)}}{|w-z|^{2(n+1)}}.$$

The case $j \neq k$ yields the kernel $\frac{(w_k - z_k)\overline{(w_j - z_j)}}{|w-z|^{2(n+1)}}$ which is a classic Calderón–Zygmund convolution kernel — homogeneous of degree $-2n$ and integrates to 0 over any sphere centered around the origin. The case $j = k$ is only slightly more complicated. Observe that if σ_{2n-1} is the surface area of the unit sphere in $\mathbb{C}^n$, then by symmetry

$$\begin{aligned}\int_{bB(0,1)} \frac{n-1}{|z|^{2n}} - \frac{n(n-1)|z_j|^2}{|z|^{2(n+1)}}\, d\sigma(z) &= (n-1)\sigma_{2n-1} - n(n-1)\int_{bB(0,1)} |z_j|^2\, d\sigma(z)\\ &= (n-1)\sigma_{2n-1} - n(n-1)\int_{bB(0,1)} \frac{1}{n}\sum_{k=1}^n |z_j|^2\, d\sigma(z) = 0.\end{aligned}$$

By homogeneity, the integral is 0 around any sphere, thus we can write

$$B_{n-1}f(z) = f(z) - \int_\Omega f(w) \wedge \vartheta_{n-1,z}\partial^*_{n-1,w} N_n(z,w),$$

where the integral is taken in the sense of (tempered) distributions. A version of this formula (written directly in terms of the Green's function) appears in [Bel92, Theorem 15.3] for domains in $\mathbb{C}$ and the Bergman projection B_0. Breaking down B_{n-1} into its constituent parts, we compute

$$\begin{aligned}-\vartheta_{n-1,z}\partial^*_{n-1,w} N_n(z,w) &= -\vartheta_{n-1,z}\partial^*_{n-1,w}\Big\{4G(z,w)\, dw \wedge d\bar{z}\Big\}\\ &= 4\vartheta_{n-1,z}\Big\{\sum_{k=1}^n (-1)^{k-1}\frac{\partial G(z,w)}{\partial \bar{w}_k}\widetilde{dw_k} \wedge d\bar{z}\Big\}\\ &= (-1)^{n-1}4\sum_{j,k=1}^n (-1)^{j+k}\frac{\partial^2 G(z,w)}{\partial z_j \partial \bar{w}_k}\widetilde{dw_k} \wedge \widetilde{d\bar{z}_j},\end{aligned}$$

from which (1.1) follows.

2.1. **The proof of Theorem 1.2, part (3).** We now restrict ourselves to the case Ω is the unit ball. The Green's function

$$G(z,w) = \Phi(z-w) - \Phi(|w|(z-\tilde{w})) = \Phi(w-z) - \Phi(|z|(w-\tilde{z})),$$

where $\tilde{w} = \frac{w}{|w|^2}$ is the reflection of w across the unit sphere. Since

$$|z|^2|w-\tilde{z}|^2 - |w-z|^2 = |z|^2|w|^2 + 1 - |w|^2 - |z|^2 = (1-|w|^2)(1-|z|^2),$$

it follows that

$$G(z,w) = \frac{(n-2)!}{4\pi^n}\left(\frac{1}{|z-w|^{2n-2}} - \frac{1}{(|z-w|^2 + (1-|w|^2)(1-|z|^2))^{n-1}}\right). \tag{2.1}$$

In this case, note that

$$\frac{\partial G(z,w)}{\partial \bar{w}_k} = \frac{(n-1)!}{4\pi^n}\left(\frac{z_k - w_k}{|z-w|^{2n}} - \frac{z_k - w_k + w_k(1-|z|^2)}{(|z-w|^2 + (1-|w|^2)(1-|z|^2))^n}\right),$$

and so $\frac{\partial G(z,w)}{\partial \bar{w}_k} \equiv 0$ whenever $w \in B(0,1)$ and $z \in \mathrm{b}B(0,1)$ (reflecting the fact that $N_n\bar{\partial}_{n-1} \in \mathrm{Dom}(\bar{\partial}^*_{n-1})$). Also,

$$\begin{aligned}\frac{\partial^2 G(z,w)}{\partial z_j \partial \bar{w}_k} = \frac{(n-1)!}{4\pi^n}\Bigg[&\frac{\delta_{jk}}{|z-w|^{2n}} - n\frac{(z_k-w_k)(\bar{z}_j-\bar{w}_j)}{|z-w|^{2n+2}} \\ &- \frac{\delta_{jk}-\bar{z}_j w_k}{(|z-w|^2+(1-|w|^2)(1-|z|^2))^n} \\ &+ n\frac{((z_k-w_k)+w_k(1-|z|^2))((\bar{z}_j-\bar{w}_j)-\bar{z}_j(1-|w|^2))}{(|z-w|^2+(1-|w|^2)(1-|z|^2))^{n+1}}\Bigg],\end{aligned}$$

from which part (3) of Theorem 1.2 follows.

3. Control geometry and the unit ball in $\mathbb{C}^2$

Observe that if $z \to \mathrm{b}B(0,1)$, then

$$B_{[\hat{k}][\hat{j}]}(z,w) = \delta_{jk}\delta_z(w) - (-1)^{j+k}\frac{1}{\pi^2}\left[\frac{\bar{z}_j w_k}{|z-w|^4} - 2\frac{\bar{z}_j(z_k-w_k)(1-|w|^2)}{|z-w|^6}\right]$$

as $z \to \mathrm{b}B(0,1)$. Let $a_{jk}(z,w) = \frac{\bar{z}_j w_k}{|z-w|^4} - 2\frac{\bar{z}_j(z_k-w_k)(1-|w|^2)}{|z-w|^6}$.

A defining function for $B(0,1)$ is $r(z) = |z|^2 - 1$. Consequently, the $(1,0)$ complex tangential vector field is $L = \bar{z}_2\frac{\partial}{\partial z_1} - \bar{z}_1\frac{\partial}{\partial z_2}$ and the complex normal is given by $S = 2z_1\frac{\partial}{\partial z_1} + 2z_2\frac{\partial}{\partial z_2}$. Observe that $[L,\bar{L}] = -\operatorname{Im} S$. If $z = (0,1)$ and $w = (w_1, 1-h)$, then $a_{1k}((0,1),w) = 0$ and

$$a_{22}\big((0,1),(w_1,1-h)\big) = \frac{1+h}{(|w_1|^2+|h|^2)^2} - \frac{4h\operatorname{Re} h}{(|w_1|^2+|h|^2)^3}$$

and

$$a_{21}\big((0,1),(w_1,1-h)\big) = -\frac{w_1}{(|w_1|^2+|h|^2)^2} + \frac{4w_1\operatorname{Re} h}{(|w_1|^2+|h|^2)^3},$$

while the Bergman kernel

$$B_0\big((0,1),(w_1,1-h)\big) = -\frac{2}{\pi^2\bar{h}^3}.$$

For the proper size estimate comparisons with $B_0(z,w)$, we recall the control metric from [NSW85] and the Bergman kernel estimates of [NRSW89, McN89]. At $(0,1)$, note that $L = \frac{\partial}{\partial z_1}$ and $S = 2\frac{\partial}{\partial z_2}$, which means that the distance from $(0,1)$ in the w_1-direction is weighted by order 1 and in the w_2-direction by order 2. In other words, $d((0,1),(w_1,1-h)) \approx |w_1| + |h|^{1/2}$. It is clear that $a_{2k}(z,w)$ observes different scaling and size estimates that $B_0(z,w)$ as $|w_1|$ appears with the same weighting as $|h|$. Once again, B_1 behaves quite differently than B_0!

4. Conclusion

This paper checks the functional analysis to show that the Bergman projection has a well-defined integral kernel and that $B_{n-1}(z,w)$ is quite computable from the Green's function $G(z,w)$. Of course, computing the Green's function for domains of interest in several complex variables (and domains in general) is a complicated task. We will return to this topic in a future paper, in particular for the $n=2$ case, as we can say much more.

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