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REPRESENTATION THEORY OF $L_k(\mathfrak{osp}(1|2))$ FROM VERTEX TENSOR CATEGORIES AND JACOBI FORMS

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Abstract. The purpose of this work is to illustrate in a family of interesting examples how to study the representation theory of vertex operator superalgebras by combining the theory of vertex algebra extensions and modular forms.

Let $L_k(\mathfrak{osp}(1|2))$ be the simple affine vertex operator superalgebra of $\mathfrak{osp}(1|2)$ at an admissible level k. We use a Jacobi form decomposition to see that this is a vertex operator superalgebra extension of $L_k(\mathfrak{sl}_2) \otimes \mathrm{Vir}(p, (p+p')/2)$ where $k+3/2 = p/(2p')$ and $\mathrm{Vir}(u, v)$ denotes the regular Virasoro vertex operator algebra of central charge $c = 1 - 6(u-v)^2/(uv)$. Especially, for a positive integer k, we get a regular vertex operator superalgebra, and this case is studied further.

The interplay of the theory of vertex algebra extensions and modular data of the vertex operator subalgebra allows us to classify all simple local (untwisted) and Ramond twisted $L_k(\mathfrak{osp}(1|2))$-modules and to obtain their super fusion rules. The latter are obtained in a second way from Verlinde's formula for vertex operator superalgebras. Finally, using again the theory of vertex algebra extensions, we find all simple modules and their fusion rules of the parafermionic coset $C_k = \mathrm{Com}(V_L, L_k(\mathfrak{osp}(1|2)))$, where V_L is the lattice vertex operator algebra of the lattice $L = \sqrt{2k}\mathbb{Z}$.

1. Introduction

Understanding the representation theory of a given vertex operator algebra, especially the tensor structure, is in general a difficult problem. However, if the given vertex operator algebra is related to a known one by a standard construction, then it is conceivable that much of the representation theory of the given vertex operator algebra can be obtained from the known one. In a series of recent works [CKL, CKLR, CKM] we have used vertex tensor categories to derive structural results about vertex operator algebras and their extensions. The starting point is actually the important work of Huang, Kirillov, and Lepowsky [HKL] saying that vertex operator algebra extensions of sufficiently nice vertex operator algebras V are in one-to-one correspondence with certain algebra objects A in the representation category $\mathcal{C}$ of V. Moreover, the (untwisted or local) modules of the extended

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vertex operator algebra are precisely objects in the category $\operatorname{Rep}^0 A$ of the local A-modules. This generalizes straightforwardly to extensions that are vertex operator superalgebras [CKL]. There is then an induction functor $\mathcal{F}$ (a tensor functor) that maps any object X of the base category $\mathcal{C}$ to a not necessarily local (super)algebra object $\mathcal{F}(X) \cong_{\mathcal{C}} A \boxtimes_{\mathcal{C}} X$; see [KO]. In a recent work [CKM], the induction functor was studied from the vertex operator algebra perspective. Most notably it was shown that $\operatorname{Rep}^0 A$ is braided equivalent to the category of extended vertex operator superalgebra modules and in addition the tensor product on $\operatorname{Rep}^0 A$ is exactly the $P(1)$-tensor product as defined in [HLZ] if mapping to local objects. The purpose of this work is to illustrate the usefulness of these recent results in very efficiently understanding the representation theories of two families of vertex operator superalgebras associated to $\mathfrak{osp}(1|2)$.

1.1. **The affine vertex operator superalgebra $L_k\left(\mathfrak{osp}(1|2)\right)$.** It is well-known that the simple affine vertex operator algebra of a simple Lie algebra $\mathfrak{g}$ of level k is regular if and only if k is a positive integer [FZ]. It is a presently unproven belief that the same statement holds for affine vertex operator superalgebras $L_k(\mathfrak{osp}(1|2n))$ and that these are the only affine vertex operator superalgebras that are regular. We note that fusion rules associated to $\widehat{\mathfrak{osp}(1|2)}$ using coinvariants have been obtained in [IK2]. This approach is believed but not known to give the vertex operator algebra fusion rules.

Let $\mathcal{L}_k$ be the simple affine vertex operator superalgebra of $\mathfrak{osp}(1|2)$ at level k and $\mathcal{L}_k^{\text{even}}$ its even component. Characters of irreducible highest-weight modules of $\widehat{\mathfrak{osp}(1|2)}$ for admissible level k are known [KW]. They converge in certain domains and allow for a meromorphic continuation to components of vector-valued Jacobi forms. It is then a Jacobi form decomposition problem to express these characters in terms of characters of $L_k(\mathfrak{sl}_2) \otimes \operatorname{Vir}(p, \Delta/2)$. Here, $\operatorname{Vir}(u, v)$ denotes the simple and rational Virasoro vertex operator algebra of central charge $1-6(u-v)^2/uv$ and the parameters are related to the level via $k+\frac{3}{2} = \frac{p}{2p'}$ and $\Delta = p+p'$. Our starting point is a character decomposition of simple highest-weight modules for $\widehat{\mathfrak{osp}(1|2)}$; see Lemma 3.1. Since characters of non-isomorphic simple modules for both $L_k(\mathfrak{sl}_2)$ and $M(p, \Delta/2)$ are linearly independent, we immediately get as a corollary that

$$\operatorname{Com}\left(L_k(\mathfrak{sl}_2), \mathcal{L}_k\right) = \operatorname{Vir}(p, \Delta/2) \qquad \text{and} \qquad \operatorname{Com}\left(\operatorname{Vir}(p, \Delta/2), \mathcal{L}_k\right) = L_k(\mathfrak{sl}_2).$$

This result appeared in the physics literature [CRS] and also in [CL].

Now let $k \in \mathbb{Z}_{>0}$. It is easy to see that $\mathcal{L}_k$ is C_2-cofinite. Moreover, since an extension of a rational and C_2-cofinite vertex operator algebra is rational, by [DH, HKL, KO] another corollary is that for k in $\mathbb{Z}_{>0}$ the vertex operator superalgebra $\mathcal{L}_k$ is rational and C_2-cofinite. We then use the induction functors from $L_k(\mathfrak{sl}_2) \otimes \operatorname{Vir}(p, \Delta/2)$ to $\mathcal{L}_k^{\text{even}}$ and to $\mathcal{L}_k$ to find a list of simple $\mathcal{L}_k^{\text{even}}$ modules. There is then a powerful result [DMNO] that relates the Frobenius-Perron dimension of the base category to the Frobenius-Perron dimension of the category of local modules for the algebra object. This dimension can be computed using the categorical S-matrix, that is, the Hopf link, but this is related to the modular S-matrix due to Huang's theorem [H1, H2]. We then use [DMNO] together with the modular data of $L_k(\mathfrak{sl}_2) \otimes \operatorname{Vir}(p, \Delta/2)$ to prove that we have found all simple $\mathcal{L}_k^{\text{even}}$-modules. This immediately gives a complete list of simple local and Ramond twisted $\mathcal{L}_k$-modules. All these results are obtained in Subsection 4.1. The properties of the induction functor derived in [CKM] then give us the fusion rules of simple $\mathcal{L}_k$-modules (Subsection 4.2) and modular S-matrix (Subsection 4.3). Alternatively, the fusion rules can be computed using Verlinde's formula for vertex operator superalgebras as derived in [CKM] as a consequence of the Verlinde formula for regular vertex operator algebras due to Huang [H1, H2].

Extensions of these methods to admissible but non-integrable levels are under investigation. For an interesting recent analysis of level $-5/4$ using methods different from ours see [RSW].

1.2. **The $L_k(\mathfrak{osp}(1|2))$ parafermion vertex operator algebra.** The $L_k(\mathfrak{osp}(1|2))$ parafermion vertex operator algebras are called graded parafermions in physics [CRS, FMW]. Let V be a regular vertex operator (super)algebra containing a lattice vertex operator algebra V_L of a positive definite lattice L as a subvertex algebra. The coset $\mathrm{Com}(V_L, V)$ is called the parafermion algebra of V. The best known case is if V is a rational affine vertex operator algebra. In that case C_2-cofiniteness [ALY, DLY, DW] and rationality [DR] have been understood for a few years. If V is the rational Bershadsky-Polyakov algebra, then these two properties have been proven indirectly [ACL]. The very recent Theorem 4.12 of [CKLR] settles this statement in generality, and so we have that $C_k := \mathrm{Com}\left(V_L, L_k\left(\mathfrak{osp}(1|2)\right)\right)$ is regular. Here, the lattice is $L = \sqrt{2k}\mathbb{Z}$. Section 4 of [CKM] then allows us to deduce a complete list of inequivalent modules $C_{\lambda,r}$. These modules are parameterized by λ in L'/L and r labelling the local $\mathcal{L}_k$-modules. Their fusion rules are

$$C_{\nu,r} \boxtimes C_{\lambda,r'} \cong \bigoplus_{r''=1}^{2k+2} {N_{r,r'}}^{r''} C_{\lambda+\nu,r''},$$

with ${N_{r,r'}}^{r''}$ the fusion structure constants of $\mathcal{L}_k$ derived earlier. One can also express characters of the $C_{\lambda,r}$ and their modular transformations immediately in terms of those of $\mathcal{L}_k$. All these results are specific instances of results in Section 4 of [CKM] and are presented in Section 5.

2. A summary of certain results for vertex operator (super)algebra extensions

For the ease of exposition, in this section we shall restrict ourselves to vertex operator algebra extensions. All the relevant theorems in [CKM] have been formulated in the super-case.

Let V be a C_2-cofinite vertex operator algebra of CFT-type (that is, V is self-contragredient, has no weight spaces of negative conformal weight, and the zeroth weight space is one dimensional) and let $\mathcal{C}$ be the category of its finite length modules. Due to [H3], $\mathcal{C}$ has a structure of a vertex tensor category in the sense of Huang-Lepowsky [HL1], in particular a braided tensor structure which is essentially all that we require in the present paper. The interested reader may see [HL2], [CKM, Sec. 3.3], and [EGNO, BK] as further references.

Denote the monoidal product on $\mathcal{C}$ by $\boxtimes$, unit object by $\mathbf{1}$ (note that $\mathbf{1} = V$, [HLZ, Sec. 12.2]), associativity isomorphisms by $\mathcal{A}_{X,Y,Z} : X \boxtimes (Y \boxtimes Z) \xrightarrow{\cong} (X \boxtimes Y) \boxtimes Z$, braiding by $\mathcal{R}_{X,Y} : X \boxtimes Y \xrightarrow{\cong} Y \boxtimes X$, and unit isomorphisms by $l_X : \mathbf{1} \boxtimes X \xrightarrow{\cong} X$ and $r_X : X \boxtimes \mathbf{1} \xrightarrow{\cong} X$. Vertex tensor categories always have a balancing isomorphism, also called twist, denoted by θ, given by the action of

$e^{2\pi i L(0)}$ ($L(0)$ is the Virasoro zero mode) satisfying $\theta_{X\boxtimes Y} = \mathcal{R}_{Y,X}\circ\mathcal{R}_{X,Y}\circ(\theta_X\boxtimes\theta_Y)$ for all objects X, Y of $\mathcal{C}$.

Definition 2.1 ([KO, Def. 1.1])**.** (A, μ, ι) (where $A \in \mathrm{Obj}(\mathcal{C})$, $\mu : A \boxtimes A \to A$, $\iota : \mathbf{1} \hookrightarrow A$) is an algebra object if: (1) μ is associative: $\mu \circ (\mathrm{Id}_A \boxtimes \mu) = \mu \circ (\mu \boxtimes \mathrm{Id}_A) \circ \mathcal{A}_{A,A,A} : A \boxtimes (A \boxtimes A) \to A$. (2) μ is commutative: $\mu \circ \mathcal{R}_{A,A} = \mu : A \boxtimes A \to A$. (3) μ is unital: $\mu \circ (\iota \boxtimes \mathrm{Id}_A) = l_A : \mathbf{1} \boxtimes A \to A$. See [CKM, Def. 2.25] for the notion of a superalgebra object, which essentially changes commutativity to a supercommutativity.

Definition 2.2 ([KO, Defs. 1.2, 1.8])**.** Let $\mathrm{Rep}\, A$ be the category whose objects are (X, μ_X) with $X \in \mathrm{Obj}(\mathcal{C})$ and $\mu_X : A \boxtimes X \to X$ such that: (1) μ_X is associative: $\mu_X \circ (\mathrm{Id}_A \boxtimes \mu_X) = \mu_X \circ (\mu \boxtimes \mathrm{Id}_X) \circ \mathcal{A}_{A,A,X} : A \boxtimes (A \boxtimes X) \to X$. (2) μ_X is unital: $\mu_X \circ (\iota \boxtimes \mathrm{Id}_X) = l_X : \mathbf{1} \boxtimes X \to X$. The morphisms in $\mathrm{Rep}\, A$ are those morphisms in $\mathcal{C}$ which commute with the algebra actions μ_X. Let $\mathrm{Rep}^0 A$ be the full subcategory of $\mathrm{Rep}\, A$ whose objects satisfy $\mu_X \circ \mathcal{R}_{X,A} \circ \mathcal{R}_{A,X} = \mu_X : A \boxtimes X \to X$. Objects of $\mathrm{Rep}^0 A$ are often called local A-modules.

Theorem 2.3 ([HKL])**.** *The algebra objects A with trivial twist, i.e., $\theta_A = \mathrm{Id}_A$, are in one-to-one correspondence with conformal embeddings $V \hookrightarrow A$, and the category $\mathrm{Rep}^0 A$ is equivalent as an abelian category to the category of untwisted modules of A as a vertex operator algebra.*

Note that the relevant theorems in [HKL] are stated for haploid algebras (i.e., when one has $\hom(\mathbf{1}, A) \cong \mathbb{C}$) when $\mathcal{C}$ is semi-simple, but these assumptions are not needed in their proof. Mild generalizations of these theorems to the superalgebra case may be found in [CKL].

Now we let A be an algebra object with $\theta_A = \mathrm{Id}_A$.

Recall from [KO, Thm. 1.5] that $\mathrm{Rep}\, A$ can be endowed with a monoidal structure, say $\boxtimes_A$, with A as unit object. Under this structure, $\mathrm{Rep}^0 A$ is a submonoidal category and moreover it is also braided in a natural way. For a generalization of this to super and not necessarily semi-simple cases, see [CKM, Sec. 2.6].

Theorem 2.4 ([CKM, Thm. 3.65])**.** *The category $\mathrm{Rep}^0 A$ endowed with the tensor product $\boxtimes_A$ is equivalent as a braided tensor category to the category of untwisted modules for A viewed as a vertex operator algebra equipped with the $P(1)$-tensor product defined as in* [HLZ, Sec. 4.1].

Note that [CKM, Thm. 3.65] is actually formulated in the generality of A being a superalgebra.

Definition 2.5 ([KO, Thm. 1.6])**.** We have the induction functor $\mathcal{F} : \mathcal{C} \to \mathrm{Rep}\, A$, where $\mathcal{F}(X) = A \boxtimes X$ and $\mu_{\mathcal{F}(X)} = (\mu \otimes \mathrm{Id}_X) \circ \mathcal{A}_{A,A,X}$.

We have the following crucial property of the induction functor which we shall use frequently:

Theorem 2.6 ([CKM, Thm. 2.59])**.** *The induction functor is monoidal, i.e.,*

$$\mathcal{F}(X \boxtimes Y) \cong \mathcal{F}(X) \boxtimes_A \mathcal{F}(Y).$$

We also recall the following useful results for the induction functor.

Theorem 2.7. *Let A be an algebra in $\mathcal{C}$ with a trivial twist and let X be a simple object of $\mathcal{C}$.*

(1) *$\mathcal{F}(X) \in \mathrm{Rep}^0 A$ if $A \boxtimes X$ is graded by a single coset of integers. This is an equivalent formulation of* [CKM, Prop. 2.65].
(2) *Suppose that V is simple, A is simple as a module over itself, and $A \cong \oplus_{i \in I} A_i$ where each A_i is a simple V-module. If $A_i \boxtimes X$ are simple and mutually inequivalent V-modules, then $\mathcal{F}(X)$ is a simple object of* $\mathrm{Rep}\, A$ (*see* [CKM, Thm. 4.4]).

Now, let V be rational, C_2-cofinite of CFT-type. Then, $\mathcal{C}$ is semi-simple, has finitely many isomorphism classes of simples, and by deep results of Huang [H1,H2], it is in fact rigid (i.e., each object X has a dual X^*; see [BK, Def. 2.1.1]) and modular (see [BK, Def. 3.1.1]). For rigid tensor categories with a twist, one can define the notion of trace; see [BK, Def. 2.3.3]. The Hopf links are defined as the following traces:

$$S^{\infty}_{X,Y} = \mathrm{trace}_{X \boxtimes Y}(\mathcal{R}_{Y,X} \circ \mathcal{R}_{X,Y}). \tag{1}$$

$\mathcal{C}$ is called modular if the S-matrix formed by letting X, Y vary over isomorphism classes of simple objects is invertible [BK, Def. 3.1.1].

Below, we shall require [KO, Thm. 4.5], which provides conditions under which $\mathrm{Rep}^0 A$ is also a modular tensor category. [CKM, Prop. 2.89] lets us deduce the relation that the induction functor $\mathcal{F}$ respects S-matrices, and, moreover, [CKM, Sec. 4.2] lets us deduce the Verlinde data. We also require the notion of Frobenius-Perron dimensions (which are recalled below), an excellent resource for which is [EGNO, Sec. 3.2].

3. Decomposition of Jacobi forms associated to $\mathfrak{osp}(1|2)$

We begin by recalling the necessary known data of the relevant vertex operator algebras and Lie superalgebras from the literature.

3.1. Characters of affine Lie superalgebra $\widehat{\mathfrak{osp}(1|2)}$ modules.

The conformal field theory pendant of the affine vertex operator superalgebra of $\widehat{\mathfrak{osp}(1|2)}$ has appeared in [ENO,ERS]. It will turn out that all we need in order to understand much of the representation theory of admissible integer level $L_k(\mathfrak{osp}(1|2))$ is a certain decomposition of the vacuum module character. Strong reconstruction theorem gives that the simple quotient of the level k vacuum Verma module of an affine Lie superalgebra carries the structure of a simple vertex operator superalgebra [FBZ]. By admissible levels we mean those where the vacuum Verma module itself is not simple and the normalized character of its simple quotient is a modular function; see [KW]. In the case of $\widehat{\mathfrak{osp}(1|2)}$ these are

$$k + \frac{3}{2} = \frac{p}{2p'}, \tag{2}$$

where $p > 1$ and p' are positive coprime integers with $p + p' \in 2\mathbb{Z}$ and p, $\frac{p+p'}{2}$ coprime. The highest-weight representations $M_{j_{r,s}}$ are indexed by isospins $j_{r,s}$, with

$$1 \le r \le p-1, \qquad 0 \le s \le p'-1, \qquad \text{and} \qquad r + s \in 2\mathbb{Z}+1.$$

From [ERS,KW], the characters of these representations are given by

$$\mathrm{ch}[M_{j_{r,s}}] = \frac{\Theta_{b_+,a}\left(\frac{z}{2p'}, \frac{\tau}{2}\right) - \Theta_{b_-,a}\left(\frac{z}{2p'}, \frac{\tau}{2}\right)}{\Pi(z,t)} \tag{3}$$

with standard Jacobi theta functions

$$\Theta_{r,s}(z,\tau) = \sum_{m\in\mathbb{Z}} w^{s(m+\frac{r}{2s})} q^{s(m+\frac{r}{2s})^2}$$

and the Weyl superdenominator

$$\begin{aligned}\Pi(z,\tau) &= \Theta_{1,3}\left(\frac{z}{2},\frac{\tau}{2}\right) - \Theta_{-1,3}\left(\frac{z}{2},\frac{\tau}{2}\right)\\ &= w^{\frac14} q^{\frac{1}{24}} \prod_{n=1}^{\infty} \frac{(1-q^n)(1-wq^n)\left(1-w^{-1}q^{n-1}\right)}{\left(1+w^{\frac12}q^n\right)\left(1+w^{-\frac12}q^{n-1}\right)}\end{aligned}$$

$$\text{with} \quad w = e^{2\pi i z}, \qquad q = e^{2\pi i\tau}, \qquad b_\pm = \pm p'r - ps, \qquad \text{and} \qquad a = pp'.$$

These characters are analytic functions in the domain $1 \le |w| \le |q^{-1}|$ and can be meromorphically continued to meromorphic Jacobi forms. Their modular transformations are a straightforward computation: we will obtain them in the positive integer level case directly from vertex tensor category theory.

3.2. **The rational Virasoro vertex operator algebra** $\mathrm{Vir}(p,u)$**.** A good reference here is [RC, IK1]. Let $u, p \in \mathbb{Z}_{\ge 2}$ be coprime. Then, the simple Virasoro vertex operator algebra at central charge

$$c = 1 - 6\frac{(u-p)^2}{up}$$

is regular [W]. Simple modules are denoted by $V_{r,s}$ for $1 \le r \le u-1$ and $1 \le s \le p-1$ and one has the relations $V_{r,s} \cong V_{u-r,p-s}$. We denote the set of inequivalent module labels by $I_{u,p}$. We define numbers by

$$N^{w\ \ t''}_{t,t'} = \begin{cases} 1 & \text{if } |t-t'|+1 \le t'' \le \min\{t+t'-1, 2w-t-t'\}, t+t'+t'' \text{ odd},\\ 0 & \text{otherwise}\end{cases}$$

that allow us to express the fusion rules as follows:

$$V_{r,s} \boxtimes_{\mathrm{Vir}} V_{r',s'} \cong \bigoplus_{r''=1}^{u-1} \bigoplus_{s''=1}^{p-1} N^{u\ \ r''}_{r,r'} N^{p\ \ s''}_{s,s'} V_{r'',s''}.$$

With p, p' as in Subsection 3.1 set $\Delta = p + p'$ and $u = \Delta/2$. Then characters are [RC, IK1]

$$\mathrm{ch}[V_{r,s}] = \frac{\left(\Theta_{2pr-\Delta s,\Delta p} - \Theta_{-2pr-\Delta s,\Delta p}\right)\left(0,\frac{\tau}{2}\right)}{\eta(q)}, \tag{4}$$

and their modular S-transformation is

$$\mathrm{ch}[V_{r,s}]\left(-\frac{1}{\tau}\right) = \sum_{(r',s')\in I_{u,p}} S^{\chi}_{(r,s),(r',s')}\,\mathrm{ch}[V_{r',s'}](\tau) \tag{5}$$

with modular S-matrix entries

$$S^{\chi}_{(r,s),(r',s')} = -2\sqrt{\frac{2}{up}}(-1)^{rs'+sr'} \sin\left(\frac{\pi p}{u} rr'\right)\sin\left(\frac{\pi u}{p} ss'\right). \tag{6}$$

The conformal weights of $V_{r,s}$ are given by $h_{r,s} = \frac{(us-pr)^2-(u-p)^2}{4up}$.

3.3. **Characters of admissible level $L_k(\mathfrak{sl}_2)$ modules.** We use [KW, CR2] as references. Let the level k be as in (2); i.e., k satisfies

$$k+2 = k+\frac{3}{2}+\frac{1}{2} = \frac{p}{2p'}+\frac{1}{2} = \frac{p+p'}{2p'} = \frac{u}{p'}.$$

The highest-weight representations of $L_k(\mathfrak{sl}_2)$ are $\mathcal{D}^+_{r,s}$ with $1 \le r \le u-1$ and $0 \le s \le p'-1$ where $\mathcal{D}^+_{r,0} ;= \mathcal{L}_{r,0}$ deserve a special name as they have finite-dimensional grade zero subspace. Let

$$\vartheta_2(z,\tau) = \sum_{n\in\mathbb{Z}} w^{\left(n+\frac{1}{2}\right)} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}, \qquad \vartheta_1(z,\tau) = -\vartheta_2\left(z+\frac{1}{2},\tau\right)$$

be the standard theta functions; then characters are

$$\mathrm{ch}[\mathcal{D}^+_{r,s}](z,\tau) = \frac{\left(\Theta_{2p'r-\Delta s,\Delta p'} - \Theta_{-2p'r-\Delta s,\Delta p'}\right)\left(\frac{z}{2p'},\frac{\tau}{2}\right)}{\mathfrak{i}\vartheta_1(w,q)}.$$

If $k \in \mathbb{Z}_{>0}$, then the $\mathcal{D}^+_{r,0} := \mathcal{L}_{r,0}$ with $1 \le r \le k+1$ of conformal dimensions $\dfrac{r^2-1}{4(k+2)}$ exhaust inequivalent simple modules up to isomorphisms, and their modular S-transformations are given by

$$\mathrm{ch}[\mathcal{L}_{r,0}]\left(0,-\frac{1}{\tau}\right) = \sum_{r'=1}^{k+1} S^{\chi,\mathfrak{sl}_2}_{r,r'}\,\mathrm{ch}[\mathcal{L}_{r',0}](0,\tau) \tag{7}$$

$$\text{with} \qquad S^{\chi,\mathfrak{sl}_2}_{r,r'} = \sqrt{\frac{2}{k+2}}\sin\left(\frac{\pi}{k+2}rr'\right).$$

Lemma 3.1. *For k as in* (2)*, we have the following character decomposition:*

$$\mathrm{ch}[M_{j_{r,s}}](w,q) = \sum_{i=1}^{\frac{\Delta}{2}-1} \mathrm{ch}[\mathcal{D}^+_{i,s}](w^{\frac{1}{2}},q)\,\mathrm{ch}[\mathrm{Vir}^{(p,\frac{\Delta}{2})}_{i,r}](q).$$

Proof. We compare left- and right-hand sides each multiplied by $\vartheta_2\left(\frac{z}{2},\tau\right)$:

$$\begin{aligned}
\Theta_{b_\pm,a}\left(\frac{z}{2p'},\frac{\tau}{2}\right)\vartheta_2\left(\frac{z}{2},\tau\right) &= \sum_{m,n\in\mathbb{Z}} w^{\frac{a}{2p'}\left(m+\frac{b_\pm}{2a}\right)+\frac{1}{2}\left(n+\frac{1}{2}\right)} q^{\frac{a}{2}\left(m+\frac{b_\pm}{2a}\right)^2+\frac{1}{2}\left(n+\frac{1}{2}\right)^2}\\
&= \sum_{m,n\in\mathbb{Z}} w^{\frac{p}{2}\left(m+\frac{b_\pm}{2a}\right)+\frac{1}{2}\left(n+\frac{1}{2}\right)} q^{\frac{pp'}{2}\left(m+\frac{b_\pm}{2pp'}\right)^2+\frac{1}{2}\left(n+\frac{1}{2}\right)^2}\\
&= \sum_{m,n\in\mathbb{Z}} w^{\frac{1}{2}\left(pm+\frac{b_\pm+p'}{2p'}+n\right)} q^{\frac{1}{2}\left(pp'm^2+mb_\pm+\frac{b_\pm^2}{4pp'}+n^2+n+\frac{1}{4}\right)}.
\end{aligned} \tag{8}$$

At this point, we make a change of variables. Let $x = pm+n$. We wish to find $y = cm+dn$ such that $pp'm^2+n^2$ has no mixed terms in x and y. Let $\Delta = c-dp$ be the determinant of this change of variables. Then $m = \Delta^{-1}(y-dx)$ and $n = \Delta^{-1}(cx-py)$. From this, we see that

$$pp'm^2+n^2 = \frac{\left(pp'd^2+c^2\right)x^2+p\left(p'+p\right)y^2-p\left(c+p'd\right)xy}{\Delta^2}. \tag{9}$$

So, we see that $c+p'd$ needs to vanish. Choosing $c=p'$ and $d=-1$, this condition is fulfilled. Therefore, $\Delta = p+p'$. The change of variables becomes $m = \Delta^{-1}(x+y)$

and $n = \Delta^{-1}(p'x - py)$. With the observations that $p' = -p \pmod{\Delta}$ and p is relatively prime to Δ, we see that this change of variables is an invertible map $\varphi : \mathbb{Z}^2 \to L = \left\{(x,y) \in \mathbb{Z}^2 \middle| x + y \in \Delta\mathbb{Z}\right\}$. So, one may simply sum over L in x and y. Simplifying (9), we now have

$$pp'm^2 + n^2 = \frac{p'}{\Delta}x^2 + \frac{p}{\Delta}y^2.$$

From this, one easily obtains the exponent for q:

$$pp'm^2 + n^2 + mb_\pm + \frac{b_\pm^2 + pp'}{4pp'} + n = \frac{p'}{\Delta}\left(x + \frac{b_\pm + p'}{2p'}\right)^2 + \frac{p}{\Delta}\left(y + \frac{b_\pm - p}{2p}\right)^2.$$

Returning to (8) and letting $x = \Delta s + i$ and $y = \Delta t - i$, for $i = 1, \ldots, \Delta$, we get

$$\begin{aligned}
\Theta_{b_\pm,a}\left(\frac{z}{2p'}, \frac{\tau}{2}\right)\vartheta_2\left(\frac{z}{2}, \tau\right) &= \sum_{(x,y)\in L} w^{\frac{1}{2}\left(x + \frac{b_\pm + p'}{2p'}\right)} q^{\frac{p'}{2\Delta}\left(x + \frac{b_\pm + p'}{2p'}\right)^2} q^{\frac{p}{2\Delta}\left(y + \frac{b_\pm - p}{2p}\right)^2} \\
&= \sum_{i=1}^{\Delta}\left(\sum_{s\in\mathbb{Z}} w^{\frac{\Delta p'}{2p'}\left(s + \frac{b_\pm + p'(2i+1)}{2\Delta p'}\right)} q^{\frac{\Delta p'}{2}\left(s + \frac{b_\pm + p'(2i+1)}{2\Delta p'}\right)^2}\right)\left(\sum_{t\in\mathbb{Z}} q^{\frac{\Delta p}{2}\left(t + \frac{b_\pm - p(2i+1)}{2\Delta p}\right)^2}\right) \\
&= \sum_{i=1}^{\Delta}\Theta_{b_\pm + p'(2i+1),\Delta p'}\left(\frac{z}{2p'}, \frac{\tau}{2}\right)\Theta_{b_\pm - p(2i+1),\Delta p}\left(0, \frac{\tau}{2}\right) \\
&= \sum_{i=1}^{\Delta}\Theta_{-ps + p'(2i\pm r+1),\Delta p'}\left(\frac{z}{2p'}, \frac{\tau}{2}\right)\Theta_{\pm p'r - p(2i+s+1),\Delta p}\left(0, \frac{\tau}{2}\right).
\end{aligned} \tag{10}$$

Note that the theta-functions in (10) are Δ-periodic in i, so the whole sum in (10) is 1-periodic in i. Next, we want to consider the difference between the two results, according to (3).

Let $X := \left(\Theta_{b_+,a}\left(\frac{z}{2p'}, \frac{\tau}{2}\right) - \Theta_{b_-,a}\left(\frac{z}{2p'}, \frac{\tau}{2}\right)\right)\vartheta_2\left(\frac{z}{2}, \tau\right)$, which expands to

$$\begin{aligned}
X = &\sum_{i=1}^{\Delta}\Theta_{-ps + p'(2i+r+1),\Delta p'}\left(\frac{z}{2p'}, \frac{\tau}{2}\right)\Theta_{p'r - p(2i+s+1),\Delta p}\left(0, \frac{\tau}{2}\right) \\
&- \sum_{i=1}^{\Delta}\Theta_{-ps + p'(2i-r+1),\Delta p'}\left(\frac{z}{2p'}, \frac{\tau}{2}\right)\Theta_{-p'r - p(2i+s+1),\Delta p}\left(0, \frac{\tau}{2}\right).
\end{aligned}$$

After applying some symmetries to the second term, such as negation of the second factor's first index, and mapping $i \mapsto -i-s-1$ we get

$$X = \sum_{i=1}^{\Delta} \left(\Theta_{-ps+p'(2i+r+1),\Delta p'}\left(\frac{z}{2p'}, \frac{\tau}{2}\right) - \Theta_{-ps-p'(2i+r+2s+1),\Delta p'}\left(\frac{z}{2p'}, \frac{\tau}{2}\right)\right) \tag{11}$$
$$\cdot\, \Theta_{p'r-p(2i+s+1),\Delta p}\left(0, \frac{\tau}{2}\right)$$

$$= \sum_{i=1}^{\Delta} \left(\Theta_{-\Delta s+p'(2i+r+s+1),\Delta p'}\left(\frac{z}{2p'}, \frac{\tau}{2}\right) - \Theta_{-\Delta s-p'(2i+r+s+1),\Delta p'}\left(\frac{z}{2p'}, \frac{\tau}{2}\right)\right) \tag{12}$$
$$\cdot\, \Theta_{p'r-p(2i+s+1),\Delta p}\left(0, \frac{\tau}{2}\right).$$

By assumption, $r+s \in 2\mathbb{Z}+1$. Thus, letting $t = \frac{r+s+1}{2}$, we have

$$X = \sum_{i=1}^{\Delta} \left(\Theta_{-\Delta s+2p'(i+t),\Delta p'}\left(\frac{z}{2p'}, \frac{\tau}{2}\right) - \Theta_{-\Delta s-2p'(i+t),\Delta p'}\left(\frac{z}{2p'}, \frac{\tau}{2}\right)\right)$$
$$\cdot\, \Theta_{p'r-p(2i+s+1),\Delta p}\left(0, \frac{\tau}{2}\right) \tag{13}$$
$$= \sum_{i=1}^{\Delta} \left(\Theta_{2p'i-\Delta s,\Delta p'}\left(\frac{z}{2p'}, \frac{\tau}{2}\right) - \Theta_{-2p'i-\Delta s,\Delta p'}\left(\frac{z}{2p'}, \frac{\tau}{2}\right)\right) \Theta_{\Delta r-2pi,\Delta p}\left(0, \frac{\tau}{2}\right).$$

For $i = \Delta, \frac{\Delta}{2}$, the first factor vanishes. We can combine summands to obtain

$$X = \sum_{i=1}^{\frac{\Delta}{2}-1} \left(\Theta_{2p'i-\Delta s,\Delta p'}\left(\frac{z}{2p'}, \frac{\tau}{2}\right) - \Theta_{-2p'i-\Delta s,\Delta p'}\left(\frac{z}{2p'}, \frac{\tau}{2}\right)\right) \Theta_{\Delta r-2pi,\Delta p}\left(0, \frac{\tau}{2}\right)$$
$$- \sum_{i=1}^{\frac{\Delta}{2}-1} \left(\Theta_{2p'i-\Delta s,\Delta p'}\left(\frac{z}{2p'}, \frac{\tau}{2}\right) - \Theta_{-2p'i-\Delta s,\Delta p'}\left(\frac{z}{2p'}, \frac{\tau}{2}\right)\right) \Theta_{\Delta r+2pi,\Delta p}\left(0, \frac{\tau}{2}\right)$$
$$= \sum_{i=1}^{\frac{\Delta}{2}-1} \left(\Theta_{2p'i-\Delta s,\Delta p'} - \Theta_{-2p'i-\Delta s,\Delta p'}\right)\left(\frac{z}{2p'}, \frac{\tau}{2}\right)$$
$$\cdot \left(\Theta_{2pi-\Delta r,\Delta p} - \Theta_{-2pi-\Delta r,\Delta p}\right)\left(0, \frac{\tau}{2}\right).$$

Comparing with the $L_k(\mathfrak{sl}_2)$ and Virasoro characters now easily gives the claim. $\square$

Since characters of modules in the category $\mathcal{O}$ of $L_k(\mathfrak{sl}_2)$ at admissible level and those of the rational Virasoro vertex algebra both determine representations, we have the analogous result on the level of modules:

$$M_{j_{r,s}} = \bigoplus_{i=1}^{\frac{\Delta}{2}-1} \mathcal{D}^+_{i,s} \otimes V_{i,r}. \tag{14}$$

Here we remark that in general, characters of simple vertex operator algebra modules are not necessarily linearly independent, and in the case $L_k(\mathfrak{sl}_2)$ for $k \in \mathbb{Q}\backslash\mathbb{Z}$ this question is subtle. In that case there are simple modules whose characters only converge to an analytic function in a certain domain (depending on the module). These characters can then be meromorphically continued to meromorphic Jacobi forms, and many different module characters will have the same meromorphic continuation. These modules are then distinguished by the domain in which the character converges to an analytic function. For each meromorphic Jacobi form in question there is a unique simple module whose character converges in the domain $1 \leq |w| \leq |q^{-1}|$ and coincides in that domain with the meromorphic Jacobi form. These interesting subtleties are discussed thoroughly in [CR1, CR2].

4. Representation theory of $\mathcal{L}_k$ for $k \in \mathbb{Z}_{>0}$

Let $k \in \mathbb{Z}_{>0}$ and consider

$$\mathcal{L}_k = \bigoplus_{i=1}^{k+1} \mathcal{L}_{i,0} \otimes V_{i,1} \qquad \text{and} \qquad \mathcal{L}_k^{\mathrm{even}} = \bigoplus_{\substack{i=1\\ i \text{ odd}}}^{k+1} \mathcal{L}_{i,0} \otimes V_{i,1}. \tag{15}$$

By Lemma 3.1, $\mathcal{L}_k$ can be endowed with the structure of $L_k(\mathfrak{osp}(1|2))$ and $\mathcal{L}_k^{\mathrm{even}}$ as its even subalgebra is simple as well by a mild extension of [DLM, Thm. 2.1] to superalgebras. By an extension of [CarM, Thm. 4.2] to the superalgebra case as in [CKLR, App. A, Rem. A.1], $\mathcal{L}_k^{\mathrm{odd}}$ is an order two simple current of $\mathcal{L}_k^{\mathrm{even}}$ where

$$\mathcal{L}_k^{\mathrm{odd}} = \bigoplus_{\substack{i=1\\ i \text{ even}}}^{k+1} \mathcal{L}_{i,0} \otimes V_{i,1}. \tag{16}$$

Let $\mathcal{C}_k$ be the modular tensor category of the vertex operator algebra $L_k(\mathfrak{sl}_2) \otimes \mathrm{Vir}(p, \frac{\Delta}{2})$. We can view $\mathcal{L}_k$ as a superalgebra object in both $\mathcal{C}_k$ and in the vertex tensor category of $\mathcal{L}_k^{\mathrm{even}}$ while $\mathcal{L}_k^{\mathrm{even}}$ is also an algebra object in $\mathcal{C}_k$. Let the corresponding induction functors be $\mathcal{F}_k, \mathcal{F}_k^{\mathrm{odd}}, \mathcal{F}_k^{\mathrm{even}}$. Our aim is to use the theory of vertex algebra extensions [CKM] recalled in Section 2 to understand the representations of $\mathcal{L}_k$.

Recall that the categorical twist θ acts on vertex operator algebra modules as $e^{2\pi i L(0)}$ with $L(0)$ the Virasoro zero-mode that provides the grading by conformal weight of modules. Conformal dimension of $\mathcal{L}_{i,0} \otimes V_{i,r}$ is

$$\frac{1}{4}\left(2i^2 - 2ir + \frac{(k+2)(r^2-1)}{2k+3}\right) \qquad (\mathrm{mod}\ \mathbb{Z}). \tag{17}$$

We have that $\theta_{\mathcal{L}_k^{\mathrm{even}}} = \mathrm{Id}_{\mathcal{L}_k^{\mathrm{even}}}$; i.e., $\mathcal{L}_k^{\mathrm{even}}$ is integer graded. Also, the calculation (20) below shows that "dimension" $\dim_{\mathcal{C}_k}(\mathcal{L}_k^{\mathrm{even}})$ is non-zero; recall also that $\mathcal{L}_k^{\mathrm{even}}$ is simple as a module over itself. These three facts ensure that as an algebra object in the modular tensor category $\mathcal{C}_k$, $\mathcal{L}_k^{\mathrm{even}}$ satisfies the conditions of [KO, Lem. 1.20] and hence those in [KO, Thm. 4.5]. We therefore deduce that the representation category of untwisted modules of $\mathcal{L}_k^{\mathrm{even}}$ (which is $\mathrm{Rep}^0\, \mathcal{L}_k^{\mathrm{even}}$) is modular. One subtlety is that the tensor product on this category defined by [KO, Thm. 1.5] and the one arising from vertex tensor categories could be different. However, this is not the case as guaranteed by [CKM, Thm. 3.65].

Now, with $1 \leq r \leq 2k+2$, we have the following (possibly non-local) modules for $\mathcal{L}_k$ via induction:

$$M_r = \mathcal{F}_k\left(\mathcal{L}_{1,0} \otimes V_{1,r}\right) = \bigoplus_{i=1}^{k+1} \mathcal{L}_{i,0} \otimes V_{i,r} = M_r^{\mathrm{even}} \oplus M_r^{\mathrm{odd}}$$

with the $\mathcal{L}_k^{\text{even}}$-modules

$$M_r^{\text{even}} = \mathcal{F}_k^{\text{even}}(\mathcal{L}_{1,0} \otimes V_{1,r}) = \bigoplus_{\substack{i=1 \\ i \text{ odd}}}^{k+1} \mathcal{L}_{i,0} \otimes V_{i,r}, \qquad M_r^{\text{odd}} = \bigoplus_{\substack{i=1 \\ i \text{ even}}}^{k+1} \mathcal{L}_{i,0} \otimes V_{i,r}.$$

All M_r and M_r^{even} are simple $\mathcal{L}_k$-, respectively, $\mathcal{L}_k^{\text{even}}-$, modules by [CKM, Prop. 4.4]. [CKM, Lem. 4.26] now implies that $M_r^{\text{odd}} \cong \mathcal{L}_k^{\text{odd}} \boxtimes_{\mathcal{L}_k^{\text{even}}} M_r^{\text{even}}$; in particular, they are simple. We now have that $\mathcal{F}_k(M_r^{\text{even}}) \cong M_r$.

M_r is local if it is graded by a coset of integers and twisted if both M_r^{even} and M_r^{odd} are each graded by a single but different coset of integers. From (17) we have that M_r is local if and only if r is odd, and otherwise it is twisted. We aim to prove that M_r exhaust all local/twisted modules (up to isomorphisms) of $\mathcal{L}_k$ and to determine their modular data and fusion rules.

Using the fact that induction is a tensor functor [CKM, Thms. 3.65 and 3.68] we have that

$$\begin{aligned}(18)\quad M_r^{\text{even}} \boxtimes_{\mathcal{L}_k^{\text{even}}} M_{r'}^{\text{even}} &= \mathcal{F}_k^{\text{even}}(\mathcal{L}_{1,0} \otimes V_{1,r}) \boxtimes_{\mathcal{L}_k^{\text{even}}} \mathcal{F}_k^{\text{even}}(\mathcal{L}_{1,0} \otimes V_{1,r'}) \\ &= \mathcal{F}_k^{\text{even}}((\mathcal{L}_{1,0} \otimes V_{1,r}) \boxtimes (\mathcal{L}_{1,0} \otimes V_{1,r'})) = \mathcal{F}_k^{\text{even}}\Big(\bigoplus_{r''=1}^{k} N_{r,r'}^{k+1\; r''} \mathcal{L}_{1,0} \otimes V_{1,r''}\Big) \\ &= \bigoplus_{r''=1}^{k} N_{r,r'}^{r''} M_{r''}^{\text{even}},\end{aligned}$$

and similarly,

$$M_r \boxtimes_{\mathcal{L}_k} M_{r'} = \bigoplus_{r''=1}^{k} N_{r,r'}^{r''} M_{r''}. \tag{19}$$

In particular, the abelian (super)categories generated by M_r and M_r^{even} are closed under taking tensor products in respective categories. Moreover, since $M_r^{\text{odd}} \cong \mathcal{L}_k^{\text{odd}} \boxtimes_{\mathcal{L}_k^{\text{even}}} M_r^{\text{even}}$, the abelian category generated by $M_r^{\text{even}\,/\,\text{odd}}$ also has the same property and, moreover, it is closed under duals:

$$\begin{aligned}(M_r^{\text{even}})^* &= \mathcal{F}_k^{\text{even}}(\mathcal{L}_{1,0} \otimes V_{1,r})^* = \mathcal{F}_k^{\text{even}}((\mathcal{L}_{1,0} \otimes V_{1,r})^*) = \mathcal{F}_k^{\text{even}}(\mathcal{L}_{1,0} \otimes V_{1,r}) \\ &= M_r^{\text{even}} \\ (M_r^{\text{odd}})^* &= (\mathcal{L}_k^{\text{odd}} \boxtimes_{\mathcal{L}_k^{\text{even}}} M_r^{\text{even}})^* \cong \mathcal{L}_k^{\text{odd}*} \boxtimes_{\mathcal{L}_k^{\text{even}}} (M_r^{\text{even}})^* \cong M_r^{\text{odd}},\end{aligned}$$

where the second equality in the first equation follows by [CKM, Prop. 2.17], and for the second equation we use the fact that $\mathcal{L}_k^{\text{even}}$ is an order two simple current. In effect, this category is a fusion category. We denote this category by $\mathcal{S}_k$ for future use.

4.1. **Frobenius-Perron dimension of $\mathcal{L}^{\text{even}}$.** We now use modular properties of characters to classify all simple inequivalent modules of $\mathcal{L}_k^{\text{even}}$ and of $\mathcal{L}_k$ as well.

The categorical dimension of a vertex operator algebra module X is the Hopf link (see (1)) $S^{\infty}_{\mathbf{1},X}$ which coincides with the modular S-matrix expression up to a constant as $S^{\infty}_{\mathbf{1},X} = \frac{S^X_{\mathbf{1},X}}{S^X_{\mathbf{1},\mathbf{1}}}$. Here $\mathbf{1}$ stands for the tensor unit, i.e., the vertex operator algebra itself. Using (6) and (7), one has

$$\dim_{\mathcal{C}_k} \mathcal{L}_k^{\mathrm{even}} = \frac{1}{\sin^2\left(\frac{\pi}{k+2}\right)} \sum_{\substack{l=1 \\ l\in 2\mathbb{Z}+1}}^{k+1} \sin^2\left(\frac{\pi l}{k+2}\right), \tag{20}$$

which we can simplify using the following lemma.

Lemma 4.1. *The following hold:*

$$\sum_{\substack{l=1 \\ l\in 2\mathbb{Z}+1}}^{k+1} \sin^2\left(\frac{\pi l}{k+2}\right) = \frac{k+2}{4} \qquad \textit{and} \qquad \sum_{l=1}^{k+1} \sin^2\left(\frac{\pi l}{k+2}\right) = \frac{k+2}{2}. \tag{21}$$

Proof. The second equation is

$$\sum_{l=1}^{k+1} \sin^2\left(\frac{\pi l}{k+2}\right) = -\frac{1}{4}\sum_{l=1}^{k+1}\left(-2 + e^{2\pi\mathrm{i}\frac{l}{k+2}} + e^{-2\pi\mathrm{i}\frac{l}{k+2}}\right) = \frac{k+1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{k+2}{2}.$$

If k is even, then the first equation follows immediately from the second one. For odd k we compute

$$\begin{aligned}\sum_{\substack{l=1 \\ l\in 2\mathbb{Z}+1}}^{k+1} \sin^2\left(\frac{\pi l}{k+2}\right) &= \sum_{r=0}^{\frac{k-1}{2}} \sin^2\left(\frac{\pi(2r+1)}{k+2}\right) \\ &= \frac{k+1}{4} - \frac{1}{4}\sum_{r=0}^{\frac{k-1}{2}}\left(e^{\frac{2\pi\mathrm{i}(2r+1)}{k+2}} + e^{-\frac{2\pi\mathrm{i}(2r+1)}{k+2}}\right)\end{aligned}$$

and

$$\sum_{r=0}^{\frac{k-1}{2}}\left(e^{\frac{2\pi\mathrm{i}(2r+1)}{k+2}} + e^{-\frac{2\pi\mathrm{i}(2r+1)}{k+2}}\right) = 2\sum_{r=0}^{k+1} e^{\frac{2\pi\mathrm{i}r}{k+2}} - \sum_{r=0}^{\frac{k+1}{2}}\left(e^{\frac{2\pi\mathrm{i}(2r)}{k+2}} + e^{-\frac{2\pi\mathrm{i}(2r)}{k+2}}\right) = -1.$$

□

The Frobenius-Perron dimension (FP for short) is the unique character of the tensor ring such that for simple modules X, $\mathrm{FP}(X) \in \mathbb{R}_{\geq 0}$ [ENO], [EGNO, Sec. 3.3]. For a modular tensor category $\mathcal{C}$

$$\mathrm{FP}(\mathcal{C}) = \sum \mathrm{FP}(X)^2 \tag{22}$$

is the FP-dimension of $\mathcal{C}$ where the sum is over all inequivalent simple objects of $\mathcal{C}$. For vertex operator algebras V, this dimension is easy to find provided there exists a unique simple module Z such that it has strictly lowest conformal dimension among all simple modules for V and provided the following limits exist [DJX]. In that case for any simple object X one has

$$\mathrm{adim}(X) := \lim_{\tau\to 0^+} \frac{\mathrm{ch}[X](\tau)}{\mathrm{ch}[V](\tau)} = \lim_{\tau\to -\infty} \frac{\mathrm{ch}[X](-1/\tau)}{\mathrm{ch}[V](-1/\tau)} = \frac{S^{\chi}_{X,Z}}{S^{\chi}_{V,Z}} \tag{23}$$

and $\mathrm{adim}(X) = \mathrm{FP}(X)$.

Recall that the $M_r^{\mathrm{even/odd}}$ form a subtensor category, called $\mathcal{S}_k$, of the full category of local modules of $\mathcal{L}_k^{\mathrm{even}}$. We will compute the $\mathrm{FP}(\mathcal{S}_k)$ and show that it agrees with $\mathrm{FP}(\mathrm{Rep}^0\, \mathcal{L}_k^{\mathrm{even}})$ so that the two categories must coincide. We also recall:

Theorem 4.2. [DMNO, Cor. 3.30] *Let A be a connected (sometimes also called haploid) étale (that is, commutative and separable) algebra in a modular tensor category $\mathcal{C}$, with $\mathrm{Rep}^0 A$ denoting the category of its local (also known as untwisted or dyslectic) modules in $\mathcal{C}$. Then*

$$\mathrm{FP}\left(\mathrm{Rep}^0(A)\right) = \frac{\mathrm{FP}(\mathcal{C})}{\mathrm{FP}_{\mathcal{C}}(A)^2}.$$

Lemma 4.3. *The Frobenius-Perron dimensions of $\mathcal{C}_k$ and $\mathcal{L}_k^{\mathrm{even}}$ are*

$$\mathrm{FP}(\mathcal{C}_k) = \frac{(k+2)^2(2k+3)}{16\sin^4\left(\frac{\pi}{k+2}\right)\sin^2\left(\frac{\pi}{2k+3}\right)} \qquad \textit{and} \qquad \mathrm{FP}_{\mathcal{C}_k}\left(\mathcal{L}_k^{\mathrm{even}}\right) = \frac{k+2}{4\sin^2\left(\frac{\pi}{k+2}\right)}.$$

Proof. We know the computation for $\mathrm{FP}(\mathcal{C}_k)$; $\mathrm{FP}_{\mathcal{C}_k}(\mathcal{L}_k^{\mathrm{even}})$ is computed similarly. Due to (22) and (23), we first determine the simple module of lowest conformal weight for $L_k(\mathfrak{sl}_2)\otimes \mathrm{Vir}(2k+3,k+2)$.

The simple module of lowest conformal weight for $L_k(\mathfrak{sl}_2)$ is $L_k(\mathfrak{sl}_2)$ itself as this is a unitary vertex operator algebra. For the Virasoro algebra $\mathrm{Vir}(2k+3,k+2)$, we will prove that the conformal weights satisfy $h_{1,2}\le h_{r,s}$ and equality holds if and only if $V_{r,s}\cong V_{1,2}$. The conformal weights of $V_{r,s}$ are given by

$$h_{r,s} = \frac{\left((k+2)s-(2k+3)r\right)^2-\left(2k+3-(k+2)\right)^2}{4(k+2)(2k+3)}$$

where $1\le r\le k+1 =: t$ and $1\le s\le 2k+2=2t$. Minimizing $h_{r,s}$ amounts to minimizing $X_{r,s} := |(k+2)s-(2k+3)r| = |t(s-2r)+(s-r)|$. We make cases based on the value of $2r-s$.

Case 1. $2r-s=0$: Since $X_{r,s}=r$, we see that the minimum here is $X_{1,2}=1$.
Case 2. $2r-s=1$: $X_{r,2r-1}=|r-1-t|$, so the minimum here is $X_{t,2t-1}=1$. Note that $V_{r,s}\cong V_{t-r+1,2t-s+1}$, so $V_{1,2}\cong V_{t,2t-1}$.
Case 3. $2r-s=n\ge 2$: $X_{r,2r-n}=|n(t+1)-r|\ge (n-1)t+n\ge t+2>2$, which is more than the minimum achieved previously.
Case 4. $2r-s=-n<0$: $X_{r,2r+n}=|t(n+1)+r|>1$, which is more than the minimum achieved previously.

We get that the simple module Z in (23) is $\mathcal{L}_{1,0}\otimes V_{1,2}$. We now have

$$\mathrm{FP}(\mathcal{C}_k) = \sum_{\substack{1\le i,r\le k+1\\ 1\le s\le 2k+2,\, s\ \mathrm{odd}}} \mathrm{FP}(\mathcal{L}_{i,0}\otimes V_{r,s})^2 = \sum_{\substack{1\le i,r\le k+1\\ 1\le s\le 2k+2,\, s\ \mathrm{odd}}} \left(\frac{S^{\chi}_{\mathcal{L}_{i,0}\otimes V_{r,s},\mathcal{L}_{1,0}\otimes V_{1,2}}}{S^{\chi}_{\mathcal{L}_{1,0}\otimes V_{1,1},\mathcal{L}_{1,0}\otimes V_{1,2}}}\right)^2$$

(24)

$$= \frac{(k+2)^2(2k+3)}{16\sin^4\left(\frac{\pi}{k+2}\right)\sin^2\left(\frac{\pi}{2k+3}\right)} \sum_{\substack{1\le i,r\le k+1\\ 1\le s\le 2k+2,\, s\ \mathrm{odd}}} \frac{16\sin^2\left(\frac{\pi i}{k+2}\right)\sin^2\left(\frac{\pi r}{k+2}\right)\sin^2\left(\frac{\pi s}{2k+3}\right)}{(2k+3)(k+2)^2},$$

where the first equality is due to (22), the second is due to (23), and the third is due to the known S-matrices for $L_k(\mathfrak{sl}_2)$ and Virasoro algebras. $\mathrm{FP}(\mathcal{C}_k)$ now follows immediately using Lemma 4.1. □

We have shown that $\mathcal{S}_k$, the full abelian subcategory formed by $M_r^{\mathrm{even/odd}}$-modules inside $\mathcal{L}_k^{\mathrm{even}}$-modules, is a fusion category. Considering the homomorphism of Grothendieck rings induced by the inclusion of $\mathcal{S}_k \hookrightarrow \mathrm{Rep}^0\,\mathcal{L}_k^{\mathrm{even}}$, from

[EGNO, Prop. 3.3.13(i)], we now immediately have that FP-dims of simples in $\mathcal{S}_k$ are equal to their FP-dims as objects of $\mathrm{Rep}^0\,\mathcal{L}_k^{\mathrm{even}}$ and hence that $\mathrm{FP}(\mathcal{S}_k) \leq \mathrm{FP}(\mathrm{Rep}^0\,\mathcal{L}_k^{\mathrm{even}})$ with equality iff the two categories are equal.

Now, with the S-matrix calculations from (27) and (28), it is clear that $X \mapsto \frac{S^{\chi}_{X,M_2^{\mathrm{even}}}}{S^{\chi}_{\mathcal{L}_k^{\mathrm{even}},M_2^{\mathrm{even}}}} \in \mathbb{R}_{>0}$ whenever $X = M_r^{\mathrm{even/odd}}$ and, moreover, this map preserves tensor products by properties of S matrices for $\mathcal{L}_k^{\mathrm{even}}$. Therefore, by uniqueness of FP dimensions, this map precisely gives us FP dimensions of modules in $\mathcal{S}_k$ (which are equal to their FP-dims considered as modules for $\mathcal{L}_k^{\mathrm{even}}$).

The following lemma is verified in a very similar manner as the previous one, using the modular S-matrix derived in the next section.

Lemma 4.4. *We have the following:*

$$\mathrm{FP}(\mathcal{S}_k) = \frac{2k+3}{\sin^2\left(\frac{\pi}{2k+3}\right)}.$$

Proof. From the S-matrices in the next section; (26), (27), and (28); and Lemma 4.1, we have that

$$\begin{aligned}\mathrm{FP}(\mathcal{S}_k) &= \sum_{1\leq r\leq 2k+2}\left(\frac{S^{\chi}_{M_r^{\mathrm{even}},M_2^{\mathrm{even}}}}{S^{\chi}_{M_1^{\mathrm{even}},M_2^{\mathrm{even}}}}\right)^2 + \left(\frac{S^{\chi}_{M_r^{\mathrm{odd}},M_2^{\mathrm{even}}}}{S^{\chi}_{M_1^{\mathrm{even}},M_2^{\mathrm{even}}}}\right)^2 \\ &= \frac{1}{\sin^2\left(\dfrac{2\pi(k+2)}{2k+3}\right)}\sum_{1\leq r\leq 2k+2} 2\cdot\sin^2\left(\frac{2r\pi(k+2)}{2k+3}\right) = \frac{2k+3}{\sin^2\left(\frac{\pi}{2k+3}\right)}.\end{aligned}$$

$\square$

Corollary 4.5. *The Frobenius-Perron dimension of the category of local $\mathcal{L}_k^{\mathrm{even}}$-modules, i.e.,* $\mathrm{FP}(\mathrm{Rep}^0\,\mathcal{L}_k^{\mathrm{even}})$*, and the one of the subtensor category given by the* $M_r^{even/odd}$*, i.e.,* $\mathrm{FP}(\mathcal{S}_k)$*, coincide.*

Proof. Theorem 4.2 and Lemma 4.3 give $\mathrm{FP}(\mathrm{Rep}^0\,\mathcal{L}_k^{\mathrm{even}}) = \frac{2k+3}{\sin^2(\frac{\pi}{2k+3})}$. This equals $\mathrm{FP}(\mathcal{S}_k)$ by Lemma 4.4. $\square$

We immediately deduce:

Corollary 4.6. *The* $M_r^{even/odd}$ *for* $1 \leq r \leq 2k+2$ *form a complete list of inequivalent simple local* $\mathcal{L}_k^{\mathrm{even}}$*-modules.*

Corollary 4.7. *Let* $1 \leq r \leq 2k+2$*. The* M_r *for* r *odd form a complete list (up to parity and isomorphisms) of simple local* $\mathcal{L}_k$*-modules and for* r *even of twisted ones.*

Remark 4.8. [CKM, Thm. 2.67] says that the induction functor is a braided tensor functor when restricted to those objects that induce to local modules for an algebra object (if inducing to modules for a superalgebra, one needs to start with an auxiliary supercategory). In our example of $\mathcal{F}_k^{\mathrm{even}}$ this is the full abelian subcategory of $\mathcal{C}_k$ formed by $\mathcal{L}_{1,0}\otimes V_{1,r}$ for $r = 1,\dots,2k+2$. Denote this category by $\mathcal{C}_{k,0}^{\mathrm{even}}$. Further, let $\mathcal{S}^{\mathrm{even}}$ be the full abelian subcategory formed by M_r^{even}-modules inside $\mathcal{L}_k^{\mathrm{even}}$-modules and let $\mathcal{S}^{\mathrm{super}}$ be the full abelian subcategory of $\mathcal{S}^{\mathrm{even}}$ formed by all M_r^{even} with r odd.

Then the same argument as in the proof of [OS, Thm. 5.1] shows that the induction functor $\mathcal{F}_k^{\text{even}}$ restricted to $\mathcal{C}_{k,0}^{\text{even}}$ is fully faithful, and thus this subcategory is braided equivalent to $\mathcal{S}^{\text{even}}$.

The argument of [OS, Thm. 5.1] also works for extensions by superalgebra objects (using then also [CKM, Lem. 2.61]), and thus one also gets a braided equivalence between an auxiliary supercategory $\mathcal{SS}^{\text{super}}$ (see [CKM, Defn. 2.11]) and the category of local $\mathcal{L}_k$-modules. As a consequence of these two equivalences of braided tensor categories, we especially have that the category of local $\mathcal{L}_k$-modules is braided equivalent to the supercategory auxiliary to the full abelian subcategory of the vertex tensor category of the Virasoro algebra $\mathrm{Vir}(p, \Delta/2)$ formed by the $V_{1,r}$ with r odd.

4.2. **Fusion rules.** Using that the induction functor is a $P(z)$-tensor functor we get the fusion rules of $\mathcal{L}_k$ for free, namely

$$M_r \boxtimes_{\mathcal{L}_k} M_{r'} \cong \bigoplus_{r''=1}^{2k+2} N_{r,r'}^{r''} M_{r''}, \tag{25}$$

which we have established in (19). Each M_r decomposes as $M_r = M_r^{\text{even}} \oplus M_r^{\text{odd}}$ as an $\mathcal{L}_k^{\text{even}}$-module. As explained in [CKM] one also assigns a parity to each vertex operator superalgebra module, and we have two choices: we can either give M_r^{even} even parity and M_r^{odd} odd or the other way around. Let us denote the first choice by M_r^+ and the second one by M_r^-. An intertwining operator is then called even if it respects the chosen parities and odd otherwise. The superdimension (sdim) of a fusion rule is accordingly the difference of the dimension of even and odd intertwining operators of the given type. From our discussion of $\mathcal{L}_k^{\text{even}}$, we have the sdim of type

$$\mathrm{sdim}\begin{pmatrix} & M_{r''}^{\epsilon''} & \\ M_r^{\epsilon} & & M_{r'}^{\epsilon'} \end{pmatrix} = N^{-\ \ (r'',\epsilon'')}_{(r,\epsilon),(r',\epsilon')} = \epsilon\epsilon'\epsilon'' N_{r,r'}{}^{r''}.$$

The fusion coefficients can also be computed using Verlinde's formula; see [CKM, Sec. 1.5].

4.3. **Modular transformations.** The results of this section follow by applying [CKM, Sec. 4.2.1]. Let $1 \le r, r' \le 2k+2$. The character and supercharacter of M_r^+ are defined as

$$\mathrm{ch}^{\pm}[M_r^+](\tau, v) = \mathrm{ch}[M_r^{\text{even}}](\tau, v) \pm \mathrm{ch}[M_r^{\text{odd}}](\tau, v).$$

Define the numbers

$$s_{r,r'} := (-1)^{r+r'}\sqrt{\frac{1}{2k+3}}\sin\left(\frac{\pi r r'(k+2)}{(2k+3)}\right). \tag{26}$$

Then, [CKM, Prop. 2.89] and [KO, Thm. 4.5] give

$$\begin{aligned} S^{\chi}_{M_r^{\text{even}},M_{r'}^{\text{even}}} &= \frac{1}{\mathcal{D}(\mathrm{Rep}^0\,\mathcal{L}_k^{\text{even}})} S^{\infty}_{M_r^{\text{even}},M_{r'}^{\text{even}}} = \frac{\dim_{\mathcal{C}_k}(\mathcal{L}_k^{\text{even}})}{\mathcal{D}(\mathcal{C}_k)} S^{\infty}_{M_r^{\text{even}},M_{r'}^{\text{even}}} \\ &= \frac{\dim_{\mathcal{C}_k}(\mathcal{L}_k^{\text{even}})}{\mathcal{D}(\mathcal{C}_k)} S^{\infty}_{\mathcal{L}_{1,0}\otimes V_{1,r},\mathcal{L}_{1,0}\otimes V_{1,r'}} = \dim_{\mathcal{C}_k}(\mathcal{L}_k^{\text{even}}) S^{\chi}_{\mathcal{L}_{1,0}\otimes V_{1,r},\mathcal{L}_{1,0}\otimes V_{1,r'}} = s_{r,r'}. \end{aligned} \tag{27}$$

The exact definition of the number $\mathcal{D}$ may be found in [KO]; however, the only thing we need is the relation used in the second equality, which is due to [KO, Thm. 4.5]. Using [CKM, Lem. 4.26] we get

$$S^{\chi}_{M_r^{\mathrm{even}},M_{r'}^{\mathrm{odd}}} = \begin{cases} s_{r,r'} & r \text{ even}, \\ -s_{r,r'} & r \text{ odd} \end{cases} \tag{28}$$

and

$$S^{\chi}_{M_r^{\mathrm{odd}},M_{r'}^{\mathrm{odd}}} = \begin{cases} s_{r,r'} & r+r' \text{ even}, \\ -s_{r,r'} & r+r' \text{ odd}. \end{cases}$$

We then have for r odd:

$$\begin{aligned} \mathrm{ch}^+[M_r^+]\left(-\frac{1}{\tau},\frac{z}{\tau}\right) &= e^{2\pi i k\frac{z^2}{\tau}} \sum_{r' \text{ even}} 2s_{r,r'}\,\mathrm{ch}^-[M_{r'}^+](\tau,z), \\ \mathrm{ch}^-[M_r^+]\left(-\frac{1}{\tau},\frac{z}{\tau}\right) &= e^{2\pi i k\frac{z^2}{\tau}} \sum_{r' \text{ odd}} 2s_{r,r'}\,\mathrm{ch}^-[M_{r'}^+](\tau,z) \end{aligned} \tag{29}$$

and for r even:

$$\begin{aligned} \mathrm{ch}^+[M_r^+]\left(-\frac{1}{\tau},\frac{z}{\tau}\right) &= e^{2\pi i k\frac{z^2}{\tau}} \sum_{r' \text{ even}} 2s_{r,r'}\,\mathrm{ch}^+[M_{r'}^+](\tau,z), \\ \mathrm{ch}^-[M_r^+]\left(-\frac{1}{\tau},\frac{z}{\tau}\right) &= e^{2\pi i k\frac{z^2}{\tau}} \sum_{r' \text{ odd}} 2s_{r,r'}\,\mathrm{ch}^+[M_{r'}^+](\tau,z). \end{aligned} \tag{30}$$

Assemble the S^{χ} into a $(4k+4)\times(4k+4)$-matrix with first $(2k+2)$ rows and columns corresponding to M_r^{even} and the remaining rows and columns with corresponding M_r^{odd}. By changing the basis of the Grothendieck ring so that $M_r^{\mathrm{even}}, M_r^{\mathrm{odd}}$ are replaced by $M_r^{\mathrm{even}} + M_r^{\mathrm{odd}}$ and $M_r^{\mathrm{even}} - M_r^{\mathrm{odd}}$ as in [CKM, eqns. (4.9), (4.10)] we form the $(4k+4)\times(4k+4)$ matrices $\widetilde{S}$ and $\widetilde{S}^{-1}$. Following the notation in [CKM], Verlinde's formula [CKM, eqn. (4.11)] reads

$$\begin{aligned} N^{+\ \ (r'',+)}_{(r,+),(r',+)} &= \delta(r+r'+r''=1 \mod 2) \sum_{t \text{ even}} \frac{\widetilde{S}_{r,t}\cdot\widetilde{S}_{r',t}\cdot(\widetilde{S}^{-1})_{t,r''}}{\widetilde{S}_{1,t}}, \\ N^{-\ \ (r'',+)}_{(r,+),(r',+)} &= \delta(r+r'+r''=1 \mod 2) \sum_{t \text{ odd}} \frac{\widetilde{S}_{r,t}\cdot\widetilde{S}_{r',t}\cdot(\widetilde{S}^{-1})_{t,r''}}{\widetilde{S}_{1,t}}. \end{aligned} \tag{31}$$

5. Parafermions

$\mathcal{L}_k$ contains the lattice vertex operator algebra V_L of the lattice $L=\sqrt{2k}\mathbb{Z}$ as a vertex operator subalgebra. Let $C_k := \mathrm{Com}\,(V_L,\mathcal{L}_k)$ be the parafermion coset. It is rational by [CKLR, Cor. 4.13] and since $C_k = \mathrm{Com}\,(V_L,\mathcal{L}_k^{\mathrm{even}})$. Every even weight module of $L_k(\mathfrak{sl}_2)$ is graded by $2L'$, and every odd weight module is graded by $L'\backslash 2L'$, so it follows especially that $\mathcal{L}_k$ is graded by L' and hence, for $1\le r\le 2k+2$,

$$M_r \cong \bigoplus_{\lambda\in L'/L} V_{\lambda+L}\otimes C_{\lambda,r}.$$

We restrict to local modules M_r; that is, r is odd. The results of [CKLR, CKL, CKM] together (see [CKM, Sec. 4.3.1]) now imply that all $C_{\lambda,r}$ are simple C_k-modules. The grading lattice being the full dual lattice L' implies ([CKM, Thm. 4.39]) that $C_{\lambda,r}\cong C_{\nu,s}$ if and only if $\lambda=\nu$ and $r=s$. Moreover, these are all inequivalent simple modules; this follows from Theorem 4.3, Lemma 4.9, and the proof of Theorem 4.12 of [CKLR]. We remark that these theorems are formulated for vertex operator algebras; however, modifying the proofs to the supersetting is not difficult. The easier alternative is to apply the results of [CKLR, CKL, CKM] to $\mathcal{L}_k^{\text{even}}$ where each simple C_k-module appears as a submodule of two distinct $\mathcal{L}_k^{\text{even}}$-modules. One of these $\mathcal{L}_k^{\text{even}}$-modules then lifts to a local $\mathcal{L}_k$-module and the other to a twisted one.

In order to relate the fusion rules for C_k with those of $\mathcal{L}_k$ (or $\mathcal{L}_k^{\text{even}}$), we view $\mathcal{L}_k$ (or $\mathcal{L}_k^{\text{even}}$) as an algebra (or superalgebra) in the tensor category of the regular vertex operator algebra $V_L \otimes C_k$ we use [CKM, Thm. 4.41]. This theorem uses the fact that the induction functor is monoidal and that the fusion for V_L is group-like. [CKM, Thm. 4.41] translated to the current notation gives us precisely

$$C_{\nu,r} \boxtimes C_{\lambda,r'} \cong \bigoplus_{r''=1}^{2k+2} {N_{r,r'}}^{r''} C_{\lambda+\nu,r''}. \tag{32}$$

From [CKM, Thm. 4.42], we have that

$$\begin{aligned}
\mathrm{ch}^+[M_r^+](\tau,u) &= \sum_{\nu\in L'/L} \frac{\theta_{L+\nu}(u,\tau)}{\eta(\tau)}\,\mathrm{ch}[C_{\nu,r}](\tau),\\
\mathrm{ch}^-[M_r^+](\tau,u) &= \sum_{\nu\in L'/L} \frac{\theta_{L+\nu}(u,\tau)}{\eta(\tau)}(-1)^{\delta(\,\nu\notin(2L')/L\,)}\,\mathrm{ch}[C_{\nu,r}](\tau),\\
\mathrm{ch}[C_{\nu,r}](\tau) &= \frac{\eta(\tau)}{2k\cdot\theta_{L+\nu}(0,\tau)} \sum_{\gamma\in L'/L} e^{-\pi i\langle\nu,\gamma\rangle}\,\mathrm{ch}^+[M_r^+](\tau,\gamma)\\
&= (-1)^{\delta(\,\nu\notin(2L')/L\,)}\frac{\eta(\tau)}{2k\cdot\theta_{L+\nu}(0,\tau)} \sum_{\gamma\in L'/L} e^{-\pi i\langle\nu,\gamma\rangle}\,\mathrm{ch}^-[M_r^+](\tau,\gamma).
\end{aligned}$$

For the T transformations, we get

$$e^{\pi i\left(\langle\lambda,\lambda\rangle-\frac{1}{12}\right)} T_{C_{\lambda,r}} = T_{M_r},$$

and finally, we can calculate the S-transformations by inducing up to $\mathcal{L}_k^{\text{even}}$-modules and then using equations (26), (27), and (28) as in [CKM, Thm. 4.42]:

$$S^{\chi,C}_{C_{\lambda,r_1},C_{\mu,r_2}}\cdot e^{2\pi i\langle\lambda,\mu\rangle}\cdot\sqrt{k/2} = \begin{cases} s_{r,r'} & \text{if } \lambda,\mu\in 2L'/L,\\ (-1)^r s_{r,r'} & \text{if } \lambda\in 2L'/L, \mu\notin 2L'/L\\ & \text{or } \mu\in 2L'/L, \lambda\notin 2L'/L,\\ (-1)^{r+r'} s_{r,r'} & \text{if } \lambda,\mu\notin 2L'/L.\end{cases}$$

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