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ESSENTIAL NORMAL AND SPUN NORMAL SURFACES IN 3-MANIFOLDS

ENSIL KANG AND J. HYAM RUBINSTEIN

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Abstract. Normal and spun normal surfaces are key tools for algorithms in 3-dimensional geometry and topology, especially concerning essential surfaces. In a recent paper of Dunfield and Garoufalidis, an interesting criterion is given for a spun normal surface to be essential in an ideal triangulation of a 3-manifold with a complete hyperbolic metric of finite volume. Their method uses ideal points of character varieties and Culler–Shalen theory. In this paper, we give a simple proof of a criterion which applies for both triangulations of closed 3-manifolds and ideal triangulations of the interior of compact 3-manifolds, giving a sufficient condition for a normal or a spun normal surface to be essential. Our criterion implies that of Dunfield and Garoufalidis. We also give a necessary and sufficient condition for a normal surface in a closed 3-manifold to be essential, using sweepouts and almost normal surface theory.

1. Introduction

A fundamental question in the topology of closed 3-manifolds concerns the existence of embedded essential (incompressible) surfaces. A closed irreducible 3-manifold is called *Haken* if it contains such a surface. In [19], it was shown that if a closed irreducible 3-manifold M is homotopy equivalent to a Haken 3-manifold N, then there is a homeomorphism between M and N. Recently Agol has solved the virtual Haken conjecture in [1], showing that any closed irreducible 3-manifold with infinite fundamental group is finitely covered by a Haken 3-manifold.

However, from an algorithmic viewpoint, it is computationally challenging to decide whether or not a given 3-manifold admits an essential surface. The fundamental result in this direction is that of Jaco and Oertel [9] that if there is such a surface, it occurs at a vertex of the projective solution space of normal surfaces, defined by Haken. (See [17] for an excellent survey of normal and spun normal surface theory). However, it is still an interesting challenge to decide if an embedded normal surface is essential; see [3] where it is shown that the Weber–Seifert hyperbolic 3-manifold is non-Haken, i.e., has no such surfaces, answering a question posed by Thurston. Also see [2], where an impressive list of essential surfaces in knot complements is achieved, using a number of innovations in normal surface theory and crushing triangulations. It is possible to combine the techniques of [2] and those

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in our final section. So instead of working with the result of cutting open along a normal surface, one could instead cut open and crush the triangulation and work with the resulting ideal triangulation. This may well be a more computationally efficient process.

In [5], an interesting criterion is given which is sufficient to guarantee that an embedded spun normal surface in an ideal triangulation is essential. The criterion is that such a surface is a vertex class of the projective solution space and it has quadrilaterals in every tetrahedron and which meet every edge of the triangulation. The authors show experimentally by searching amongst ideal triangulations of a given 3-manifold that such normal surfaces can often be found, hence giving interesting essential surfaces at various boundary slopes in, e.g., knot and link complements. The main tool used in [5] is the ideal points of character varieties, following [4].

In this paper, our aim is to extend the criterion of [5] by first applying to embedded spun normal surfaces which are not necessarily vertices of the projective solution space and second to embedded normal surfaces in triangulations of closed 3-manifolds.

We give two versions of our results in the case of closed 3-manifolds. The second version uses sweepout technology and may be more useful for implementation. See [7], [6] for a detailed discussion of sweepouts and a more refined algorithm to decide when two embedded normal surfaces represent isotopic essential surfaces.

For general results about embedded spun normal surfaces, including existence theory for essential surfaces, see [11–13, 17, 20]. In [14], we will give a version of [9] for spun normal surfaces, showing that if there is an embedded essential spun normal surface at a particular boundary slope for a manifold with an ideal triangulation with one ideal vertex, then there is such a surface at a vertex of the projective solution space.

2. Essential spun normal and normal surfaces

In this section, we will give our main result which describes a criterion for an embedded normal or spun normal surface to be essential in a given triangulation. This is an extension of the criterion of [5] as shown in the example presented in the next section. Throughout this paper, all normal and spun normal surfaces will be embedded.

Definition 1. Suppose that M is a compact 3-manifold with incompressible boundary. A properly embedded surface F is essential if it is incompressible and ∂-incompressible. By this we mean the induced maps $\pi_1(F) \to \pi_1(M)$ and $\pi_1(F, \partial F) \to \pi_1(M, \partial M)$ are both injections.

Definition 2. Let M be the interior of a compact 3-manifold with an ideal triangulation $\Im$. A spun normal surface S in M is an embedded surface formed from elementary disks in the tetrahedra consisting of finitely many quadrilaterals and infinitely many triangular disks. A nonempty intersection of S with a regular neighborhood of an ideal vertex forms an infinite cylinder, which is made of an infinite number of copies of trivial normal tori, by cutting and pasting along a family of parallel essential simple closed curves. The remainder of S outside these cylinders which spiral towards ideal vertices is compact and called the core.

Definition 3. The complexity of an embedded surface which is transverse to a triangulation $\Im$ of a 3-manifold is given by the lexicographically ordered pair (w, l) where w is the weight, i.e., the number of intersections of the surface with the edges, and l is the number of loops of intersection of the surface with the faces of the triangulation.

Theorem 1. *Suppose that M is either a closed 3-manifold or the interior of a compact 3-manifold with incompressible tori or Klein bottles as boundary components. Let $\Im$ be a finite or ideal triangulation of M, respectively, and let Σ be a connected normal or spun normal surface. The following is a sufficient condition for Σ being essential:*

(1) *Every tetrahedron of $\Im$ contains a quadrilateral of Σ, and*

(2) *suppose Σ' is a connected normal or spun normal surface which has Euler characteristic greater than that of Σ. Then any surface normally isotopic to Σ' intersects Σ.*

Proof. Let Σ be chosen to satisfy conditions (1) and (2). If Σ is 1-sided, we replace it by the boundary $\tilde{\Sigma}$ of a small regular neighbourhood. This gives a new normal or spun normal surface satisfying conditions (1) and (2). Moreover, it is well known that Σ is essential if and only if $\tilde{\Sigma}$ is essential. So we may assume from now on that Σ is 2-sided.

The first step is to remove a small open regular neighbourhood of each vertex, chosen disjoint from Σ, and then split along Σ to form a 3-manifold M^*. There is a decomposition of M^* into cells and copies of products of a triangular disk and a half open interval $[0, 1)$ adjacent to the normal tori linking ideal vertices where all cells are truncated triangular prisms, which we call *truncated prisms*, or products of triangular or quadrilateral disks and closed intervals, which we call *product cells* (see Figure 1).

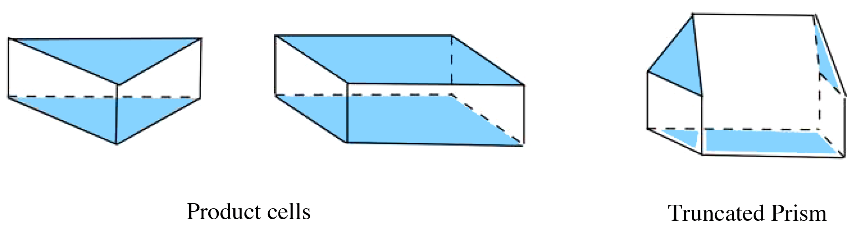

FIGURE 1. Cell types after truncating and splitting along Σ: shadowed faces are in $\Sigma \subset \partial M^*$ if the triangulation is ideal or in Σ or a vertex linking normal 2-sphere in the closed case

Each boundary face of such cells is called an *external* face if it is in ∂M^* or an *internal* face if in intM^*. We call an arc in an internal face a *vertical arc* or simply *vertical* if the ends are in two different edges in ∂M^*, a *horizontal arc* or simply *horizontal* if the ends are in two different edges of intM^*, a *mixed arc* or simply *mixed* if one end is in ∂M^* and the other in intM^*, and *trivial* otherwise (See Figure 2.). We also describe loops in the interior of internal faces as trivial.

Suppose there is a compressing disk D for Σ. By the usual transversality argument, reducing the complexity of the compressing disk at each stage, we can remove all trivial loops, together with trivial and mixed arcs of intersection of D

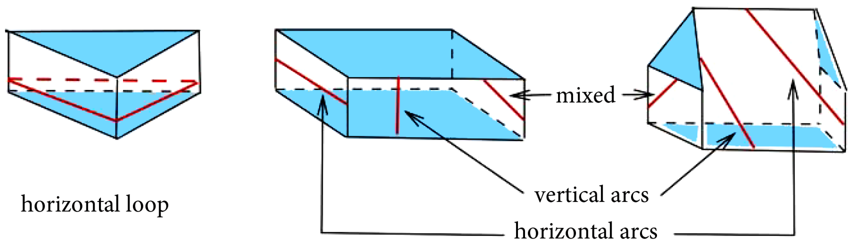

FIGURE 2. Arc types in internal faces

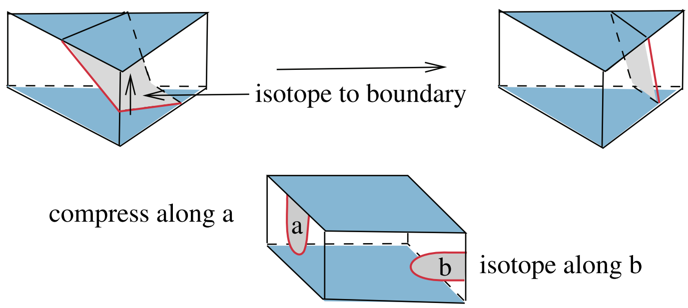

FIGURE 3. Simplifying intersections of D with product regions

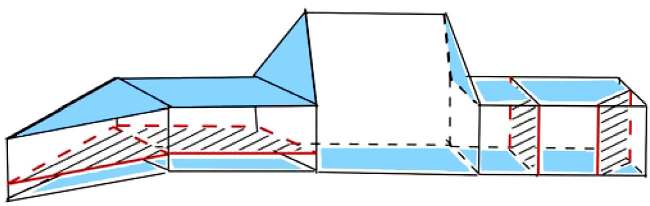

FIGURE 4. Intersections of D with product cells

with product cells (See Figure 3.). Subsequently, a least complexity choice of D will meet the vertical boundaries of product cells in vertical arcs and horizontal simple closed curves on the internal boundary annuli (see Figure 4). Note that such a least complexity disk D will also have no mixed arc of intersection cutting off a vertex in an internal quadrilateral or hexagonal face of a truncated prism.

The key idea is then to study how the compressing disk D meets each product cell or truncated prism. There are two types of intersection (which we call horizontal and vertical) of the intersection disks of D with a product cell and three types (which we call type 1, type 2, and type 3) with a truncated prism (see Figure 4, Figure 5, and Figure 6).

We first show that there are no vertical arcs in the intersection of D with faces of product cells and truncated prisms. Suppose that there is such an arc. Since the ends of a vertical arc are in ∂M^*, such an arc bisects D into two bigons. Choose an outermost bigon B in D cut off by some vertical arc α, possibly with horizontal and mixed arcs inside the bigon B, but no other vertical arcs except for α. Consider

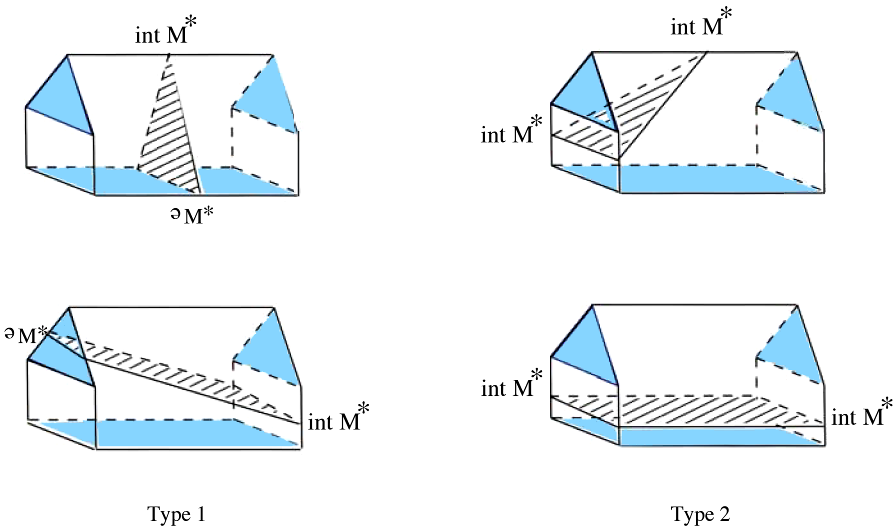

FIGURE 5. Intersection disks of type 1 and type 2

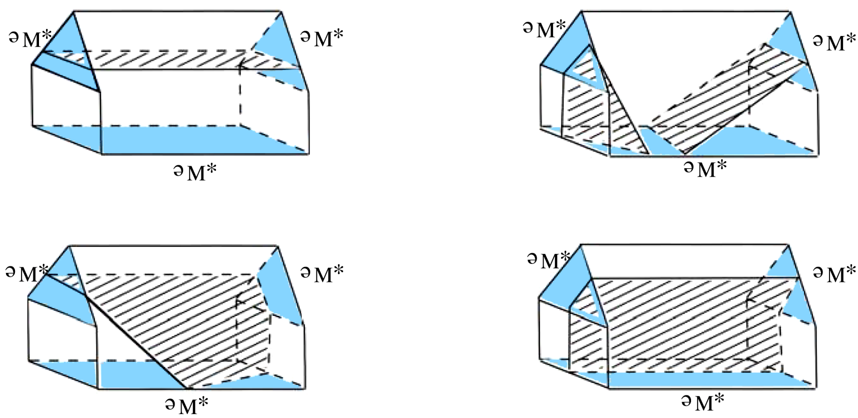

FIGURE 6. Intersection disks of type 3 in a truncated prism

the intersection disk d of B (so also of D) with the cell containing the vertical arc α (see Figure 7). We can immediately rule out d being vertical in a product cell or of type 3 in truncated prism, as in both cases there would be additional vertical arcs in B other than α. But all other intersection disk types have no vertical arcs, so we conclude that D has no vertical arc intersections.

Therefore it follows that every intersection disk of D with a cell must be horizontal if the cell is a product, or of type 1 or type 2 if the cell is a truncated prism. Now we claim that compressing Σ along D results in a normal surface Σ' with $\chi(\Sigma') > \chi(\Sigma)$ and Σ' can be normally isotoped off Σ, contrary to hypothesis.

This is easy to see, since we can directly check that the result of such a compression in all the cells met by D gives a new normal surface. If ∂D is nonseparating, then we can take Σ' as the result of compression along D. If ∂D is separating,

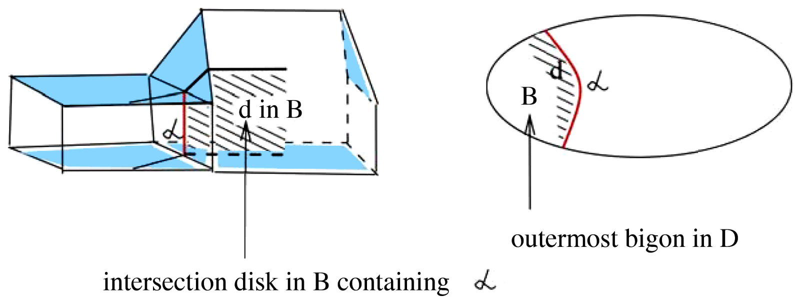

FIGURE 7. Outermost vertical arc and bigon in D

we can pick Σ' as either component of the result of compression along D. So the conclusion is that Σ is incompressible, and this completes the proof. □

Corollary 1. *Suppose that M is a closed 3-manifold or the interior of a compact 3-manifold with incompressible tori or Klein bottles as boundary components. Assume that $\Im$ is a finite or ideal triangulation of M, respectively, and let Σ be a normal or spun normal surface. If every tetrahedron of $\Im$ contains a quadrilateral of Σ and $[\Sigma]$ is a vertex class in the projective solution space for Q normal and spun normal surface theory, then Σ is essential.*

Proof. Suppose that a normal or spun normal surface Σ satisfies the conditions of the corollary. We claim it satisfies condition (2) of Theorem 1.

Assume that there is a normal or spun normal surface Σ^* which is disjoint from Σ. Note that since Σ has at least one quadrilateral in each tetrahedron of $\Im$ and Σ^* is disjoint from Σ, the quadrilateral types of Σ^* must form a subset of the quadrilateral types of Σ. Then it is clear that a multiple of $[\Sigma]$ must contain all the quadrilaterals of $[\Sigma^*]$. Hence we can write $n[\Sigma] = [\Sigma^*] + [\Sigma']$ for some normal or spun normal class $[\Sigma']$. But since $[\Sigma]$ is a vertex class, $[\Sigma'] = [0]$. So $[\Sigma^*] = [\Sigma]$, and this completes the proof of the corollary. □

Remark 1. Note that in [5], the conditions to imply that a spun normal surface is essential are that every tetrahedron and edge are met by a quadrilateral of the spun normal surface and that the surface is a vertex solution. Corollary 1 clearly extends the result of [5] to apply to closed as well as ideal triangulations and does not require the condition that all edges are met by quadrilaterals. (However, it is easy to see that if there is an edge not met by quadrilaterals, then the surface cannot be a vertex solution, by the same argument as in the corollary. So this condition is actually redundant.)

3. EXAMPLE

Consider the six tetrahedron triangulation of the 3-torus M, shown in Figure 8 as a 3-cube with opposite faces identified in the usual way. There is an embedded normal torus Σ which has a quadrilateral in each tetrahedron. Σ has a further 12 triangular disks shown as Areas 1 and 2 in Figure 8.

This is clearly essential and by Poincaré duality represents a class in $H_2(M)$ dual to a cohomology class which evaluates to one on each of the homology classes of the coordinate edge loops e_1, e_2, e_3. We claim first that $[\Sigma]$ is not a vertex class.

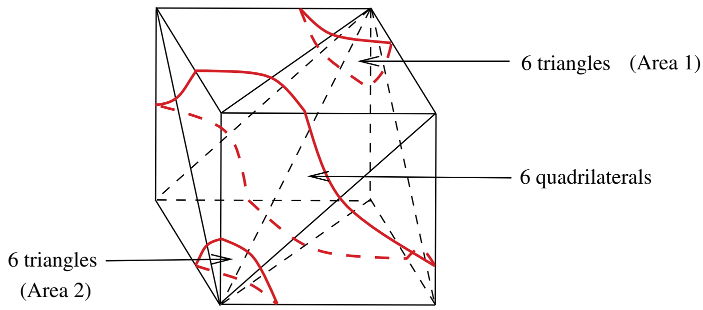

FIGURE 8. Σ in 3-torus triangulated with 6 tetrahedra

To see this, note that each of the bisecting tori T_i, for $1 \leq i \leq 3$, are readily proved to be vertex solutions for the projective solution space (see Figure 9).

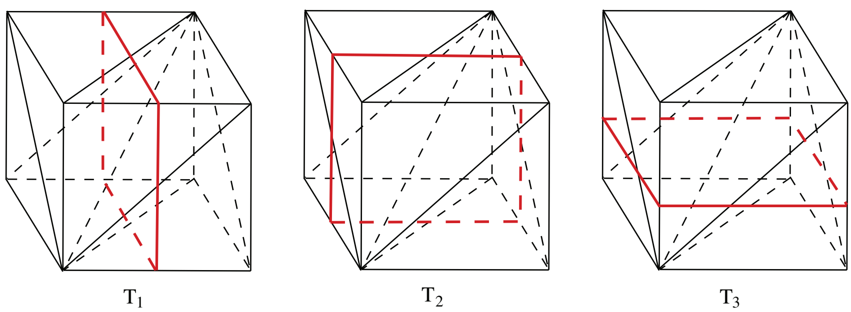

FIGURE 9. Bisecting tori each having 2 quadrilaterals and 4 triangles

But then $[\Sigma] = [T_1] + [T_2] + [T_3]$ (see Figure 10), and so $[\Sigma]$ is not a vertex solution. Clearly there is a facet $\mathcal{F}$ of the projective solution space which has vertices including $[T_i]$, for $1 \leq i \leq 3$, and $[\Sigma]$ is an interior point of $\mathcal{F}$.

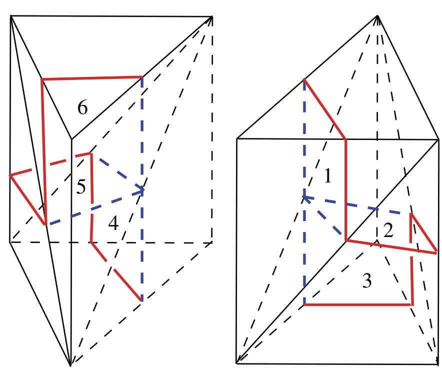

FIGURE 10. 6 quadrilaterals in Σ

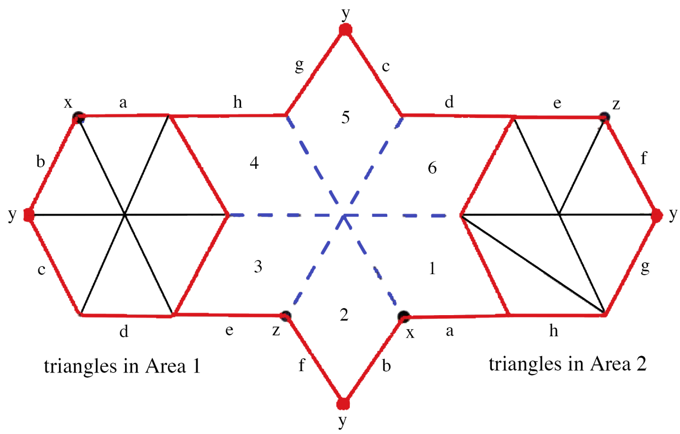

FIGURE 11. Σ with 6 quadrilaterals and 12 triangles

Finally we claim that there are no normal surfaces disjoint from Σ other than Σ. Suppose that there was such a normal surface $\Sigma^{'}$. Clearly its quadrilateral disk types would have to be a subset of those of Σ so that $\Sigma^{'}$ has normal class in $\mathcal{F}$.

Note that each of the 3 edges coming from the boundary of the 3-cube in Figure 10 are met by precisely 2 quadrilaterals from Σ. Here we are only considering the original edges of the 3-cube, not the diagonals of the faces. (The 12 edges are identified to form 3 edges in the triangulation of the 3-torus.) By the Q-matching equations of [18] along the edges, if $[\Sigma^{'}]$ has one of the quadrilateral types meeting one of these 3 edges, it must contain the other quadrilateral type meeting the edge. Moreover, the multiplicity of each of these two quadrilateral types must be the same. Hence we conclude that there is an equation of the form $[\Sigma^{'}] = k_1[T_1]+k_2[T_2]+k_3[T_3]$ for some nonnegative integers k_1, k_2, and k_3, since these two quadrilateral types are precisely those in one of the three tori $[T_i]$, for $i = 1, 2, 3$. But this then implies that $[\Sigma^{'}]$ is a multiple m of a torus $T^{'}$, since we can cut and paste $k_1[T_1], k_2[T_2]$, and $k_3[T_3]$ to form a normal torus $T^{'}$ with multiplicity m equal to the greatest common divisor of k_1, k_2, and k_3.

But then the homology class of $T^{'}$ is different from Σ unless $k_1 = k_2 = k_3$. If these tori are in different classes, they have nontrivial intersection, contrary to the assumption. If they are in the same class, then $\Sigma^{'}$ is a multiple of Σ.

So this shows that the conditions of Theorem 1 are strictly weaker than those of [5].

4. SWEEPOUTS

In [15], [16], [7], sweepouts are used to produce almost normal surfaces. Employing this technology, we can give a necessary and sufficient condition for a normal surface in a closed triangulated 3-manifold M to be essential. See [6] for a more refined result, namely an algorithm to decide if two normal surfaces are isotopic or nonisotopic essential surfaces.

Definition 4. An embedded surface Σ' in a closed triangulated 3-manifold M is almost normal if it satisfies one of the following two conditions:

- Σ' is obtained from an embedded normal surface by attaching a small tube inside a tetrahedron Δ running between two normal disks. Moreover, the tube is unknotted, i.e., there is a subarc λ of an edge, which meets Σ' only at the endpoints of λ and there is an arc μ running along Σ' crossing the tube once so that $\partial\lambda = \partial\mu$ and $\lambda \cup \mu$ bound an embedded disk in Δ.
- There is a properly embedded octagon in one of the tetrahedra Δ whose boundary consists of 8 normal arcs in the faces of Δ and the remainder of Σ' consists of normal triangular and quadrilateral disks in the tetrahedra of M.

We now sketch the main ideas of the next result.

Definition 5. A pair $(\Sigma, \Sigma^{'})$ of disjoint closed embedded 2-sided surfaces in a closed triangulated 3-manifold is called *a*djacent if Σ is normal, $\Sigma^{'}$ is almost normal, and $\Sigma, \Sigma^{'}$ cobound a product region R containing no normal or almost normal surfaces except those normally isotopic to components of ∂R, i.e., Σ or $\Sigma^{'}$.

To simplify the discussion, we will assume that M has a 0-efficient 1 vertex triangulation, as in [10]. Any closed irreducible 3-manifold has such a triangulation [10], where irreducible means any embedded 2-sphere bounds a 3-ball.

Let $\tilde{\Sigma}$ denote the boundary of a small regular neighbourhood of Σ when Σ is a 1-sided embedded normal surface. We replace Σ by $\tilde{\Sigma}$ in our discussion if Σ is 1-sided, since it is sufficient to decide if $\tilde{\Sigma}$ is essential, and so we only need to consider 2-sided surfaces. Let M^* be the result of splitting M open along Σ. Then M^* has one or two components, depending on whether Σ is nonseparating or separating, respectively.

Our key idea is that if Σ is normal and *not* essential, then there must be an adjacent pair $(\Sigma, \Sigma^{'})$. So we immediately see that if there is no such adjacent pair, then Σ is essential; see Corollary 2. However, to make this into a necessary and sufficient criterion for a normal surface Σ to be essential, we iterate the process and look at maximal sequences of adjacent pairs starting with Σ.

Theorem 2. *Suppose that M is a closed irreducible 3-manifold and Σ is a 2-sided normal surface in M. Then Σ is essential if and only if one of the two alternate sets of the following conditions are satisfied:*

1. *M is not a surface bundle over a circle with fiber Σ.*
2. *There are two (possibly empty) sequences of disjoint almost normal and normal surfaces $\Sigma_i^{'}, \Sigma_i$ for $i = 1, 2, \dots, s$, each homeomorphic to Σ.*
3. *Each of the pairs $(\Sigma_i^{'}, \Sigma_i)$, $(\Sigma_i, \Sigma_{i+1}^{'})$, and $(\Sigma, \Sigma_1^{'})$ is adjacent, and all the product regions the pairs cobound have disjoint interiors.*
4. *There are no additional almost normal surfaces $\Sigma^{''}$ homeomorphic to Σ with the property that $\Sigma_s, \Sigma^{''}$ are adjacent.*

The second possibility is:

1. *M is a surface bundle over a circle with fiber Σ.*
2. *There is one nonempty sequence of disjoint almost normal and normal surfaces $\Sigma_i^{'}, \Sigma_i$ for $i = 1, 2, \dots, s$ each homeomorphic to Σ and $\Sigma_s = \Sigma$.*
3. *Each of the pairs $(\Sigma_i^{'}, \Sigma_i), (\Sigma_i, \Sigma_{i+1}^{'})$, and $(\Sigma, \Sigma_1^{'})$ is adjacent, and all the product regions the pairs cobound have disjoint interiors.*

Proof. We first prove that these conditions imply that Σ is essential. The argument is by contradiction. Suppose that Σ is not essential but satisfies the conditions of the theorem. Then Σ has a compressing disk. Note that all the surfaces $\Sigma, \ldots, \Sigma_s, \Sigma_s^{'}$ are topologically parallel, i.e., cobound product regions with each other. Moreover, clearly M is not a surface bundle over a circle with fiber Σ. So we know that either Σ is separating or nonseparating but not a fiber of a bundle structure, and the first set of conditions (1), (2), (3), and (4) are satisfied.

By adding a thickening of a maximal collection of compressing disks to a regular neighbourhood of Σ_s, we can find a region R in each component of M^* with the following properties:

- ∂R contains Σ_s but R contains none of the other surfaces $\Sigma_i, \Sigma_i^{'}$.
- ∂R possibly contains some components which are essential surfaces in M^*.
- R is a handlebody or compression body.

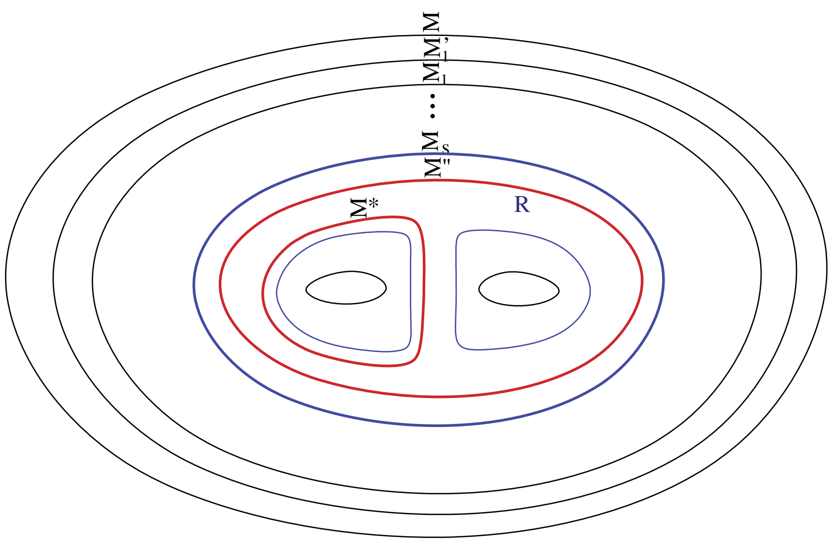

FIGURE 12. Sequence of adjacent pairs $(\Sigma_i^{'}, \Sigma_i)$

Next, we use the sweepout method of [15] or [16] or [7] across R. Notice that all the components of ∂R can be chosen to be normal surfaces, since the extra boundary components are essential surfaces in M^* and so can be normalised. The approach in [7] applies to this situation of sweeping across a handlebody or compression body with normal boundary surfaces. The conclusion is that there is some minimax complexity surface amongst all sweepouts, which is almost normal. Let Σ^* denote this almost normal minimax surface (see Figure 12).

By a simple iteration of this process, we can deduce that a (possibly new) almost normal surface $\Sigma^{''}$ is generated which is homeomorphic to Σ and is in fact adjacent to Σ_s. For if Σ^* is obtained by compressing Σ_s, then it is incompressible in the smaller compression body region R^* between Σ^* and Σ_s. So we can push Σ^* into R^* and normalise it to form a normal surface $\hat{\Sigma}$ parallel to Σ^*. But now we can run a new sweepout in the compression body between Σ_s and $\hat{\Sigma}$ to form a new almost normal surface Σ^{**}. After a finite number of steps, this process must terminate, by Haken–Kneser finiteness (see [8]). But the only way it can terminate is if an almost normal surface is constructed which normalises to Σ_s. (For if at any step an almost normal surface is formed, for which normalisation towards Σ_s produces a normal surface different from Σ_s, then a further sweepout step is possible.) We can now take this as our choice for Σ''.

Notice that since normalisation of Σ'' towards Σ_s gives Σ_s, there are no embedded normal (or almost normal) surfaces in the product region between Σ_s and Σ''. For such a surface would be a barrier to normalisation. (See [10] for a discussion of barriers for normalisation.) So we have found an almost normal surface in R adjacent to Σ_s, contrary to the assumption and which completes the proof sketch.

We now consider the converse, namely, that if Σ is essential, it satisfies the conditions of the theorem. Consider first the case that M is a surface bundle over a circle with fiber Σ. Then the existence of the sequence of adjacent pairs of normal and almost normal copies of Σ follows by the standard sweepout method; see [7].

Next assume that M is not a surface bundle over a circle with fiber Σ. The argument in this case is by contradiction, i.e., assume that the conditions fail but Σ is essential. Suppose we have found zero, one or two sequences of normal and almost normal surfaces for M^* satisfying the conditions (2) and (3) of the theorem, but condition (4) fails. We can also assume our sequences are as long as possible, so no further adjacent pairs can be added. By assumption, there is an almost normal surface Σ'' disjoint from our sequence, so that Σ'' is homeomorphic to Σ or $\tilde{\Sigma}$ and bounds a region with Σ_s so that the pair (Σ_s, Σ'') are adjacent.

But now we can push off Σ'' away from the product region cobounded with Σ_s to reduce complexity, i.e., by performing the usual normalisation moves. Since Σ and the surface Σ'' parallel to either Σ or $\tilde{\Sigma}$ are essential, normalisation must produce a new normal surface homeomorphic to Σ or $\tilde{\Sigma}$. So we have extended our sequence with a new adjacent pair, satisfying conditions (2) and (3) of the theorem.

Therefore there is no maximal sequence satisfying conditions (2) and (3). But an elementary application of Haken–Kneser finiteness methods shows that the process of building such a sequence must terminate after a finite number of steps, depending only on the triangulation (see [8]). So this completes the proof of the theorem. □

Corollary 2. *Suppose that M is a closed 3-manifold with a 0-efficient triangulation and Σ is a closed embedded normal surface. If Σ is 2-sided, assume there are no almost normal surfaces disjoint from Σ which are homeomorphic to Σ. Similarly if Σ is 1-sided, assume there are no disjoint almost normal surfaces homeomorphic to $\tilde{\Sigma}$. Then Σ is essential.*

Proof. The case when Σ is 2-sided follows immediately from Theorem 2. Suppose next that Σ is 1-sided. Consider the boundary $\tilde{\Sigma}$ of a small regular neighbourhood of Σ. We can apply Theorem 2 to $\tilde{\Sigma}$ and deduce it is essential. But then as is well known this also implies that Σ is essential. □

Remark 2. Note that Corollary 2 implies Theorem 1 for the case of closed 3-manifolds with 0-efficient triangulations. Suppose M is closed, triangulated, and contains an embedded normal surface Σ which has a quadrilateral in every tetrahedron and there are no normal surfaces disjoint from Σ with Euler characteristic greater than that of Σ. We claim that there are no almost normal surfaces disjoint from Σ, homeomorphic to Σ or $\tilde{\Sigma}$ depending on whether we are in the 2-sided or 1-sided cases. For if there is a tubed such almost normal surface, then removing the tube gives a normal surface with Euler characteristic greater than Σ or $\tilde{\Sigma}$ and disjoint from Σ, contrary to the assumption. On the other hand, there cannot be an almost normal surface with an octagon, which is disjoint from Σ, since any octagon in a tetrahedron Δ must intersect the quadrilaterals of Σ in Δ.

Remark 3. Theorem 2 gives a new way of testing whether or not normal surfaces Σ are essential. So together with the result of [9], to decide if a closed irreducible 3-manifold is Haken, i.e., contains an essential surface, the finite number of vertex solutions of the projective solution space can be checked to see whether or not any are essential. If we consider the cell structure of a component of M^*, the result of cutting open along Σ or $\tilde{\Sigma}$, then by studying normal (almost normal) surface theory in this space, we can check the conditions of Theorem 2. In [6] this method is much further developed to give an algorithm to decide if two normal surfaces represent isotopic or nonisotopic essential surfaces.

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Department of Mathematics, College of Natural Sciences, Chosun University, Kwangju, 501-759, South Korea
Email address: ekang@chosun.ac.kr

School of Mathematics and Statistics, University of Melbourne Parkville, Peter Hall Building, Parkville, Victoria, Australia
Email address: joachim@unimelb.edu.au