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GENERIC LINEAR PERTURBATIONS

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ABSTRACT. In his celebrated paper *Generic projections*, John Mather has shown that almost all linear projections from a submanifold of a vector space into a subspace are transverse with respect to a given modular submanifold. In this paper, an improvement of Mather's result is stated. Namely, we show that almost all linear perturbations of a smooth mapping from a submanifold of $\mathbb{R}^m$ into $\mathbb{R}^\ell$ yield a transverse mapping with respect to a given modular submanifold. Moreover, applications of this result are given.

1. INTRODUCTION

Throughout this paper, ℓ, m, and n stand for positive integers. In this paper, unless otherwise stated, all manifolds and mappings belong to class C^∞ and all manifolds are without boundary.

An n-dimensional manifold is denoted by N. Let $\pi : \mathbb{R}^m \to \mathbb{R}^\ell$ be a linear mapping.

In a celebrated paper [17], for a given embedding $f : N \to \mathbb{R}^m$, a composition $\pi \circ f : N \to \mathbb{R}^\ell$ $(m > \ell)$ is investigated and the following assertions (M1)–(M5) are obtained for a generic mapping. All of (M1)–(M5) follow from the main result (Theorem 1 in Section 2) proved by Mather.

- (M1) If $(n, \ell) = (n, 1)$, then a generic function $\pi \circ f : N \to \mathbb{R}$ is a Morse function.
- (M2) If $(n, \ell) = (2, 2)$, then a generic mapping $\pi \circ f : N \to \mathbb{R}^2$ is an excellent map in the sense defined by Whitney in [20].
- (M3) If $(n, \ell) = (2, 3)$, then the only singularities of the image of a generic mapping $\pi \circ f : N \to \mathbb{R}^3$ are normal crossings and pinch points.
- (M4) A generic mapping $\pi \circ f : N \to \mathbb{R}^\ell$ is transverse with respect to the Thom–Boardman varieties (for the definition of Thom–Boardman varieties, refer to [1], [2], [16], [19]).
- (M5) If (n, ℓ) is in the nice range of dimensions (for the definition of nice range of dimensions, refer to [15]), then a generic mapping $\pi \circ f : N \to \mathbb{R}^\ell$ is locally infinitesimally stable (for the definition of local infinitesimal stability, see Section 2). If, moreover, N is compact, then a generic mapping $\pi \circ f : N \to \mathbb{R}^\ell$ is stable (for the definition of stability, see Section 2).

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Let $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ be the space consisting of linear mappings of $\mathbb{R}^m$ into $\mathbb{R}^\ell$. For a given embedding $f : N \to \mathbb{R}^m$, a property of mappings $\pi \circ f : N \to \mathbb{R}^\ell$ will be said to be true for a *generic mapping* if there exists a subset Σ with Lebesgue measure zero of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, $\pi \circ f : N \to \mathbb{R}^\ell$ has the property.

The main aim of this paper is to prove Theorem 2 in Section 2, which is an improvement of Theorem 1 in Section 2, shown by Mather ([17]).

Let U be an open subset of $\mathbb{R}^m$ and let $F : U \to \mathbb{R}^\ell$ be a mapping. For any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$, set F_π as follows:

$$F_\pi = F + \pi.$$

Here, the mapping π in $F_\pi = F + \pi$ is restricted to U. For a given embedding $f : N \to U$ and a given mapping $F : U \to \mathbb{R}^\ell$, a property of mappings $F_\pi \circ f : N \to \mathbb{R}^\ell$ will be said to be true for a *generic mapping* if there exists a subset Σ with Lebesgue measure zero of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, $F_\pi \circ f : N \to \mathbb{R}^\ell$ has the property. For a given embedding $f : N \to U$, from Theorem 2, the following assertions hold:

(I1) If $(n, \ell) = (n, 1)$, then a generic function $F_\pi \circ f : N \to \mathbb{R}$ is a Morse function.
(I2) If $(n, \ell) = (2, 2)$, then a generic mapping $F_\pi \circ f : N \to \mathbb{R}^2$ is an excellent map.
(I3) If $(n, \ell) = (2, 3)$, then the only singularities of the image of a generic mapping $F_\pi \circ f : N \to \mathbb{R}^3$ are normal crossings and pinch points.
(I4) A generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is transverse with respect to the Thom–Boardman varieties.
(I5) If (n, ℓ) is in the nice range of dimensions, then a generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is locally infinitesimally stable. If, moreover, N is compact, then a generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is stable.

Assertion (M5) (resp., (I5)) above implies assertions (M1), (M2), and (M3) (resp., (I1), (I2), and (I3)). Both assertions (M4) and (M5) (resp., (I4) and (I5)) follow from Theorem 1 (resp., Theorem 2) of Section 2. Moreover, in the special case of $F = 0$, $U = \mathbb{R}^m$, and $m > \ell$, assertions (I1)–(I5) are exactly the same as assertions (M1)–(M5), respectively. Note that in the case $m \leq \ell$, a generic mapping $\pi \circ f : N \to \mathbb{R}^\ell$ is an embedding. On the other hand, in the same case, a generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is not necessarily an embedding.

The original motivation for this work is to investigate the stability of quadratic mappings of $\mathbb{R}^m$ into $\mathbb{R}^\ell$ of a special type called "generalized distance-squared mappings" (for the precise definition of generalized distance-squared mappings, see Section 4). In [12] (resp., [11]), the generalized distance-squared mappings in the case $(m, \ell) = (2, 2)$ (resp., $(m, \ell) = (k + 1, 2k + 1)$) have been investigated, where k is a positive integer. As an application of (I5), if (m, ℓ) is in the nice range of dimensions, then it is shown that a generic generalized distance-squared mapping of $\mathbb{R}^m$ into $\mathbb{R}^\ell$ is locally infinitesimally stable (see Corollary 4 and Remark 2 in Section 4).

Notice that for example, the references [4] and [18] are also important papers related to generic projections. In [4], an improvement of Mather's result is given by replacing a given embedding $f : N \to \mathbb{R}^m$ by a given stable mapping $f : N \to \mathbb{R}^m$ (see Theorem 2.2 in [4]). On the other hand, in this paper, an improvement of

Mather's result is given by replacing generic projections by generic linear perturbations.

In Section 2, some standard definitions and the important notion of a "modular" submanifold (Definition 1) defined in [17] are reviewed, and the main theorem (Theorem 2) in this paper is stated. Section 3 is devoted to the proof of Theorem 2. In Section 4, the motivation to investigate the stability of generalized distance-squared mappings is given in detail, and as applications of the main theorem, results containing Corollary 4 are stated.

2. Preliminaries and the statement of the main result

Let N and P be manifolds and let $J^r(N,P)$ be the space of r-jets of mappings of N into P. For a given mapping $g : N \to P$, the mapping $j^r g : N \to J^r(N,P)$ is defined by $q \mapsto j^r g(q)$. Let $C^\infty(N,P)$ be the set of C^∞ mappings of N into P, and the topology on $C^\infty(N,P)$ is the Whitney C^∞ topology (for the definition of Whitney C^∞ topology, see for example [8]). Given $g, h \in C^\infty(N,P)$, we say that g is $\mathcal{A}$-*equivalent* to h if there exist diffeomorphisms $\Phi : N \to N$ and $\Psi : P \to P$ such that $g = \Psi \circ h \circ \Phi^{-1}$. Then, g is said to be *stable* if the $\mathcal{A}$-equivalence class of g is open in $C^\infty(N,P)$.

Let s be a positive integer. Define $N^{(s)}$ as follows:

$$N^{(s)} = \{(q_1, \ldots, q_s) \mid q_i \neq q_j (1 \leq i < j \leq s)\}.$$

Let ${}_sJ^r(N,P)$ be the space consisting of elements $(j^r g(q_1), \ldots, j^r g(q_s)) \in J^r(N,P)^s$ satisfying $(q_1, \ldots, q_s) \in N^{(s)}$. Since $N^{(s)}$ is an open submanifold of N^s, the space ${}_sJ^r(N,P)$ is also an open submanifold of $J^r(N,P)^s$. For a given mapping $g : N \to P$, the mapping ${}_sj^r g : N^{(s)} \to {}_sJ^r(N,P)$ is defined by $(q_1, \ldots, q_s) \mapsto (j^r g(q_1), \ldots, j^r g(q_s))$.

Let W be a submanifold of ${}_sJ^r(N,P)$. For a given mapping $g : N \to P$, we say that ${}_sj^r g : N^{(s)} \to {}_sJ^r(N,P)$ is *transverse* to W if for any $q \in N^{(s)}$, ${}_sj^r g(q) \notin W$, or in the case of ${}_sj^r g(q) \in W$, the following holds:

$$d({}_sj^r g)_q(T_q N^{(s)}) + T_{{}_sj^r g(q)} W = T_{{}_sj^r g(q)}\, {}_sJ^r(N,P).$$

A mapping $g : N \to P$ will be said to be *transverse with respect to* W if ${}_sj^r g : N^{(s)} \to {}_sJ^r(N,P)$ is transverse to W.

Following Mather ([17]), we can partition P^s as follows. Given any partition π of $\{1, \ldots, s\}$, let P^π denote the set of s-tuples $(y_1, \ldots, y_s) \in P^s$ such that $y_i = y_j$ if and only if two positive integers i and j are in the same member of the partition π.

Let Diff N denote the group of diffeomorphisms of N. There is a natural action of Diff $N \times$ Diff P on ${}_sJ^r(N,P)$ such that for a mapping $g : N \to P$, the equality $(h, H) \cdot {}_sj^r g(q) = {}_sj^r(H \circ g \circ h^{-1})(q')$ holds, where $q = (q_1, \ldots, q_s)$ and $q' = (h(q_1), \ldots, h(q_s))$. A subset W of ${}_sJ^r(N,P)$ is said to be *invariant* if it is invariant under this action.

We recall the following identification $(*)$ from [17]. Let $q = (q_1, \ldots, q_s) \in N^{(s)}$, let $g : U \to P$ be a mapping defined in a neighborhood U of $\{q_1, \ldots, q_s\}$ in N, and let $z = {}_sj^r g(q)$, $q' = (g(q_1), \ldots, g(q_s))$. Let ${}_sJ^r(N,P)_q$ and ${}_sJ^r(N,P)_{q,q'}$ denote the fibers of ${}_sJ^r(N,P)$ over q and over (q, q'), respectively. Let $J^r(N)_q$ denote the $\mathbb{R}$-algebra of r-jets at q of functions on N. Namely,

$$J^r(N)_q = {}_sJ^r(N, \mathbb{R})_q.$$

Set $g^*TP = \bigcup_{\widetilde{q}\in U} T_{g(\widetilde{q})}P$, where TP is the tangent bundle of P. Let $J^r(g^*TP)_q$ denote the $J^r(N)_q$-module of r-jets at q of sections of the bundle g^*TP. Let $\mathfrak{m}_q$ be the ideal in $J^r(N)_q$ consisting of jets of functions which vanish at q. Namely,

$$\mathfrak{m}_q = \{{}_sj^r h(q) \in {}_sJ^r(N,\mathbb{R})_q \mid h(q_1) = \cdots = h(q_s) = 0\}.$$

Let $\mathfrak{m}_q J^r(g^*TP)_q$ be the set consisting of finite sums of products of an element of $\mathfrak{m}_q$ and an element of $J^r(g^*TP)_q$. Namely, we get

$$\mathfrak{m}_q J^r(g^*TP)_q = J^r(g^*TP)_q \cap \{{}_sj^r\xi(q) \in {}_sJ^r(N,TP)_q \mid \xi(q_1) = \cdots = \xi(q_s) = 0\}.$$

Then, it is seen that the following canonical identification of $\mathbb{R}$ vector spaces $(*)$ holds:

$$(*) \qquad T({}_sJ^r(N,P)_{q,q'})_z = \mathfrak{m}_q J^r(g^*TP)_q.$$

Let W be a nonempty submanifold of ${}_sJ^r(N,P)$. Choose $q = (q_1,\ldots,q_s) \in N^{(s)}$ and $g : N \to P$, and let $z = {}_sj^r g(q)$ and $q' = (g(q_1),\ldots,g(q_s))$. Suppose that the choice is made so that $z \in W$. Set $W_{q,q'} = \widetilde{\pi}^{-1}(q,q')$, where $\widetilde{\pi} : W \to N^{(s)} \times P^s$ is defined by $\widetilde{\pi}({}_sj^r\widetilde{g}(\widetilde{q})) = (\widetilde{q}, (\widetilde{g}(\widetilde{q}_1),\ldots,\widetilde{g}(\widetilde{q}_s)))$ and $\widetilde{q} = (\widetilde{q}_1,\ldots,\widetilde{q}_s) \in N^{(s)}$. Suppose that $W_{q,q'}$ is a submanifold of ${}_sJ^r(N,P)$. Then, under the identification $(*)$, the tangent space $T(W_{q,q'})_z$ can be identified with a vector subspace of $\mathfrak{m}_q J^r(g^*TP)_q$. We denote this vector subspace by $E(g,q,W)$.

Definition 1. We say that a submanifold W of ${}_sJ^r(N,P)$ is *modular* if conditions (α) and (β) below are satisfied:

- (α) The set W is an invariant submanifold of ${}_sJ^r(N,P)$ and lies over P^π for some partition π of $\{1,\ldots,s\}$.
- (β) For any $q \in N^{(s)}$ and any mapping $g : N \to P$ such that ${}_sj^r g(q) \in W$, the subspace $E(g,q,W)$ is a $J^r(N)_q$-submodule.

Now, suppose that $P = \mathbb{R}^\ell$. The main theorem of [17] is the following.

Theorem 1 ([17]). *Let N be a manifold of dimension n. Let f be an embedding of N into $\mathbb{R}^m$. If W is a modular submanifold of ${}_sJ^r(N,\mathbb{R}^\ell)$ and $m > \ell$, then there exists a subset Σ with Lebesgue measure zero of $\mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell)$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell) - \Sigma$, $\pi \circ f : N \to \mathbb{R}^\ell$ is transverse with respect to W.*

Then, the main theorem in this paper is the following.

Theorem 2. *Let N be a manifold of dimension n. Let f be an embedding of N into an open subset U of $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. If W is a modular submanifold of ${}_sJ^r(N,\mathbb{R}^\ell)$, then there exists a subset Σ with Lebesgue measure zero of $\mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell)$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell) - \Sigma$, $F_\pi \circ f : N \to \mathbb{R}^\ell$ is transverse with respect to W.*

It follows that the Thom–Boardman varieties are modular by Mather (see [16] and [17]). Hence, we have the following as a corollary of Theorem 2.

Corollary 1. *Let N be a manifold of dimension n. Let f be an embedding of N into an open subset U of $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. Then, there exists a subset Σ with Lebesgue measure zero of $\mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell)$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell) - \Sigma$, $F_\pi \circ f : N \to \mathbb{R}^\ell$ is transverse with respect to the Thom–Boardman varieties.*

Let S be a finite subset of N and let y be a point of P. Let $g : (N, S) \to (P, y)$ be a map-germ. A map-germ $\xi : (N, S) \to (TP, \xi(S))$ such that $\Pi \circ \xi = g$ is called a *vector field along* g, where $\Pi : TP \to P$ is the canonical projection. Let $\theta(g)_S$ be the set consisting of vector fields along g. Set $\theta(N)_S = \theta(\mathrm{id}_N)_S$ and $\theta(P)_y = \theta(\mathrm{id}_P)_y$, where $\mathrm{id}_N : (N, S) \to (N, S)$ and $\mathrm{id}_P : (P, y) \to (P, y)$ are the identify map-germs. The mapping $tg : \theta(N)_S \to \theta(g)_S$ is defined by $tg(\xi) = Tg \circ \xi$, where $Tg : TN \to TP$ is the derivative mapping of g. The mapping $\omega g : \theta(P)_y \to \theta(g)_S$ is defined by $\omega g(\eta) = \eta \circ g$. Then, a mapping $g : N \to P$ is said to be *locally infinitesimally stable* if the following holds for every $y \in P$ and every finite subset $S \subset g^{-1}(y)$ ([14] and [17]):

$$tg(\theta(N)_S) + \omega g(\theta(P)_y) = \theta(g)_S.$$

By the same way as in the proof of Theorem 3 of [17], we have the following as a corollary of Theorem 2.

Corollary 2. *Let N be a manifold of dimension n. Let f be an embedding of N into an open subset U of $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. If a dimension pair (n, ℓ) is in the nice dimensions, then there exists a subset Σ with Lebesgue measure zero of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the composition $F_\pi \circ f : N \to \mathbb{R}^\ell$ is locally infinitesimally stable.*

Remark 1.

(1) In the special case that $F = 0$, $U = \mathbb{R}^m$, and $m > \ell$, Theorem 2 is Theorem 1.
(2) The set Σ in Mather's theorem (Theorem 1) depends only on $f : N \to \mathbb{R}^m$ and a modular submanifold W of ${}_sJ^r(N, \mathbb{R}^\ell)$. On the other hand, Σ in the main theorem of this paper (Theorem 2) depends on $F : U \to \mathbb{R}^\ell$, too.
(3) Suppose that the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is proper in Corollary 2. Then, the local infinitesimal stability of $F_\pi \circ f$ implies the stability of it (see [14]).
(4) We explain the advantage that the domain of the mapping F is not $\mathbb{R}^m$ but an open set U. Suppose that $U = \mathbb{R}$. Let $F : \mathbb{R} \to \mathbb{R}$ be the mapping defined by $x \mapsto |x|$. Since F is not differentiable at $x = 0$, we cannot apply Theorem 2 to the mapping $F : \mathbb{R} \to \mathbb{R}$.
    On the other hand, if $U = \mathbb{R} - \{0\}$, then Theorem 2 can be applied to the restriction $F|_U$.

3. Proof of Theorem 2

Let $(\alpha_{ij})_{1 \le i \le \ell, 1 \le j \le m}$ be a representing matrix of a linear mapping $\pi : \mathbb{R}^m \to \mathbb{R}^\ell$. Set $F_\alpha = F_\pi$, and we have

$$F_\alpha(x) = \left(F_1(x) + \sum_{j=1}^{m} \alpha_{1j} x_j, \ldots, F_\ell(x) + \sum_{j=1}^{m} \alpha_{\ell j} x_j \right),$$

where $\alpha = (\alpha_{11}, \ldots, \alpha_{1m}, \ldots, \alpha_{\ell 1}, \ldots, \alpha_{\ell m}) \in (\mathbb{R}^m)^\ell$, $F = (F_1, \ldots, F_\ell)$, and $x = (x_1, \ldots, x_m)$. For a given embedding $f : N \to U$, a mapping $F_\alpha \circ f : N \to \mathbb{R}^\ell$ is as follows:

$$F_\alpha \circ f = \left(F_1 \circ f + \sum_{j=1}^{m} \alpha_{1j} f_j, \ldots, F_\ell \circ f + \sum_{j=1}^{m} \alpha_{\ell j} f_j \right),$$

where $f = (f_1, \dots, f_m)$. Since there is the natural identification $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$, in order to prove Theorem 2, it is sufficient to show that there exists a subset Σ with Lebesgue measure zero of $(\mathbb{R}^m)^\ell$ such that for any $\alpha \in (\mathbb{R}^m)^\ell - \Sigma$, the mapping ${}_sj^r(F_\alpha \circ f) : N^{(s)} \to {}_sJ^r(N, \mathbb{R}^\ell)$ is transverse to the given modular submanifold W.

Let $H_\Lambda : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be the linear isomorphism defined by

$$H_\Lambda(X_1, \dots, X_\ell) = (X_1, \dots, X_\ell)\Lambda,$$

where $\Lambda = (\lambda_{ij})_{1\le i\le \ell, 1\le j\le \ell}$ is an $\ell \times \ell$ regular matrix. The composition of H_Λ and $F_\alpha \circ f$ is as follows:

$$H_\Lambda \circ F_\alpha \circ f = \left(\sum_{k=1}^{\ell}\left(F_k \circ f + \sum_{j=1}^{m} \alpha_{kj} f_j\right)\lambda_{k1}, \dots, \sum_{k=1}^{\ell}\left(F_k \circ f + \sum_{j=1}^{m} \alpha_{kj} f_j\right)\lambda_{k\ell}\right)$$

$$= \left(\sum_{k=1}^{\ell}\left(F_k\circ f\right)\lambda_{k1} + \sum_{j=1}^{m}\left(\sum_{k=1}^{\ell}\lambda_{k1}\alpha_{kj}\right) f_j, \dots, \sum_{k=1}^{\ell}\left(F_k\circ f\right)\lambda_{k\ell} + \sum_{j=1}^{m}\left(\sum_{k=1}^{\ell}\lambda_{k\ell}\alpha_{kj}\right) f_j\right).$$

Set $\mathrm{GL}(\ell) = \{B \mid B : \ell \times \ell \text{ matrix}, \det B \neq 0\}$. Let $\varphi : \mathrm{GL}(\ell) \times (\mathbb{R}^m)^\ell \to \mathrm{GL}(\ell) \times (\mathbb{R}^m)^\ell$ be the mapping as follows:

$$\begin{aligned}
&\varphi(\lambda_{11}, \lambda_{12}, \dots, \lambda_{\ell\ell}, \alpha_{11}, \alpha_{12}, \dots, \alpha_{\ell m}) \\
&= \Bigg(\lambda_{11}, \lambda_{12}, \dots, \lambda_{\ell\ell}, \sum_{k=1}^{\ell}\lambda_{k1}\alpha_{k1}, \sum_{k=1}^{\ell}\lambda_{k2}\alpha_{k1}, \dots, \sum_{k=1}^{\ell}\lambda_{k\ell}\alpha_{k1}, \\
&\sum_{k=1}^{\ell}\lambda_{k1}\alpha_{k2}, \sum_{k=1}^{\ell}\lambda_{k2}\alpha_{k2}, \dots, \sum_{k=1}^{\ell}\lambda_{k\ell}\alpha_{k2}, \dots, \\
&\qquad\qquad \sum_{k=1}^{\ell}\lambda_{k1}\alpha_{km}, \sum_{k=1}^{\ell}\lambda_{k2}\alpha_{km}, \dots, \sum_{k=1}^{\ell}\lambda_{k\ell}\alpha_{km}\Bigg).
\end{aligned}$$

For the proof of Theorem 2, it is the key to show that φ is a C^∞ diffeomorphism. In order to show that φ is a C^∞ diffeomorphism, for any point $(\Lambda', \alpha') \in \mathrm{GL}(\ell) \times (\mathbb{R}^m)^\ell$ of the target space of φ, we will find (Λ, α) satisfying $\varphi(\Lambda, \alpha) = (\Lambda', \alpha')$, where $\Lambda = (\lambda_{11}, \lambda_{12}, \dots, \lambda_{\ell\ell})$, $\Lambda' = (\lambda'_{11}, \lambda'_{12}, \dots, \lambda'_{\ell\ell})$, $\alpha = (\alpha_{11}, \alpha_{12}, \dots, \alpha_{\ell m})$, and $\alpha' = (\alpha'_{11}, \alpha'_{12}, \dots, \alpha'_{m\ell})$. Hence, it is sufficient to find (Λ, α) satisfying

$$\begin{aligned}
\lambda_{ij} &= \lambda'_{ij} \ (1 \le i \le \ell, 1 \le j \le \ell), \\
\sum_{k=1}^{\ell}\lambda_{ki}\alpha_{kj} &= \alpha'_{ji} \ (1 \le i \le \ell, 1 \le j \le m).
\end{aligned}$$

Therefore, for any j $(1 \le j \le m)$, we get

$$\sum_{k=1}^{\ell}\lambda'_{k1}\alpha_{kj} = \alpha'_{j1}, \ \sum_{k=1}^{\ell}\lambda'_{k2}\alpha_{kj} = \alpha'_{j2}, \dots, \ \sum_{k=1}^{\ell}\lambda'_{k\ell}\alpha_{kj} = \alpha'_{j\ell}.$$

Thus, for any j $(1 \le j \le m)$, we have the following:

$$\begin{pmatrix} \lambda'_{11} & \cdots & \lambda'_{\ell 1} \\ \vdots & \ddots & \vdots \\ \lambda'_{1\ell} & \cdots & \lambda'_{\ell\ell} \end{pmatrix} \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{\ell j} \end{pmatrix} = \begin{pmatrix} \alpha'_{j1} \\ \vdots \\ \alpha'_{j\ell} \end{pmatrix}.$$

Since the matrix

$$\begin{pmatrix} \lambda'_{11} & \cdots & \lambda'_{\ell 1} \\ \vdots & \ddots & \vdots \\ \lambda'_{1\ell} & \cdots & \lambda'_{\ell\ell} \end{pmatrix}$$

is regular, for any j $(1 \leq j \leq m)$, $\alpha_{1j}, \ldots, \alpha_{\ell j}$ can be expressed by rational functions of $\lambda'_{11}, \ldots, \lambda'_{\ell\ell}$, $\alpha'_{j1}, \ldots, \alpha'_{j\ell}$. Therefore, there exists the inverse mapping φ^{-1} and we see that φ^{-1} is of class C^∞. Hence, the mapping φ is a C^∞ diffeomorphism.

Next, let $\widetilde{f} : U \to \mathbb{R}^{m+\ell}$ be the mapping as follows:

$$\widetilde{f}(x_1, \ldots, x_m) = (F_1(x_1, \ldots, x_m), \ldots, F_\ell(x_1, \ldots, x_m), x_1, \ldots, x_m).$$

It is clearly seen that $\widetilde{f}$ is an embedding. Since $f : N \to U$ is an embedding, the mapping $\widetilde{f} \circ f : N \to \mathbb{R}^{m+\ell}$ is also an embedding:

$$\widetilde{f} \circ f = (F_1 \circ f, \ldots, F_\ell \circ f, f_1, \ldots, f_m).$$

In order to prove Theorem 2, the following lemma is important. The following lemma is the special case of Theorem 1.

Lemma 1 ([17])**.** *Let N be a manifold of dimension n. Let $\widetilde{f} \circ f : N \to \mathbb{R}^{m+\ell}$ be an embedding. If W is a modular submanifold of ${}_sJ^r(N, \mathbb{R}^\ell)$, then there exists a subset Σ with Lebesgue measure zero of $\mathcal{L}(\mathbb{R}^{m+\ell}, \mathbb{R}^\ell)$ such that for any $\Pi \in \mathcal{L}(\mathbb{R}^{m+\ell}, \mathbb{R}^\ell) - \Sigma$, the mapping ${}_sj^r(\Pi \circ \widetilde{f} \circ f) : N^{(s)} \to {}_sJ^r(N, \mathbb{R}^\ell)$ is transverse to W.*

From Lemma 1, there exists a subset Σ with Lebesgue measure zero of $\mathcal{L}(\mathbb{R}^{m+\ell}, \mathbb{R}^\ell)$ such that for any $\Pi \in \mathcal{L}(\mathbb{R}^{m+\ell}, \mathbb{R}^\ell) - \Sigma$, the mapping ${}_sj^r(\Pi \circ (\widetilde{f} \circ f)) : N^{(s)} \to {}_sJ^r(N, \mathbb{R}^\ell)$ is transverse to W.

There is the natural identification $\mathcal{L}(\mathbb{R}^{m+\ell}, \mathbb{R}^\ell) = \mathbb{R}^{\ell(m+\ell)}$. Thus, we identify the target space $\mathrm{GL}(\ell) \times (\mathbb{R}^m)^\ell$ of the mapping φ with an open submanifold of $\mathcal{L}(\mathbb{R}^{m+\ell}, \mathbb{R}^\ell)$. Since the intersection $(\mathrm{GL}(\ell) \times (\mathbb{R}^m)^\ell) \cap \Sigma$ is a subset with Lebesgue measure zero of $\mathrm{GL}(\ell) \times (\mathbb{R}^m)^\ell$ and the mapping φ^{-1} is of class C^∞, it follows that $\varphi^{-1}((\mathrm{GL}(\ell) \times (\mathbb{R}^m)^\ell) \cap \Sigma)$ is a subset with Lebesgue measure zero of $\mathrm{GL}(\ell) \times (\mathbb{R}^m)^\ell$. For any $(\Lambda, \alpha) \in \mathrm{GL}(\ell) \times (\mathbb{R}^m)^\ell$, let $\Pi_{(\Lambda,\alpha)} : \mathbb{R}^{m+\ell} \to \mathbb{R}^\ell$ be the linear mapping defined by $\varphi(\Lambda, \alpha)$ as follows:

$$\begin{aligned} &\Pi_{(\Lambda,\alpha)}(X_1, \ldots, X_{m+\ell}) \\ = \; &(X_1, \ldots, X_{m+\ell}) \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1\ell} \\ \vdots & \ddots & \vdots \\ \lambda_{\ell 1} & \cdots & \lambda_{\ell\ell} \\ \displaystyle\sum_{k=1}^{\ell} \lambda_{k1}\alpha_{k1} & \cdots & \displaystyle\sum_{k=1}^{\ell} \lambda_{k\ell}\alpha_{k1} \\ \vdots & \ddots & \vdots \\ \displaystyle\sum_{k=1}^{\ell} \lambda_{k1}\alpha_{km} & \cdots & \displaystyle\sum_{k=1}^{\ell} \lambda_{k\ell}\alpha_{km} \end{pmatrix}. \end{aligned}$$

Then, we have the following:

$$
\begin{aligned}
&\Pi_{(\Lambda,\alpha)} \circ \widetilde{f} \circ f \\
&= \left(\sum_{k=1}^{\ell} \Big(F_k \circ f \Big) \lambda_{k1} + \sum_{j=1}^{m} \Big(\sum_{k=1}^{\ell} \lambda_{k1} \alpha_{kj} \Big) f_j, \ldots, \right. \\
&\qquad \left. \sum_{k=1}^{\ell} \Big(F_k \circ f \Big) \lambda_{k\ell} + \sum_{j=1}^{m} \Big(\sum_{k=1}^{\ell} \lambda_{k\ell} \alpha_{kj} \Big) f_j \right) \\
&= H_\Lambda \circ F_\alpha \circ f.
\end{aligned}
$$

Therefore, for any $(\Lambda, \alpha) \in \mathrm{GL}(\ell) \times (\mathbb{R}^m)^\ell - \varphi^{-1}((\mathrm{GL}(\ell) \times (\mathbb{R}^m)^\ell) \cap \Sigma)$, it follows that ${}_sj^r(\Pi_{(\Lambda,\alpha)} \circ \widetilde{f} \circ f)$ $(= {}_sj^r(H_\Lambda \circ F_\alpha \circ f))$ is transverse to W. Since the mapping H_Λ is a diffeomorphism, we see that ${}_sj^r(F_\alpha \circ f)$ is transverse to W.

Let $\widetilde{\Sigma}$ be a subset consisting of $\alpha \in (\mathbb{R}^m)^\ell$ such that ${}_sj^r(F_\alpha \circ f)$ is not transverse to W. In order to prove Theorem 2, it is sufficient to show that $\widetilde{\Sigma}$ is a subset with Lebesgue measure zero of $(\mathbb{R}^m)^\ell$. Suppose that $\widetilde{\Sigma}$ is not a subset with Lebesgue measure zero of $(\mathbb{R}^m)^\ell$. Then, $\mathrm{GL}(\ell) \times \widetilde{\Sigma}$ is not a subset with Lebesgue measure zero of $\mathrm{GL}(\ell) \times (\mathbb{R}^m)^\ell$. For any $(\Lambda, \alpha) \in \mathrm{GL}(\ell) \times \widetilde{\Sigma}$, since ${}_sj^r(F_\alpha \circ f)$ is not transverse to W and the mapping H_Λ is a diffeomorphism, ${}_sj^r(H_\Lambda \circ F_\alpha \circ f)$ is not transverse to W. This contradicts to the claim that $\varphi^{-1}((\mathrm{GL}(\ell) \times (\mathbb{R}^m)^\ell) \cap \Sigma)$ is a subset with Lebesgue measure zero of $\mathrm{GL}(\ell) \times (\mathbb{R}^m)^\ell$. ☐

4. Applications of the main result

4.1. **Introduction of generalized distance-squared mappings.** In this subsection, the definition of generalized distance-squared mappings and the motivation to investigate the mappings are given. Moreover, for the sake of the reader's convenience, the main properties of generalized distance-squared mappings are also reviewed (for more details on properties of generalized distance-squared mappings, refer to [9], [10], [11], [12]).

Let i, j be positive integers, and let $p_i = (p_{i1}, p_{i2}, \ldots, p_{im})$ $(1 \le i \le \ell)$ (resp., $A = (a_{ij})_{1 \le i \le \ell, 1 \le j \le m}$) be points of $\mathbb{R}^m$ (resp., an $\ell \times m$ matrix with nonzero entries). Set $p = (p_1, p_2, \ldots, p_\ell) \in (\mathbb{R}^m)^\ell$. Let $G_{(p,A)} : \mathbb{R}^m \to \mathbb{R}^\ell$ be the mapping defined by

$$
G_{(p,A)}(x) = \left(\sum_{j=1}^{m} a_{1j}(x_j - p_{1j})^2, \sum_{j=1}^{m} a_{2j}(x_j - p_{2j})^2, \ldots, \sum_{j=1}^{m} a_{\ell j}(x_j - p_{\ell j})^2 \right),
$$

where $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$. The mapping $G_{(p,A)}$ is called a *generalized distance-squared mapping*, and the ℓ-tuple of points $p = (p_1, p_2, \ldots, p_\ell) \in (\mathbb{R}^m)^\ell$ is called the *central point* of the generalized distance-squared mapping $G_{(p,A)}$.

A *distance-squared mapping* D_p (resp., *Lorentzian distance-squared mapping* L_p) is the mapping $G_{(p,A)}$ satisfying that each entry of A is 1 (resp., $a_{i1} = -1$ and $a_{ij} = 1$ $(j \ne 1)$).

In [9] (resp., [10]), a classification result on distance-squared mappings D_p (resp., Lorentzian distance-squared mappings L_p) is given.

In [12], a classification result on generalized distance-squared mappings of the plane into the plane is given. If the rank of A is equal to two, then a generalized distance-squared mapping having a generic central point is a stable mapping of which any singular point is a fold point except one cusp point (for details on fold points and cusp points, refer to [20]). If the rank of A is equal to one, then a generalized distance-squared mapping having a generic central point is $\mathcal{A}$-equivalent to the normal form of a definite fold mapping $(x_1, x_2) \to (x_1, x_2^2)$. Since the normal form of a definite fold mapping is proper, it is easily shown that the mapping is stable by Mather's characterization theorem of stable proper mappings given in [14].

In [11], a classification result on generalized distance-squared mappings of $\mathbb{R}^{m+1}$ into $\mathbb{R}^{2m+1}$ is given. If the rank of A is equal to $m+1$, then a generalized distance-squared mapping having a generic central point is $\mathcal{A}$-equivalent to the mapping called the normal form of Whitney umbrella as follows:

$$(x_1, \ldots, x_{m+1}) \mapsto (x_1^2, x_1x_2, \ldots, x_1x_{m+1}, x_2, \ldots, x_{m+1}).$$

The normal form of Whitney umbrella is proper and stable. If the rank of A is less than $m+1$, then a generalized distance-squared mapping having a generic central point is $\mathcal{A}$-equivalent to the inclusion as follows:

$$(x_1, \ldots, x_{m+1}) \mapsto (x_1, \ldots, x_{m+1}, 0, \ldots, 0).$$

The inclusion is proper and stable.

Quadratic polynomial mappings have been investigated for particular pairs of dimensions (m, ℓ) in different areas of mathematics, and there exists a vast literature on the subject. For example, in [6] (resp., [7]), a classification of quadratic polynomial mappings of the plane into the plane (resp., of the plane into the n-dimensional space) is given. In [5], nets of quadrics are investigated.

Hence, as a research on quadratic polynomial mappings, it is natural to investigate the stability of generalized distance-squared mappings on submanifolds and the stability of generalized distance-squared mappings of $\mathbb{R}^m \to \mathbb{R}^\ell$ in the case that (m, ℓ) is neither $(2, 2)$ nor $(k+1, 2k+1)$, where k is a positive integer.

We have another original motivation. Height functions and distance-squared functions have been investigated in detail so far, and they are useful tools in the applications of singularity theory to differential geometry (for example, see [3] and [13]). A mapping in which each component is a height function is nothing but a projection. Projections as well as height functions or distance-squared functions have been investigated so far. In [17], the stability of projections on submanifolds is investigated.

On the other hand, a mapping in which each component is a distance-squared function is a distance-squared mapping. Moreover, the notion of a generalized distance-squared mapping is an extension of that of a distance-squared mapping. Therefore, we investigate the generalized distance-squared mappings as well as projections on submanifolds from the view point of the stability (see Corollary 4 and Remark 2).

4.2. **Applications of Theorem 2 to $G_{(p,A)} : \mathbb{R}^m \to \mathbb{R}^\ell$.** As an application of Theorem 2, we have the following.

Proposition 1. *Let N be a manifold of dimension n. Let $f : N \to \mathbb{R}^m$ be an embedding. Let $A = (a_{ij})_{1\le i\le \ell, 1\le j\le m}$ be an $\ell \times m$ matrix with nonzero entries. If W is a modular submanifold of ${}_sJ^r(N, \mathbb{R}^\ell)$, then there exists a subset Σ with Lebesgue measure zero of $(\mathbb{R}^m)^\ell$ such that for any $p = (p_1, \ldots, p_\ell) \in (\mathbb{R}^m)^\ell - \Sigma$, $G_{(p,A)} \circ f : N \to \mathbb{R}^\ell$ is transverse with respect to W.*

Proof. Let $H : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be the diffeomorphism of the target for deleting constant terms. The composition $H \circ G_{(p,A)} : \mathbb{R}^m \to \mathbb{R}^\ell$ is given as follows:

$$H \circ G_{(p,A)}(x) = \left(\sum_{j=1}^{m} a_{1j}x_j^2 - 2\sum_{j=1}^{m} a_{1j}p_{1j}x_j, \ldots, \sum_{j=1}^{m} a_{\ell j}x_j^2 - 2\sum_{j=1}^{m} a_{\ell j}p_{\ell j}x_j \right),$$

where $x = (x_1, \ldots, x_m)$.

Let $\psi : (\mathbb{R}^m)^\ell \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ be the mapping defined by

$$\psi(p_{11}, p_{12}, \ldots, p_{\ell m}) = -2(a_{11}p_{11}, a_{12}p_{12}, \ldots, a_{\ell m}p_{\ell m}).$$

Note that there exists the natural identification $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$. Since $a_{ij} \neq 0$ for any i, j ($1 \leq i \leq \ell$, $1 \leq j \leq m$), it is clearly seen that ψ is a C^∞ diffeomorphism.

Set $F_i(x) = \sum_{j=1}^{m} a_{ij}x_j^2$ ($1 \leq i \leq \ell$) and $F = (F_1, \ldots, F_\ell)$. From Theorem 2, there exists a subset Σ with Lebesgue measure zero of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping ${}_sj^r(F_\pi \circ f) : N^{(s)} \to {}_sJ^r(N, \mathbb{R}^\ell)$ is transverse to W. Since $\psi^{-1} : \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \to (\mathbb{R}^m)^\ell$ is a C^∞ mapping, $\psi^{-1}(\Sigma)$ is a subset with Lebesgue measure zero of $(\mathbb{R}^m)^\ell$. For any $p \in (\mathbb{R}^m)^\ell - \psi^{-1}(\Sigma)$, we have $\psi(p) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$. Hence, for any $p \in (\mathbb{R}^m)^\ell - \psi^{-1}(\Sigma)$, the mapping ${}_sj^r(H \circ G_{(p,A)} \circ f) : N^{(s)} \to {}_sJ^r(N, \mathbb{R}^\ell)$ is transverse to W. Then, since $H : \mathbb{R}^\ell \to \mathbb{R}^\ell$ is a diffeomorphism, ${}_sj^r(G_{(p,A)} \circ f) : N^{(s)} \to {}_sJ^r(N, \mathbb{R}^\ell)$ is transverse to W. $\square$

As in Section 2, from Proposition 1, we get two applications.

Corollary 3. *Let N be a manifold of dimension n. Let $f : N \to \mathbb{R}^m$ be an embedding. Let $A = (a_{ij})_{1\leq i\leq \ell, 1\leq j\leq m}$ be an $\ell \times m$ matrix with non-zero entries. Then, there exists a subset Σ with Lebesgue measure zero of $(\mathbb{R}^m)^\ell$ such that for any $p = (p_1, \ldots, p_\ell) \in (\mathbb{R}^m)^\ell - \Sigma$, $G_{(p,A)} \circ f : N \to \mathbb{R}^\ell$ is transverse with respect to the Thom–Boardman varieties.*

Corollary 4. *Let N be a manifold of dimension n. Let $f : N \to \mathbb{R}^m$ be an embedding. Let $A = (a_{ij})_{1\leq i\leq \ell, 1\leq j\leq m}$ be an $\ell \times m$ matrix with nonzero entries. If a dimension pair (n, ℓ) is in the nice dimensions, then there exists a subset Σ with Lebesgue measure zero of $(\mathbb{R}^m)^\ell$ such that for any $p = (p_1, \ldots, p_\ell) \in (\mathbb{R}^m)^\ell - \Sigma$, the composition $G_{(p,A)} \circ f : N \to \mathbb{R}^\ell$ is locally infinitesimally stable.*

Remark 2.

1. Suppose that the mapping $G_{(p,A)} \circ f : N \to \mathbb{R}^\ell$ is proper in Corollary 4. Then, the local infinitesimal stability of $G_{(p,A)} \circ f$ implies the stability of it (see [14]).
2. Suppose that $N = \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}^m$ is the identify. From Corollary 4, it is clearly seen that if (m, ℓ) is in the nice dimensions, then there exists a subset Σ with Lebesgue measure zero of $(\mathbb{R}^m)^\ell$ such that for any $p = (p_1, \ldots, p_\ell) \in (\mathbb{R}^m)^\ell - \Sigma$, the mapping $G_{(p,A)} : \mathbb{R}^m \to \mathbb{R}^\ell$ is locally infinitesimally stable. This is an application of (I5) in Section 1.

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