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MASSERA'S THEOREM IN QUANTUM CALCULUS

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ABSTRACT. In this paper, we present versions of Massera's theorem for linear and nonlinear q-difference equations and present some examples to illustrate our results.

1. INTRODUCTION

Quantum calculus plays an important role in applications, since it is a powerful tool to better describe several physical phenomena such as cosmic strings and black holes, conformal quantum mechanics, nuclear and high energy physics, fractional quantum Hall effect, and high-T_c superconductors. Thermostatics of q-bosons and q-fermions can well be described by using basic numbers and employing the q-calculus based on the Jackson derivative. See, for instance, [12–15, 20] and the references therein.

Also, it is a known fact that many physical phenomena have periodic properties, and the better understanding of these properties allows several improvements in the investigation of these phenomena. Therefore, motivated by this, recently Bohner and Chieochan in [2] introduced in the literature the concept of periodicity in quantum calculus, and several results were proved using this concept (see [1–4, 6–8]).

On the other hand, the classical Massera theorem for q-difference equations was not presented in the literature until now. This theorem is very important and states that under certain conditions, the existence of a bounded solution of a periodic equation is necessary and sufficient to ensure the existence of a periodic solution of this equation. Versions of this theorem for several types of equations such as functional differential equations, dynamic equations on time scales, and ordinary differential equations were presented by several authors. See [9–11, 16–19, 21] and the references therein.

In this paper, we study a version of Massera's theorem for a linear q-difference equation of the form

$$x^{\Delta}(t) = a(t)x(t) + \frac{b(t)}{t} \tag{1.1}$$

using the Brouwer fixed point theorem. Also, we prove that if $b(t) \neq 0$ for some $t \in q^{\mathbb{N}_0}$ and the functions $a, b : q^{\mathbb{N}_0} \to \mathbb{R}$ are ω-periodic with $a \in \mathcal{R}$, then the equation

$$x^{\Delta}(t) = a(t)x(t) + b(t) \tag{1.2}$$

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has no ω-periodic solutions. Moreover, we present a version of Massera's theorem for nonlinear q-difference equations of the form

$$x^{\Delta}(t) = f(t, tx(t)), \tag{1.3}$$

and we provide some examples to illustrate our main results.

2. Quantum calculus

In this section, our goal is to present some basic concepts concerning the theory of quantum calculus. All definitions and results of this section can be found in [2, 5–8, 13]. Throughout the paper, we let $q > 1$.

We start by presenting the quantum derivative of a function $f : q^{\mathbb{N}_0} \to \mathbb{R}$.

Definition 2.1 (See [13])**.** The expression

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{(q-1)t}, \quad \sigma(t) = qt, \quad t \in q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\}$$

is called the *q-derivative* (or *Jackson derivative*) of the function $f : q^{\mathbb{N}_0} \to \mathbb{R}$.

Remark 2.2. Note that

$$\lim_{q \to 1} f^{\Delta}(t) = \frac{\mathrm{d}f(t)}{\mathrm{d}t}$$

if $f : \mathbb{R} \to \mathbb{R}$ is differentiable.

In what follows, we present some important properties of the quantum derivative.

Theorem 2.3. *If $\alpha, \beta \in \mathbb{R}$ and $f, g : q^{\mathbb{N}_0} \to \mathbb{R}$ are q-differentiable, then*

$$(\alpha f + \beta g)^{\Delta}(t) = \alpha f^{\Delta}(t) + \beta g^{\Delta}(t),$$
$$(fg)^{\Delta}(t) = f(qt)g^{\Delta}(t) + g(t)f^{\Delta}(t) = f(t)g^{\Delta}(t) + g(qt)f^{\Delta}(t),$$

and

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(qt)}$$

for all $t \in q^{\mathbb{N}_0}$.

Let us denote for simplicity the quantum intervals by $[a,b]_{q^{\mathbb{N}_0}}$, $[a,b)_{q^{\mathbb{N}_0}}$, and $(a,b]_{q^{\mathbb{N}_0}}$ to represent $[a,b] \cap q^{\mathbb{N}_0}$, $[a,b) \cap q^{\mathbb{N}_0}$, and $(a,b] \cap q^{\mathbb{N}_0}$, respectively.

The concept of the definite integral of a function is defined as follows.

Definition 2.4. Let $f : q^{\mathbb{N}_0} \to \mathbb{R}$, and let $a, b \in q^{\mathbb{N}_0}$ be such that $a < b$. The definite integral of the function f is given by

$$\int_a^b f(t)\Delta t = (q-1) \sum_{t \in [a,b) \cap q^{\mathbb{N}_0}} t f(t).$$

Remark 2.5. As an immediate consequence of Definition 2.4, we have that if $m, n \in \mathbb{N}_0$ with $m < n$ and $f : q^{\mathbb{N}_0} \to \mathbb{R}$, then

$$\int_{q^m}^{q^n} f(t)\Delta t = (q-1) \sum_{k=m}^{n-1} q^k f(q^k).$$

Definition 2.6. We say that a function $p : q^{\mathbb{N}_0} \to \mathbb{R}$ is *regressive* provided that

$$1 + (q-1)tp(t) \neq 0 \quad \text{for all} \quad t \in q^{\mathbb{N}_0}$$

holds. The set of all regressive functions will be denoted by $\mathcal{R}$.

Definition 2.7. If $p \in \mathcal{R}$, then the exponential function is defined by

$$e_p(t,s) = \prod_{k=\log_q s}^{\log_q t - 1} \left(1 + (q-1)q^k p(q^k)\right) \quad \text{for} \quad t, s \in q^{\mathbb{N}_0} \quad \text{with} \quad t > s.$$

If $t = s$, then we define $e_p(t,s) = 1$, and if $t < s$, then we define $e_p(t,s) = \frac{1}{e_p(s,t)}$.

Theorem 2.8 (Semigroup property [5, Theorem 2.36(v)]). *If $p \in \mathcal{R}$ and $t, s, r \in q^{\mathbb{N}_0}$, then*

$$e_p(t,s)e_p(s,r) = e_p(t,r). \tag{2.1}$$

Theorem 2.9 (Variation of constants [5, Theorem 2.77]). *Let $p \in \mathcal{R}$, $f : q^{\mathbb{N}_0} \to \mathbb{R}$, $t_0 \in q^{\mathbb{N}_0}$, and $y_0 \in \mathbb{R}$. The unique solution of the initial value problem*

$$y^{\Delta}(t) = p(t)y + f(t), \quad y(t_0) = y_0$$

is given by

$$y(t) = e_p(t,t_0)y_0 + \int_{t_0}^{t} e_p(t,\sigma(s))f(s)\Delta s.$$

In the sequel, we give the definition of an ω-periodic function on $q^{\mathbb{N}_0}$.

Definition 2.10 (See [2, Definition 3.1]). Let $\omega \in \mathbb{N}$. A function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is called *ω-periodic* if

$$q^{\omega} f(q^{\omega} t) = f(t) \quad \text{for all} \quad t \in q^{\mathbb{N}_0}.$$

3. Massera's theorem for linear q-difference equations

In this section, our goal is to prove Massera's theorem for linear q-difference equations of form (1.1), where $a, b : q^{\mathbb{N}_0} \to \mathbb{R}$ are ω-periodic functions. We first present some auxiliaries results.

Lemma 3.1 (See [8, Lemma 2.15]). *If $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is regressive and ω-periodic, then*

$$e_f(q^{\omega} t, q^{\omega} s) = e_f(t,s) \quad \textit{for all} \quad t, s \in q^{\mathbb{N}_0} \tag{3.1}$$

and

$$e_f(q^{\omega} t, t) = e_f(q^{\omega} s, s) \quad \textit{for all} \quad t, s \in q^{\mathbb{N}_0}. \tag{3.2}$$

Lemma 3.2 (Chain rule). *If $x : q^{\mathbb{N}_0} \to \mathbb{R}$ and $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is defined by*

$$f(t) = x(q^{\omega} t),$$

then

$$f^{\Delta}(t) = q^{\omega} x^{\Delta}(q^{\omega} t).$$

Proof. Notice that

$$\begin{aligned} f^{\Delta}(t) &= \frac{f(qt)-f(t)}{(q-1)t} \\ &= \frac{x(q^{\omega}qt)-x(q^{\omega}t)}{(q-1)t} \\ &= q^{\omega}\frac{x(qq^{\omega}t)-x(q^{\omega}t)}{(q-1)q^{\omega}t} \\ &= q^{\omega}x^{\Delta}(q^{\omega}t), \end{aligned}$$

obtaining the desired result. □

Theorem 3.3. *If $a, b : q^{\mathbb{N}_0} \to \mathbb{R}$ are ω-periodic with $a \in \mathcal{R}$ and $b(t) \neq 0$ for some $t \in q^{\mathbb{N}_0}$, then* (1.2) *has no ω-periodic solution.*

Proof. Assume x is a solution of (1.2) and define $f(t) = q^{\omega}x(q^{\omega}t) - x(t)$ for $t \in q^{\mathbb{N}_0}$. Then, by Lemma 3.2, we get

$$\begin{aligned} f^{\Delta}(t) &= q^{2\omega}x^{\Delta}(q^{\omega}t) - x^{\Delta}(t) \\ &= q^{2\omega}[a(q^{\omega}t)x(q^{\omega}t) + b(q^{\omega}t)] - [a(t)x(t) + b(t)] \\ &= a(t)q^{\omega}x(q^{\omega}t) + q^{\omega}b(t) - a(t)x(t) - b(t) \\ &= a(t)f(t) + (q^{\omega}-1)b(t). \end{aligned}$$

Since $b(t) \neq 0$ for some $t \in q^{\mathbb{N}_0}$, $f(t) = 0$ for all $t \in q^{\mathbb{N}_0}$ is not possible. Therefore, we get the desired result. □

Theorem 3.4. *If $a, b : q^{\mathbb{N}_0} \to \mathbb{R}$ are ω-periodic with $a \in \mathcal{R}$ and $q^{\omega}x(q^{\omega}) = x(1)$ for some solution x of* (1.1)*, then x is ω-periodic.*

Proof. Let x be a solution of (1.1) satisfying $q^{\omega}x(q^{\omega}) = x(1)$. Define $f(t) = q^{\omega}x(q^{\omega}t) - x(t)$ so that $f(1) = 0$ and

$$\begin{aligned} f^{\Delta}(t) &= q^{2\omega}x^{\Delta}(q^{\omega}t) - x^{\Delta}(t) \\ &= q^{2\omega}\left[a(q^{\omega}t)x(q^{\omega}t) + \frac{b(q^{\omega}t)}{q^{\omega}t}\right] - \left[a(t)x(t) + \frac{b(t)}{t}\right] \\ &= a(t)f(t) + \frac{b(t)}{t} - \frac{b(t)}{t} = a(t)f(t). \end{aligned}$$

Hence, $f(t) \equiv 0$, which implies that $q^{\omega}x(q^{\omega}t) = x(t)$ for every $t \in q^{\mathbb{N}_0}$, proving the result. □

Lemma 3.5. *If $a, b : q^{\mathbb{N}_0} \to \mathbb{R}$ are ω-periodic with $a \in \mathcal{R}$, then*

$$q^{\omega}\int_{q^{\omega}t}^{q^{2\omega}t} e_a(q^{\omega}t, \sigma(s))\frac{b(s)}{s}\Delta s = \int_{t}^{q^{\omega}t} e_a(t, \sigma(s))\frac{b(s)}{s}\Delta s$$

for all $t \in q^{\mathbb{N}_0}$.

Proof. Notice that

$$
\begin{aligned}
&q^{\omega} \int_{q^{\omega} t}^{q^{2\omega} t} e_a(q^{\omega} t, \sigma(s)) \frac{b(s)}{s} \Delta s \\
&\overset{(t=q^n)}{=} q^{\omega} \sum_{k=n+\omega}^{n+2\omega-1} e_a(q^{n+\omega}, q^{k+1}) \frac{b(q^k)}{q^k}(q-1)q^k \\
&= q^{\omega} \sum_{k=n}^{n+\omega-1} e_a(q^{n+\omega}, q^{k+1+\omega}) \frac{b(q^{k+\omega})}{q^{k+\omega}}(q-1)q^{k+\omega} \\
&\overset{(3.1)}{=} \sum_{k=n}^{n+\omega-1} e_a(q^n, q^{k+1}) \frac{b(q^k)}{q^k}(q-1)q^k \\
&= \int_t^{q^{\omega} t} e_a(t, \sigma(s)) \frac{b(s)}{s} \Delta s,
\end{aligned}
$$

obtaining the desired result. □

Lemma 3.6. *If $a, b : q^{\mathbb{N}_0} \to \mathbb{R}$ are ω-periodic with $a \in \mathcal{R}$, then*

$$\int_1^t e_a(t, \sigma(s)) \frac{b(s)}{s} \Delta s = q^{\omega} \int_{q^{\omega}}^{q^{\omega} t} e_a(q^{\omega} t, \sigma(s)) \frac{b(s)}{s} \Delta s.$$

Proof. We have

$$
\begin{aligned}
\int_1^t e_a(t, \sigma(s)) \frac{b(s)}{s} \Delta s &\overset{(t=q^n)}{=} \sum_{k=0}^{n-1} e_a(q^n, q^{k+1}) \frac{b(q^k)}{q^k}(q-1)q^k \\
&= \sum_{k=\omega}^{n+\omega-1} e_a(q^n, q^{k+1-\omega}) \frac{b(q^{k-\omega})}{q^{k-\omega}}(q-1)q^{k-\omega} \\
&\overset{(3.1)}{=} \sum_{k=\omega}^{n+\omega-1} e_a(q^{n+\omega}, q^{k+1}) q^{\omega} \frac{b(q^k)}{q^k}(q-1)q^k \\
&= q^{\omega} \int_{q^{\omega}}^{q^{\omega} t} e_a(q^{\omega} t, \sigma(s)) \frac{b(s)}{s} \Delta s,
\end{aligned}
$$

proving the result. □

Definition 3.7. A function $x : q^{\mathbb{N}_0} \to \mathbb{R}$ is called *q-bounded* if there exists $K > 0$ such that

$$t|x(t)| \leq K \quad \text{for all} \quad t \in q^{\mathbb{N}_0}.$$

Definition 3.8. We say that (1.1) is *ω-periodic* whenever $a, b : q^{\mathbb{N}_0} \to \mathbb{R}$ are ω-periodic functions with $a \in \mathcal{R}$.

In the sequel, we present our main result of this section. It is a version of Massera's theorem for linear q-difference equations of the form (1.1). In its proof, we use the following well-known fixed point result.

Theorem 3.9 (Brouwer's fixed point theorem)**.** *For any continuous function f mapping a compact convex set into itself, there exists an x_0 in that set satisfying $f(x_0) = x_0$.*

Theorem 3.10. *The ω-periodic linear q-difference equation* (1.1) *has an ω-periodic solution if and only if it has a q-bounded solution.*

Proof. First, let us assume (1.1) has an ω-periodic solution x. Define

$$K := \max_{0\le k\le \omega-1} q^k|x(q^k)|.$$

Let $t\in q^{\mathbb{N}_0}$. Then there exist $n,k\in\mathbb{N}_0$ with $0\le k\le \omega-1$ such that $t=q^{n\omega+k}$. Thus

$$t|x(t)| = q^k|q^{n\omega}x(q^{n\omega}q^k)| = q^k|x(q^k)|\le K.$$

Hence, x is q-bounded.

Now, assume (1.1) has a q-bounded solution $\tilde{x}$. Then there exists $K>0$ such that $t|\tilde{x}(t)|\le K$ for all $t\in q^{\mathbb{N}_0}$. Define

$$\Omega := \left\{x_0\in\mathbb{R}:\ |x_0|\le K,\ t|x(t,x_0)|\le K \text{ for all } t\in q^{\mathbb{N}_0}\right\},$$

where $x(\cdot,x_0)$ is the unique solution of (1.1) with $x(1)=x_0$ (see Theorem 2.9), i.e.,

$$x(t,x_0) = e_a(t,1)x_0 + \int_1^t e_a(t,\sigma(s))\frac{b(s)}{s}\Delta s.$$

Since $\tilde{x}(1)\in\Omega$, we have $\Omega\ne\emptyset$. Since $\Omega\subset\mathbb{R}$ is closed and bounded, it is compact. Now, we will show that Ω is convex. Let $x_1,x_2\in\Omega$ and $0\le\alpha\le 1$. Then

$$|\alpha x_1+(1-\alpha)x_2|\le \alpha|x_1|+(1-\alpha)|x_2|\le \alpha K+(1-\alpha)K = K$$

and

$$\begin{aligned} t|x(t,\alpha x_1+(1-\alpha)x_2)| &= t\left|e_a(t,1)(\alpha x_1+(1-\alpha)x_2)+\int_1^t e_a(t,\sigma(s))\frac{b(s)}{s}\Delta s\right| \\ &\le t\left|\alpha\left[e_a(t,1)x_1+\int_1^t e_a(t,\sigma(s))\frac{b(s)}{s}\Delta s\right]\right. \\ &\qquad \left. +(1-\alpha)\left[e_a(t,1)x_2+\int_1^t e_a(t,\sigma(s))\frac{b(s)}{s}\Delta s\right]\right| \\ &= t|\alpha x(t,x_1)+(1-\alpha)x(t,x_2)| \\ &\le \alpha t|x(t,x_1)|+(1-\alpha)t|x(t,x_2)| \\ &\le \alpha K+(1-\alpha)K = K \end{aligned}$$

for all $t\in q^{\mathbb{N}_0}$. So $\alpha x_1+(1-\alpha)x_2\in\Omega$, and hence Ω is convex. Now define $P:\Omega\to\mathbb{R}$ by

$$P(x_0) := q^\omega x(q^\omega,x_0) = q^\omega e_a(q^\omega,1)x_0 + q^\omega\int_1^{q^\omega} e_a(q^\omega,\sigma(s))\frac{b(s)}{s}\Delta s.$$

Since P is linear, it is continuous. Let $x_0\in\Omega$. Since $t|x(t,x_0)|\le K$ for all $t\in q^{\mathbb{N}_0}$, we have

$$|P(x_0)| = q^\omega|x(q^\omega,x_0)|\le K.$$

Moreover, using Lemma 3.6, we get

$$\begin{aligned}
x(t, P(x_0)) &= e_a(t,1)P(x_0) + \int_1^t e_a(t,\sigma(s))\frac{b(s)}{s}\Delta s \\
&= e_a(t,1)\left\{q^\omega e_a(q^\omega,1)x_0 + q^\omega \int_1^{q^\omega} e_a(q^\omega,\sigma(s))\frac{b(s)}{s}\Delta s\right\} \\
&\quad + \int_1^t e_a(t,\sigma(s))\frac{b(s)}{s}\Delta s \\
&\overset{(3.1)}{\underset{(3.2)}{=}} q^\omega e_a(t,1)e_a(q^\omega t,t)x_0 + q^\omega e_a(q^\omega t, q^\omega)\int_1^{q^\omega} e_a(q^\omega,\sigma(s))\frac{b(s)}{s}\Delta s \\
&\quad + q^\omega \int_{q^\omega}^{q^\omega t} e_a(q^\omega t,\sigma(s))\frac{b(s)}{s}\Delta s \\
&\overset{(2.1)}{=} q^\omega\left\{e_a(q^\omega t,1)x_0 + \int_1^{q^\omega t} e_a(q^\omega t,\sigma(s))\frac{b(s)}{s}\Delta s\right\} \\
&= q^\omega x(q^\omega t, x_0),
\end{aligned}$$

and thus

$$t|x(t,P(x_0))| = q^\omega t|x(q^\omega t, x_0)| \le K$$

for all $t \in q^{\mathbb{N}_0}$. Hence $P(x_0) \in \Omega$. Thus $P : \Omega \to \Omega$. By Theorem 3.9, P has a fixed point in Ω, i.e., there exists $\tilde{x}_0 \in \Omega$ with

$$x(1,\tilde{x}_0) = \tilde{x}_0 = P(\tilde{x}_0) = q^\omega x(q^\omega, \tilde{x}_0).$$

By Theorem 3.4, $\tilde{x} = x(\cdot, \tilde{x}_0)$ is an ω-periodic solution of (1.1). □

Remark 3.11. The periodic solution $\tilde{x}$ in the last line of the proof of Theorem 3.10 is unique provided

$$q^\omega e_a(q^\omega, 1) \ne 1, \tag{3.3}$$

and it is then given by

$$\tilde{x}(t) = \lambda \int_t^{q^\omega t} e_a(t,\sigma(s))\frac{b(s)}{s}\Delta s, \tag{3.4}$$

where

$$\lambda = \frac{q^\omega e_a(q^\omega,1)}{1 - q^\omega e_a(q^\omega,1)}.$$

Proof. Since $\tilde{x}$ is a solution of (1.1), Theorem 2.9 implies

$$\tilde{x}(t) = e_a(t,1)\tilde{x}(1) + \int_1^t e_a(t,\sigma(s))\frac{b(s)}{s}\Delta s.$$

Since $\tilde{x}$ is ω-periodic, we have

$$\tilde{x}(t) = q^\omega \tilde{x}(q^\omega t) = q^\omega e_a(q^\omega t, t)\tilde{x}(t) + q^\omega \int_t^{q^\omega t} e_a(q^\omega t,\sigma(s))\frac{b(s)}{s}\Delta s,$$

and thus (by (2.1) and (3.2))

$$(1 - q^\omega e_a(q^\omega,1))\tilde{x}(t) = q^\omega e_a(q^\omega,1)\int_t^{q^\omega t} e_a(t,\sigma(s))\frac{b(s)}{s}\Delta s,$$

from which the result follows due to (3.3). □

Remark 3.12. Lemma 3.5 shows directly that $\tilde{x}$ given by (3.4) is ω-periodic.

Example 3.13. Consider the linear q-difference equation

$$x^\Delta(t) = \frac{x(t)}{t} + \frac{1}{t^2}. \tag{3.5}$$

Here, $a(t) = b(t) = \frac{1}{t}$ are 1-periodic. By Theorem 3.4, the solution x of (3.5) satisfying

$$qx(q) = x(1) \tag{3.6}$$

is 1-periodic. Since $x(q) = qx(1) + q - 1$, (3.6) happens if and only if

$$q^2x(1) + q(q-1) = x(1), \quad \text{i.e.,} \quad x(1) = -\frac{q}{1+q}. \tag{3.7}$$

Thus, the solution x of (3.5) satisfying the initial condition (3.7) is 1-periodic. By Theorem 3.10, (3.5) also has a q-bounded solution. In fact, it is easy to see that

$$\tilde{x}(t) = -\frac{q}{1+q}\frac{1}{t}$$

is a q-bounded (with $K = q/(1+q)$) and 1-periodic solution of (3.5).

4. Massera's theorem for nonlinear q-difference equations

In this section, our goal is to prove a version of Massera's theorem for nonlinear q-difference equations. Throughout, we assume that initial value problems for (1.3) are uniquely solvable. We consider (1.3) subject to the hypotheses:

(H_1) $f : q^{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to the second variable.
(H_2) For each fixed $x \in \mathbb{R}$, F defined by $F(t) = tf(t,x)$ is ω-periodic, that is,

$$q^{2\omega} f(tq^\omega, x) = f(t,x) \quad \text{for all} \quad t \in q^{\mathbb{N}_0} \quad \text{and} \quad x \in \mathbb{R}.$$

Let us start by proving some auxiliary results.

Lemma 4.1. *Assume (H_2). Then the following assertions hold:*

(i) *If x is a solution of (1.3), then so is y defined by $y(t) = q^\omega x(q^\omega t)$.*
(ii) *Equation (1.3) has an ω-periodic solution if and only if there exist a solution x of (1.3) and $t_0 \in q^{\mathbb{N}_0}$ with*

$$q^\omega x(q^\omega t_0) = x(t_0).$$

Proof. Let us start by proving (i). Let x be a solution of (1.3) and define $y(t) := q^\omega x(q^\omega t)$. Then

$$\begin{aligned} y^\Delta(t) &= q^{2\omega} x^\Delta(q^\omega t) \\ &= q^{2\omega} f(tq^\omega, tq^\omega x(q^\omega t)) \\ &= f(t, ty(t)), \end{aligned}$$

proving our result.

Now, let us prove (ii). Suppose (1.3) has an ω-periodic solution x. Then, clearly for any $t_0 \in q^{\mathbb{N}_0}$, we get $q^\omega x(q^\omega t_0) = x(t_0)$. On the other hand, assume that x is a solution of (1.3) satisfying $q^\omega x(q^\omega t_0) = x(t_0)$ for some $t_0 \in q^{\mathbb{N}_0}$. Then, by (i), $y(t) = q^\omega x(q^\omega t)$ is also a solution of (1.3) which satisfies $y(t_0) = x(t_0)$. Therefore, by the uniqueness of solutions, it follows that $y(t) = x(t)$ for every $t \in q^{\mathbb{N}_0}$, so x is ω-periodic. ∎

In the proof of our main result below, we also use the hypothesis

(H$_3$) For all $t \in q^{\mathbb{N}_0}$,

$$x < y \quad \text{always implies} \quad x + (q-1)tf(t, tx) < y + (q-1)tf(t, ty).$$

Lemma 4.2. *Assume* (H_3). *Then*

$$x(1) < y(1) \quad \textit{implies} \quad x(t) < y(t) \quad \textit{for all} \quad t \in q^{\mathbb{N}_0}.$$

Proof. By induction, $x(1) < y(1)$ is given. Now suppose that $x(t) < y(t)$ holds. Then

$$\begin{aligned} x(qt) &= x(t) + (q-1)tx^{\Delta}(t) \\ &= x(t) + (q-1)tf(t, tx(t)) \\ &< y(t) + (q-1)tf(t, ty(t)) \\ &= y(t) + (q-1)ty^{\Delta}(t) = y(qt), \end{aligned}$$

obtaining the desired result. □

Theorem 4.3. *Assume* (H_1)–(H_3). *If* (1.3) *has a q-bounded solution, then it has an ω-periodic solution.*

Proof. Let x be a q-bounded solution of (1.3). Hence there exists $K > 0$ with $t|x(t)| \le K$ for all $t \in q^{\mathbb{N}_0}$. Define the sequence of functions $\{x_n\}$ by $x_n(t) = q^{\omega n} x(tq^{\omega n})$ on $q^{\mathbb{N}_0}$ for $n \in \mathbb{N}_0$. Since $x_{n+1}(t) = q^{\omega} x_n(q^{\omega} t)$ for all $n \in \mathbb{N}_0$, Lemma 4.1(i) shows that each x_n, $n \in \mathbb{N}$, is a solution of (1.3). Moreover, since $t|x_n(t)| = q^{\omega n} t|x(q^{\omega n} t)| \le K$, each x_n is q-bounded.

First, assume $x(1) = x_1(1)$. Then $x(1) = q^{\omega} x(q^{\omega})$, and by Lemma 4.1(ii), x is ω-periodic. Next, assume $x(1) < x_1(1)$. Then by Lemma 4.2, $x(t) < x_1(t)$ for all $t \in q^{\mathbb{N}_0}$. Hence

$$x(q^{\omega n} t) < x_1(q^{n\omega} t) \quad \text{for all} \quad t \in q^{\mathbb{N}_0}$$

so that

$$x_n(t) = q^{n\omega} x(q^{n\omega} t) < q^{n\omega} x_1(q^{n\omega} t) = q^{(n+1)\omega} x(q^{(n+1)\omega} t) = x_{n+1}(t) \tag{4.1}$$

for all $t \in q^{\mathbb{N}_0}$. Thus, for each $t \in q^{\mathbb{N}_0}$, $\{x_n(t)\}_{n=1}^{\infty}$ is increasing and bounded, and so we have

$$\lim_{n\to\infty} x_n(t) = \tilde{x}(t) \quad \text{pointwise for} \quad t \in q^{\mathbb{N}_0},$$

where $\tilde{x}$ is some function defined on $q^{\mathbb{N}_0}$. We have

$$\begin{aligned} \tilde{x}^{\Delta}(t) &= \frac{\tilde{x}(qt) - \tilde{x}(t)}{(q-1)t} = \lim_{n\to\infty} \frac{x_n(qt) - x_n(t)}{(q-1)t} = \lim_{n\to\infty} x_n^{\Delta}(t) \\ &= \lim_{n\to\infty} f(t, tx_n(t)) = f(t, t\tilde{x}(t)) \end{aligned}$$

for each $t \in q^{\mathbb{N}_0}$, as f is continuous in the second variable. So $\tilde{x} : q^{\mathbb{N}_0} \to \mathbb{R}$ solves (1.3). Moreover,

$$q^{\omega} \tilde{x}(q^{\omega} t) = \lim_{n\to\infty} q^{\omega} x_n(q^{\omega} t) = \lim_{n\to\infty} x_{n+1}(t) = \tilde{x}(t)$$

for each $t \in q^{\mathbb{N}_0}$, so $\tilde{x}$ is ω-periodic. Finally, the last case, $x(1) > x_1(1)$, leads to an ω-periodic solution in the same way. □

Remark 4.4. If we replace the condition (H$_3$) by

(H$_4$) $f(t, x) > 0$ for all $t \in q^{\mathbb{N}_0}$ and $x \in \mathbb{R}$, then it is not difficult to prove that the sequence $\{x_n\}$ obtained in the same way as in the proof of Theorem 4.3 is increasing and therefore satisfies the inequality (4.1). The rest of the proof follows in the same way as the proof of Theorem 4.3.

Example 4.5. If $a, b : q^{\mathbb{N}_0} \to \mathbb{R}$ are ω-periodic with $a \in \mathcal{R}$ and $f(t,x) = \frac{a(t)}{t}x + \frac{b(t)}{t}$, then (1.3) is the same as (1.1). Moreover, since

$$\begin{aligned} q^{2\omega} f(tq^{\omega}, x) &= q^{2\omega}\frac{a(tq^{\omega})}{tq^{\omega}}x + q^{2\omega}\frac{b(q^{\omega}t)}{q^{\omega}t} \\ &= \frac{q^{\omega}a(q^{\omega}t)}{t} + \frac{q^{\omega}b(q^{\omega}t)}{t} \\ &= \frac{a(t)}{t}x + \frac{b(t)}{t} = f(t,x), \end{aligned}$$

f is seen to satisfy (H_2).

Example 4.6. Let $q = 2$. Consider the nonlinear q-difference equation

$$x^{\Delta}(t) = \frac{x(t)(1 - tx(t))}{t(1 + tx(t))}. \tag{4.2}$$

Then (4.2) is of the form (1.3) with

$$f(t,x) = \frac{x(1-x)}{t^2(1+x)}.$$

Since

$$4f(2t,x) = 4\frac{x(1-x)}{(2t)^2(1+x)} = f(t,x),$$

f satisfies (H_2) with $\omega = 1$. By Lemma 4.1(ii), (4.2) has a 1-periodic solution if a solution x satisfies

$$2x(2) = x(1). \tag{4.3}$$

Since $x(2) = x(1) + \frac{x(1)(1-x(1))}{1+x(1)}$, (4.3) happens if and only if (without loss of generality, assume $x(1) \neq 0$)

$$2 + 2\frac{1 - x(1)}{1 + x(1)} = 1, \quad \text{i.e.,} \quad x(1) = 3. \tag{4.4}$$

It is clear that (4.2) has a unique solution satisfying the initial condition (4.4). Thus, by Lemma 4.1(ii), (4.2) has a 1-periodic solution. By Theorem 4.3, (4.2) also has a q-bounded solution. In fact, one can see that

$$\tilde{x}(t) = \frac{3}{t}$$

is a q-bounded (with $K = 3$) and 1-periodic solution of (4.2). Note that

$$\tilde{x}^{\Delta}(t) = -\frac{3}{2t^2}$$

and

$$\frac{\tilde{x}(t)(1 - t\tilde{x}(t))}{t(1 + t\tilde{x}(t))} = \frac{\frac{3}{t}\left(1 - t\frac{3}{t}\right)}{t\left(1 + t\frac{3}{t}\right)} = -\frac{6}{4t^2} = -\frac{3}{2t^2}.$$

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