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# $C$-CYCLICAL MONOTONICITY AS A SUFFICIENT CRITERION FOR OPTIMALITY IN THE MULTIMARGINAL MONGE–KANTOROVICH PROBLEM

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Abstract. This paper establishes that a generalization of $c$-cyclical monotonicity from the Monge–Kantorovich problem with two marginals gives rise to a sufficient condition for optimality also in the multimarginal version of that problem. To obtain the result, the cost function is assumed to be bounded by a sum of integrable functions. The proof rests on ideas from martingale transport.

## 1. Introduction and result

Let $X_1, \dots, X_d$ be Polish spaces, and let $\mu_1, \dots, \mu_d$ be probability measures on their Borel-$\sigma$-fields. By $\mathcal{M}(\mu_1, \dots, \mu_d)$ we denote the set of probability measures on the space $E = X_1 \times \cdots \times X_d$ with marginal distributions $\mu_1, \dots, \mu_d$. Writing $p_i$ for the canonical projections $E \to X_i$, a measure $\mu$ on $E$ is in $\mathcal{M}(\mu_1, \dots, \mu_d)$ if and only if

$$p_i(\mu) = \mu_i \quad \text{for } i = 1, \dots, d.$$

These measures are called transports or transport plans. Given a measurable cost function $c : E \to \mathbb{R}$, the cost of a transport $\mu$ is the integral $\int c \, d\mu$. The multimarginal Monge–Kantorovich problem is to minimize the cost amongst transports, i.e., to solve

$$\text{(mmMK)} \qquad \underset{\mu \in \mathcal{M}(\mu_1, \dots, \mu_d)}{\text{minimize}} \int c \, d\mu.$$

There is a huge literature for the case $d = 2$, *the* Monge–Kantorovich problem; see, e.g., [Vil03], [Vil09], or [AG12] for an overview. The literature on the case $d > 2$ is more recent and less voluminous. For an overview the reader is referred to [Pas15].

For $d = 2$, a characterization of optimal transport plans is given by the concept of $c$-cyclical monotonicity (see [Vil09, Ch. 5]): under fairly weak assumptions on the cost function, a transport is optimal if and only if it is $c$-cyclically monotone. A transport is $c$-cyclically monotone if it is concentrated on a $c$-cyclically monotone set $\Gamma \subseteq X_1 \times X_2 = X \times Y$, i.e., a set $\Gamma$ such that for any pairs $(x_1, y_1), \dots, (x_n, y_n) \in \Gamma$

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one has, with $y_{n+1} = y_1$,

$$\sum_{i=1}^{n} c(x_i, y_i) \leq \sum_{i=1}^{n} c(x_i, y_{i+1}). \tag{1}$$

Does such a characterization also hold for the case $d > 2$?

We start with a definition with a built-in minitheorem that is well known for $d = 2$ and similarly easy to show for $d > 2$.[1]

**Definition 1.1.** A set $\Gamma \subseteq E$ is $c$-cyclically monotone if it fulfills any of the two following equivalent conditions:

(i) for any $n$ and any points $\big(x_1^{(1)}, \dots, x_d^{(1)}\big), \dots, \big(x_1^{(n)}, \dots, x_d^{(n)}\big) \in \Gamma$ and permutations $\sigma_2, \dots, \sigma_d : \{1, \dots, n\} \to \{1, \dots, n\}$, one has
$$\sum_{i=1}^{n} c\big(x_1^{(i)}, \dots, x_d^{(i)}\big) \leq \sum_{i=1}^{n} c\big(x_1^{(i)}, x_2^{(\sigma_2(i))}, \dots, x_d^{(\sigma_d(i))}\big);$$

(ii) any finite measure $\alpha$ concentrated on finitely many points in $\Gamma$ is a cost-minimizing transport plan between its marginals; i.e., if $\alpha'$ has the same marginals as $\alpha$, then
$$\int c \, d\alpha \leq \int c \, d\alpha'.$$

A weaker notion of $c$-monotonicity allowing only comparisons of two points in (i) was shown to be a necessary condition for optimality in [Pas12]; see also [CDPDM15]. The necessity of $c$-cyclical monotonicity in the sense of (i) is included in the results of [BG14, Zae15]. Cyclical monotonicity was also discussed in [KP14], where cost functions that satisfy the twist condition on cyclically monotone or on splitting sets are shown to have a unique Monge solution of (mmMK), but the exact connection between splitting sets and cyclically monotone sets remains an open question. It is answered here as a byproduct in Proposition 2.5.

The question of sufficiency of cyclical monotonicity of a transport plan for optimality was open, although there was an early result in [KS94] for quadratic costs in the case $d = 3$. The situation is hence somewhat similar to the two-marginals case, where the sufficiency of $c$-cyclical monotonicity was open for some time and is now known to require more regularity of the cost function; see [AP03, Pra08, ST09, BGMS09, BC10, Bei15].

In order to prove the sufficiency of $c$-cyclical monotonicity for optimality, we assume $c$ to be continuous and *bounded by a sum of integrable functions.* This means that there are functions $f_i \in L_1(\mu_i)$ such that
$$c(x_1, \dots, x_d) \leq f_1(x_1) + \dots + f_d(x_d) \quad \text{for all} \quad x_1, \dots, x_d.$$

Essentially the same condition was employed by Kellerer in [Kel84] to show the existence of dual maximizers. It will be used here to obain the desired integrability properties of the $c$-splitting functions defined and constructed in the next section as the crucial step to the following theorem.

[1] In order to show (ii) from (i) it is enough to deal with measures $\alpha$ and $\alpha'$ that assume only rational values. One multiplies both $\int c \, d\alpha$ and $\int c \, d\alpha'$ with the integer $\tau$ which is defined as the product of all the denominators appearing in the values of $\alpha$ and $\alpha'$. It is then possible to write $\tau \int c \, d\alpha$ as a sum of the form $\sum_{i=1}^{n} c(x_1^{(i)}, \dots, x_d^{(i)})$, and because of the assumptions on $\alpha$ and $\alpha'$ one can find permutations to write $\tau \int c \, d\alpha'$ as $\sum_{i=1}^{n} c(x_1^{(i)}, x_2^{(\sigma_2(i))}, \dots, x_d^{(\sigma_d(i))})$.

**Theorem 1.2.** *Let $c$ be a continuous cost-function $E \to [0,\infty)$ which is bounded by a sum of integrable functions. Let $\mu$ be a $c$-cyclically monotone transport plan in $\mathcal{M}(\mu_1,\dots,\mu_d)$. Then $\mu$ is optimal.*

## 2. Proof of Theorem 1.2

The proof of Theorem 1.2 takes the proof for the case $d = 2$ in [ST09] as a blueprint: we show that $c$-cyclically monotone sets are *c-splitting* sets. Optimality then follows easily from the assumptions on $c$. We exploit ideas found in [BJ16], where a notion of finite optimality is introduced as a generalization of $c$-cyclical monotonicity to the martingale-transport problem (with two marginals). The compactness argument to show that $c$-cyclically monotone sets are $c$-splitting is an adapted version of the argument in [BJ16] to show that finitely optimal sets are "$c$-good". It is perhaps worth mentioning that, although the arguments from [BJ16] can be adapted to the multimarginal Monge–Kantorovich problem, it is an open question whether this is also possible for the multimarginal martingale problem.

**Definition 2.1.** A set $G \subseteq E$ is called $c$-splitting if there exist $d$ functions $\varphi_i : X_i \to [-\infty,\infty)$ such that

$$\varphi_1(x_1) + \varphi_2(x_2) + \cdots + \varphi_d(x_d) \le c(x_1, x_2, \dots, x_d)$$

holds for all $(x_1, x_2, \dots, x_d) \in E$, and

$$\varphi_1(x_1) + \varphi_2(x_2) + \cdots + \varphi_d(x_d) = c(x_1, x_2, \dots, x_d)$$

holds for all $(x_1,\dots,x_d) \in G$. We call the functions $(\varphi_1,\dots,\varphi_d)$ a $(G,c)$-splitting tuple.

The definition of splitting tuples comes without regularity assumptions on the functions $\varphi_i$. If the functions in a $(G,c)$-splitting tuple are measurable, we call it a measurable tuple. The next lemma shows that for continuous $c$ measurability comes at no cost.

**Lemma 2.2.** *If $G$ is a $c$-splitting set and $c$ is continuous, then there is a measurable $(G,c)$-splitting tuple.*

*Proof.* There is a $c$-splitting tuple $(\varphi_1,\dots,\varphi_d)$ by assumption. Set

$$\tilde\varphi_1(x_1^0) = \inf_{x_2,\dots,x_d} \big\{ c(x_1^0, x_2, \dots, x_d) - \varphi_2(x_2) - \cdots - \varphi_d(x_d) \big\}.$$

If $\tilde\varphi_1,\dots,\tilde\varphi_i$ are already defined, set

$$\begin{aligned}\tilde\varphi_{i+1}(x_{i+1}^0) = \inf_{x_1,\dots,x_i,x_{i+2},\dots,x_d} \big\{ & c(x_1,\dots,x_i,x_{i+1}^0,x_{i+2},\dots,x_d) \\ & - \tilde\varphi_1(x_1) - \cdots - \tilde\varphi_i(x_i) \\ & - \varphi_{i+2}(x_{i+2}) - \cdots - \varphi_d(x_d) \big\}.\end{aligned}$$

The functions $\tilde\varphi_1,\dots,\tilde\varphi_d$ are measurable (in fact, upper semicontinuous) and constitute a $(G,c)$-splitting tuple. □

**Lemma 2.3.** *If $G$ is $c$-cyclically monotone and finite, then it is $c$-splitting.*

*Proof.* Immediate application of the definition of $c$-cyclical monotonicity and LP duality; cf. [BJ16]. □

**Lemma 2.4.** *Let $c$ be continous, let $G$ be a $c$-splitting set, and let $x^0 = (x_1^0, \dots, x_d^0) \in G$. Then there exists a measurable $(G, c)$-splitting tuple $(\varphi_1, \dots, \varphi_d)$, such that*

$$\varphi_i(x_i) \le c(x_1^0, \dots, x_{i-1}^0, x_i, x_{i+1}^0, \dots, x_d^0) \quad \textit{for all} \quad x_i \in X_i, i = 1, \dots, d.$$

*Proof.* By the assumptions there is a measurable $(G, c)$-splitting-tuple $(\tilde{\varphi}_1, \dots, \tilde{\varphi}_d)$. As $x^0 \in G$, we have

$$\sum_{i=1}^d \tilde{\varphi}_i(x_i^0) = c(x^0).$$

Hence, the values $\tilde{\varphi}_i(x_i^0)$ are all in $\mathbb{R}$. Now define

$$\begin{aligned} \varphi_1 &: x_1 \mapsto \tilde{\varphi}_1(x_1) + \tilde{\varphi}_2(x_2^0) + \dots + \tilde{\varphi}_d(x_d^0), \\ \varphi_i &: x_i \mapsto \tilde{\varphi}_i(x_i) - \tilde{\varphi}_i(x_i^0), \quad \text{for } i = 2, \dots, d. \end{aligned}$$

We have $\sum_{i=1}^d \varphi_i(x_i) = \sum_{i=1}^d \tilde{\varphi}_i(x_i)$, and hence $(\varphi_1, \dots, \varphi_d)$ is a $(G, c)$-splitting tuple with $\varphi_1(x_1^0) = c(x^0) \ge 0$ and $\varphi_i(x_i^0) = 0$ for $i = 2, \dots, d$. We hence have

$$\begin{aligned} \varphi_1(x_1) &\le c(x_1, x_2^0, \dots, x_d^0) \quad \text{for all } x_1 \in X_1, \\ \varphi_i(x_i) &\le c(x_1^0, \dots, x_{i-1}^0, x_i, x_{i+1}^0, \dots, x_d^0) - \varphi_1(x_1^0) \\ &\le c(x_1^0, \dots, x_{i-1}^0, x_i, x_{i+1}^0, \dots, x_d^0) \quad \text{for all } x_i \in X_i. \end{aligned}$$

□

**Proposition 2.5.** *Every $c$-cyclically monotone set $\Gamma$ is $c$-splitting.*

*Proof.* (The result is trivial if $\Gamma$ is empty.)

We fix an element $x^0 \in \Gamma$. Define the functions $c_i : X_i \to [0, \infty)$:

$$c_i : x_i \mapsto c(x_1^0, \dots, x_{i-1}^0, x_i, x_{i+1}^0, \dots, x_d^0).$$

For each finite subset $G$ of $\Gamma$, set $G' = G \cup \{x^0\}$. By the previous two lemmas, for each such $G'$ there is a $(G', c)$-splitting tuple with the components of the tuple bounded from above by $c_1, \dots, c_d$, respectively. Now we define

$$\begin{aligned} \mathcal{G}_G = \big\{\varphi \equiv (\varphi_1, \dots, \varphi_d) : \varphi &\text{ is a } (G', c)\text{-splitting tuple with} \\ &\varphi_i(x_i) \le c_i(x_i) \text{ for all } x_i \in X_i, \quad i = 1, \dots, d\big\}. \end{aligned}$$

The sets $\mathcal{G}_{G'}$ are nonempty by our previous considerations. Note that they are closed in the topology of pointwise convergence on the *compact* function space $\overline{\mathbb{R}}^{X_1} \times \dots \times \overline{\mathbb{R}}^{X_d}$. Also, the sets $\mathcal{G}_{G'}$ have the finite intersection property: this is clear from

$$\mathcal{G}_{(G_1 \cup G_2)'} \subseteq \mathcal{G}_{G_1'} \cap \mathcal{G}_{G_2'}.$$

Consequently, the set

$$\mathcal{G} = \bigcap_{G \subseteq \Gamma,\ G \text{ finite}} \mathcal{G}_{G'}$$

is nonempty. It is easy to check that each of the tuples in $\mathcal{G}$ is $(\Gamma, c)$-splitting. □

*Proof of Theorem* 1.2. $\mu$ is concentrated on a $c$-cyclically monotone, and hence a $c$-splitting set $\Gamma$. By the assumption on $c$, for any $x^0 = (x_1^0, \dots, x_d^0)$ in $\Gamma$ the functions

$$c_i : x_i \mapsto c(x_1^0, \dots, x_{i-1}^0, x_i, x_{i+1}^0, \dots, x_d^0)$$

are in $L_1(\mu_i)$. By Lemma 2.4, there is a measurable $(\Gamma, c)$-splitting tuple $(\varphi_1, \dots, \varphi_d)$ such that

$$\varphi_i(x_i) \le c_i(x_i) \quad \text{for all } x_i \in X_i, \quad i = 1, \dots, d.$$

Hence, the functions $\varphi_i$ are all integrable against $\mu_i$, with the value of the integral in $[-\infty, \infty)$. Now take any $\mu' \in \mathcal{M}(\mu_1, \dots, \mu_d)$. We have, as $\mu$ is concentrated on the $c$-splitting set $\Gamma$, and $(\varphi_1, \dots, \varphi_d)$ is $(\Gamma, c)$-splitting,

$$\int c \, d\mu = \sum \int \varphi_i \, d\mu_i \leq \int c \, d\mu'. \qquad \square$$

## References


[AG12] Luigi Ambrosio and Nicola Gigli, *A user's guide to optimal transport*, Modelling and optimisation of flows on networks, Lecture Notes in Math., vol. 2062, Springer, Heidelberg, 2013, pp. 1–155, DOI 10.1007/978-3-642-32160-3_1. MR3050280

[AP03] Luigi Ambrosio and Aldo Pratelli, *Existence and stability results in the $L^1$ theory of optimal transportation*, Optimal transportation and applications (Martina Franca, 2001), Lecture Notes in Math., vol. 1813, Springer, Berlin, 2003, pp. 123–160, DOI 10.1007/978-3-540-44857-0_5. MR2006307

[BC10] Stefano Bianchini and Laura Caravenna, *On optimality of c-cyclically monotone transference plans* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **348** (2010), no. 11-12, 613–618, DOI 10.1016/j.crma.2010.03.022. MR2652484

[Bei15] Mathias Beiglböck, *Cyclical monotonicity and the ergodic theorem*, Ergodic Theory Dynam. Systems **35** (2015), no. 3, 710–713, DOI 10.1017/etds.2013.75. MR3334900

[BG14] M. Beiglböck and C. Griessler, *An optimality principle with applications in optimal transport. ArXiv e-prints*, April 2014.

[BGMS09] Mathias Beiglböck, Martin Goldstern, Gabriel Maresch, and Walter Schachermayer, *Optimal and better transport plans*, J. Funct. Anal. **256** (2009), no. 6, 1907–1927, DOI 10.1016/j.jfa.2009.01.013. MR2498564

[BJ16] Mathias Beiglböck and Nicolas Juillet, *On a problem of optimal transport under marginal martingale constraints*, Ann. Probab. **44** (2016), no. 1, 42–106, DOI 10.1214/14-AOP966. MR3456332

[CDPDM15] Maria Colombo, Luigi De Pascale, and Simone Di Marino, *Multimarginal optimal transport maps for one-dimensional repulsive costs*, Canad. J. Math. **67** (2015), no. 2, 350–368, DOI 10.4153/CJM-2014-011-x. MR3314838

[Kel84] Hans G. Kellerer, *Duality theorems for marginal problems*, Z. Wahrsch. Verw. Gebiete **67** (1984), no. 4, 399–432, DOI 10.1007/BF00532047. MR761565

[KP14] Young-Heon Kim and Brendan Pass, *A general condition for Monge solutions in the multi-marginal optimal transport problem*, SIAM J. Math. Anal. **46** (2014), no. 2, 1538–1550, DOI 10.1137/130930443. MR3190751

[KS94] M. Knott and C. S. Smith, *On a generalization of cyclic monotonicity and distances among random vectors*, Linear Algebra Appl. **199** (1994), 363–371, DOI 10.1016/0024-3795(94)90359-X. MR1274425

[Pas12] Brendan Pass, *On the local structure of optimal measures in the multi-marginal optimal transportation problem*, Calc. Var. Partial Differential Equations **43** (2012), no. 3-4, 529–536, DOI 10.1007/s00526-011-0421-z. MR2875651

[Pas15] Brendan Pass, *Multi-marginal optimal transport: theory and applications*, ESAIM Math. Model. Numer. Anal. **49** (2015), no. 6, 1771–1790, DOI 10.1051/m2an/2015020. MR3423275

[Pra08] A. Pratelli, *On the sufficiency of c-cyclical monotonicity for optimality of transport plans*, Math. Z. **258** (2008), no. 3, 677–690, DOI 10.1007/s00209-007-0191-7. MR2369050

[ST09] Walter Schachermayer and Josef Teichmann, *Characterization of optimal transport plans for the Monge–Kantorovich problem*, Proc. Amer. Math. Soc. **137** (2009), no. 2, 519–529, DOI 10.1090/S0002-9939-08-09419-7. MR2448572

[Vil03] Cédric Villani, *Topics in optimal transportation*, Graduate Studies in Mathematics, vol. 58, American Mathematical Society, Providence, RI, 2003. MR1964483

[Vil09] Cédric Villani, *Optimal transport*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 338, Springer-Verlag, Berlin, 2009. Old and new. MR2459454

[Zae15] D. A. Zaev, *On the Monge-Kantorovich problem with additional linear constraints* (Russian, with Russian summary), Mat. Zametki **98** (2015), no. 5, 664–683, DOI 10.4213/mzm10896; English transl., Math. Notes **98** (2015), no. 5-6, 725–741. MR3438523

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