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RESONANCES NEAR THRESHOLDS IN SLIGHTLY TWISTED WAVEGUIDES

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ABSTRACT. We consider the Dirichlet Laplacian in a straight three dimensional waveguide with non-rotationally invariant cross section, perturbed by a twisting of small amplitude. It is well known that such a perturbation does not create eigenvalues below the essential spectrum. However, around the bottom of the spectrum, we provide a meromorphic extension of the weighted resolvent of the perturbed operator and show the existence of exactly one pole near this point. Moreover, we obtain the asymptotic behavior of this resonance as the size of the twisting goes to 0. We also extend the analysis to the upper eigenvalues of the transversal problem, showing that the number of resonances is bounded by the multiplicity of the eigenvalue and obtaining the corresponding asymptotic behavior.

1. Introduction

Let ω be a not radially symmetric bounded domain in $\mathbb{R}^2$ with C^2 boundary. Set $\Omega := \omega \times \mathbb{R}$ and $(x_1, x_2, x_3) =: (x_t, x_3)$. Define H_0 as the Laplacian in Ω with Dirichlet boundary conditions, and denote by $-\Delta_\omega$ the Laplacian in ω with Dirichlet boundary conditions. Since ω is bounded, the spectrum of the operator $-\Delta_\omega$ is a discrete sequence of values converging to infinity, denoted by $\{\lambda_n\}_{n=1}^{\infty}$. Then, the spectrum of H_0 is given by

$$\sigma(H_0) = \bigcup_{n=1}^{\infty} [\lambda_n, \infty) = [\lambda_1, \infty)$$

and is purely absolutely continuous.

Geometric deformations of such a straight waveguide have been widely studied in recent years and have numerous applications in quantum transport in nanotubes. The spectrum of the Dirichlet Laplacian in waveguides provides information about the quantum transport of spinless particles with *hardwall* boundary conditions. In particular, the existence of eigenvalues describes the occurrence of *bound states* corresponding to trapped trajectories created by the geometric deformations. For a review we refer to [12], where bending against twisting is discussed, and to [9] for a general differential approach. Without being exhaustive we recall some well-known situations: a local bending of the waveguide creates eigenvalues below the essential spectrum, as also do a local enlarging of its width ([6, 9]). On the contrary, it has

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been proved, under general assumptions, that a twisting of the waveguide does not change the spectrum ([8]), and in particular a twisting going to 0 at infinity will not introduce discrete eigenvalues ([8, 9]). In such a situation it is natural to introduce the notion of resonance and to analyze the effect of the twisting on the resonances near the real axis. There already exist studies of resonances in waveguides: resonances in a thin curved waveguide ([7, 13]) or more recently in a straight waveguide with an electric potential, perturbed by a twisting ([11]). However, in both cases the resonances appear as perturbations of embedded eigenvalues of a reference operator and follow the *Fermi golden rule* (see [10] for references and for an overview on such resonances). As we will see, in our case the origin of the resonances will rather be due to the presence of thresholds appearing as branch points created by a 1d Laplacian. Our analysis will be close to the studies of energies near 0 for the 1d Laplacian (see for instance [1, 5, 14]). A similar phenomenon of threshold resonances was already studied for a magnetic Hamiltonian in [2], where the thresholds are eigenvalues of infinite multiplicity of some transversal problem.

In this article we will consider a small twisting of the waveguide: Let $\varepsilon : \mathbb{R} \to \mathbb{R}$ be a non-zero function of class C^1 with exponential decay; i.e., for some $\alpha > 2(\lambda_2 - \lambda_1)^{1/2}$ (this hypothesis can be relaxed; see Remark 2.1), ε satisfies

$$\varepsilon(x_3) = O(e^{-\alpha\langle x_3\rangle}), \quad \varepsilon'(x_3) = O(e^{-\alpha\langle x_3\rangle}), \tag{1.1}$$

where $\langle x_3\rangle := (1 + x_3^2)^{1/2}$. For $\delta > 0$, take θ_δ such that $\theta'_\delta(x_3) = \delta\varepsilon(x_3)$. Then, we define Ω_δ as the waveguide obtained by twisting Ω with θ_δ; i.e., we define

$$\Omega_\delta := \{(r_{\theta_\delta(x_3)}(x_t), x_3), (x_t, x_3) \in \Omega\},$$

where r_θ is the rotation of angle θ in $\mathbb{R}^2$. Set

$$W(\delta) := -\delta\partial_\varphi\varepsilon\partial_3 - \delta\partial_3\varepsilon\partial_\varphi - \delta^2\varepsilon^2\partial_\varphi^2 = -2\delta\varepsilon\partial_\varphi\partial_3 - \delta\varepsilon'\partial_\varphi - \delta^2\varepsilon^2\partial_\varphi^2,$$

with the notation ∂_φ for $x_1\partial_2 - x_2\partial_1$. Then, it is standard (see for instance [9, Section 2]) that the Dirichlet Laplacian in Ω_δ is unitarily equivalent to the operator

$$H(\delta) := H_0 + W(\delta),$$

defined in Ω with a Dirichlet boundary condition. Since the perturbation is a second order differential operator, $H(\delta)$ is not a relatively compact perturbation of H_0. However the resolvent difference $H(\delta)^{-1} - H_0^{-1}$ is compact ([4, Section 4.1]), and therefore $H(\delta)$ and H_0 have the same essential spectrum. Moreover, the spectrum of $H(\delta)$ coincides with $[\lambda_1, +\infty)$; see [8].

In this article we will show that around λ_1 there exists, for δ small enough, a meromorphic extension of the weighted resolvent of $H(\delta)$ with respect to the variable $k := \sqrt{z - \lambda_1}$, where z is the spectral parameter (with the convention $\sqrt{-1} = i$). In other words, the resolvent $(H(\delta) - z)^{-1}$ acting on a weighted space of functions with exponential decay along the tube, which is first defined for z in $\mathbb{C} \setminus [\lambda_1, +\infty)$ (i.e., $\mathrm{Im}k > 0$), admits a meromorphic extension in a neighborhood of λ_1 in the 2-sheeted Riemann surface of $\sqrt{z - \lambda_1}$.

We will identify the resonances around λ_1 with the poles of this meromorphic extension in the parameter k. We will prove in Theorem 3.2 that in a neighborhood independent of δ, there is exactly one pole $k(\delta)$, whose behavior as $\delta \to 0$ is explicit:

$$k(\delta) = -i\mu\delta^2 + O(\delta^3), \tag{1.2}$$

where $\mu > 0$ is given by (3.3) below and, moreover, $k(\delta)$ is on the imaginary axis.

The fact that $k(\delta)$ is on the negative imaginary axis means that in the spectral variable the resonance is on the second sheet of the 2-sheeted Riemann surface, far from the real axis (it is sometimes called an anti-bound state [15]). In particular such a resonance cannot be detected using dilations (a dilation of angle larger than π would be needed) and is different in nature from those created by perturbations of embedded eigenvalues. Besides, a difficulty comes from the non-relative compactness of the perturbation $W(\delta)$. This problem will be overcome by exploiting the smallness of the perturbation and the locality of our problem.

Our analysis provides an analogous result for higher thresholds, namely, in Theorem 4.1 of Section 4 we prove that around each λ_{q_0} there are at most m_0 resonances (for all δ small enough), where m_0 is the multiplicity of λ_{q_0} as eigenvalue of $-\Delta_\omega$. Moreover, under an additional assumption, each one of these resonances has an asymptotic behavior of the form (1.2), where the constant μ is an eigenvalue of an $m_0 \times m_0$ explicit matrix (symmetric but not necessarily Hermitian). Although Theorem 4.1 may be viewed as a generalization of Theorem 3.2, we preferred to push forward the proof for the first threshold for the following reasons: it is easier to follow, it contains all the main ingredients needed for the proof in the upper thresholds, and the eigenvalues of $-\Delta_\omega$ are generically simple as we know the first eigenvalue is.

Remark 1.1. Independently of the size of the perturbation $W(\delta)$, a more global definition of resonances would be possible by showing that a generalized determinant (as in [3] or in [16, Definition 4.3]) is well defined on $\mathbb{C} \setminus [0, +\infty)$ and admits an analytic extension. Then the resonances would be defined as the zeros of this determinant on an infinite-sheeted Riemann surface (as in [2, Definitions 1-2]).

2. Preliminary decomposition of the free resolvent

Let us describe the singularities of the free resolvent. Setting $D_3 := -i\partial_3$, we have that

$$H_0 - \lambda_1 = (-\Delta_\omega - \lambda_1) \otimes I_{x_3} + I_{x_t} \otimes D_3^2. \tag{2.1}$$

For $k \in \mathbb{C}^+ := \{k \in \mathbb{C};\ \operatorname{Im} k > 0\}$, define

$$R_0(k) := (H_0 - \lambda_1 - k^2)^{-1}$$

and R similarly for $H(\delta)$. If for $n \in \mathbb{N}$, π_n is the orthogonal projection onto the space $\ker(-\Delta_\omega - \lambda_n)$, using (2.1) we have that

$$R_0(k) = (H_0 - \lambda_1 - k^2)^{-1} = \sum_{q \geq 1} \pi_q \otimes (D_3^2 + (\lambda_q - \lambda_1) - k^2)^{-1}. \tag{2.2}$$

The integral kernel of $(D_3^2 - k^2)^{-1}$ is explicitly given by

$$\frac{i}{2k} e^{i\,k|x_3 - x_3'|}. \tag{2.3}$$

Let η be an exponential weight of the form $\eta(x_3) = e^{-N\langle x_3 \rangle}$, for $(\lambda_2 - \lambda_1)^{1/2} < N < \alpha/2$. Also, for $a \in \mathbb{C}$ and $r > 0$ set $B(a, r) := \{z \in \mathbb{C}; |a - z| < r\}$. Then, as in [2, Lemma 1] it can be seen that the operator-valued function $k \mapsto (R_0(k) : \eta L^2(\Omega) \to \eta^{-1} L^2(\Omega))$, initially defined on $\mathbb{C}^+$, has a meromorphic extension in

$B(0,r)$ for any $0 < r < (\lambda_2 - \lambda_1)^{1/2}$, with a unique pole, of multiplicity one, at $k = 0$. More precisely,

$$\eta R_0(k)\eta = \frac{1}{k}\pi_1 \otimes \alpha_0 + A_0(k), \tag{2.4}$$

where α_0 is the rank one operator $\alpha_0 = \frac{i}{2}|\eta\rangle\langle\eta|$ and $k \mapsto (A_0(k)\colon\ L^2(\Omega) \to L^2(\Omega))$ is the analytic operator-valued function

$$A_0(k) := \pi_1 \otimes r_1(k) + \sum_{q\geq 2} \pi_q \otimes \eta(D_3^2 + (\lambda_q - \lambda_1) - k^2)^{-1}\eta, \tag{2.5}$$

with r_1 being the operator in $L^2(\mathbb{R})$ with integral kernel given by

$$i\eta(x_3)\frac{(e^{i\,k|x_3 - x_3'|} - 1)}{2k}\eta(x_3'). \tag{2.6}$$

Clearly, for $0 < r < (\lambda_2 - \lambda_1)^{1/2}$, the family of operators $A_0(k)$ is uniformly bounded on $B(0,r)$.

Remark 2.1. Note that the condition $\alpha > 2(\lambda_2 - \lambda_1)^{\frac{1}{2}}$ on the function ε enters here in order to have analytic properties in the ball $B(0,r)$, $0 < r < (\lambda_2 - \lambda_1)^{1/2}$. This assumption can be relaxed to $\alpha > 0$, but the results would be restricted to $B(0,r)$ with $0 < r < \frac{\alpha}{2}$.

In order to define and study the resonances, we will consider a suitable meromorphic extension of $R(k)$, using the identity

$$\eta R(k)\eta = \eta R_0(k)\eta \left(\mathrm{Id} + \eta^{-1}W(\delta)R_0(k)\eta\right)^{-1}. \tag{2.7}$$

Since $H(\delta)$ has no eigenvalue below λ_1 (see [8]), the above relation is initially well defined and analytic for $k \in \mathbb{C}^+$. It is necessary then to understand under which conditions this formula can be used to define such an extension. Since we cannot apply directly the meromorphic Fredholm theory ($W(\delta)$ is not H_0-compact), we will need to show explicitly that $\left(\mathrm{Id} + \eta^{-1}W(\delta)R_0(k)\eta\right)^{-1}$ is meromorphic in some region around zero.

Let ψ_1 be such that $-\Delta_\omega\psi_1 = \lambda_1\psi_1$, $\|\psi_1\|_{L^2(\omega)} = 1$ (then $\pi_1 = |\psi_1\rangle\langle\psi_1|$), and define

$$\Phi_\delta := -\frac{i}{2}((\partial_\varphi\psi_1 \otimes \eta^{-1}\varepsilon') + \delta(\partial_\varphi^2\psi_1 \otimes \eta^{-1}\varepsilon^2)). \tag{2.8}$$

Proposition 2.2. Let $0 < r < (\lambda_2 - \lambda_1)^{1/2}$. There exists $\delta_0 > 0$ such that for any $0 < \delta \leq \delta_0$ and $k \in B(0,r) \setminus \{0\}$,

$$\eta^{-1}W(\delta)R_0(k)\eta = \frac{\delta}{k}K_0 + \delta T(\delta,k),$$

where K_0 is the rank one operator

$$K_0 := |\Phi_\delta\rangle\langle\psi_1 \otimes \eta|, \tag{2.9}$$

and $B(0,r) \ni k \mapsto (T(\delta,k)\colon\ L^2(\Omega) \to L^2(\Omega))$ is an analytic operator-valued function. Moreover,

$$\sup_{0<\delta\leq\delta_0,\, k\in B(0,r)} ||T(\delta,k)|| < \infty. \tag{2.10}$$

Proof. Thanks to (2.4),

$$\eta^{-1}W(\delta)R_0(k)\eta = \frac{1}{k}\eta^{-1}W(\delta)\eta^{-1}(\pi_1 \otimes \alpha_0) + \eta^{-1}W(\delta)\eta^{-1}A_0(k). \tag{2.11}$$

Since the range of the operator $\eta^{-1}\alpha_0 = \frac{i}{2}|1\rangle\langle\eta$ is spanned by constant functions, we have $\partial_3\eta^{-1}\alpha_0 = 0$, and therefore

$$\eta^{-1}W(\delta)\eta^{-1}(\pi_1 \otimes \alpha_0) = \frac{i}{2}|\eta^{-1}(-\delta\varepsilon'\partial_\varphi - \delta^2\varepsilon^2\partial_\varphi^2)\eta^{-1}(\psi_1 \otimes \eta)\rangle\langle\psi_1 \otimes \eta| = \delta K_0.$$

We now treat the last term of (2.11): Setting $\delta T(\delta, k) = \eta^{-1}W(\delta)\eta^{-1}A_0(k)$ we get

$$\begin{aligned}
T(\delta,k)&\\
=&2\Big(\partial_\varphi\pi_1 \otimes \eta^{-1}\varepsilon\partial_3\eta^{-1}r_1(k) + \sum_{q\geq 2}\partial_\varphi\pi_q \otimes \eta^{-1}\varepsilon\partial_3(D_3^2 + (\lambda_q - \lambda_1) - k^2)^{-1}\eta\Big)\\
&-\Big(\partial_\varphi\pi_1 \otimes \eta^{-1}\varepsilon'\eta^{-1}r_1(k) + \sum_{q\geq 2}\partial_\varphi\pi_q \otimes \eta^{-1}\varepsilon'(D_3^2 + (\lambda_q - \lambda_1) - k^2)^{-1}\eta\Big)\\
&-\delta\Big(\partial_\varphi^2\pi_1 \otimes \eta^{-1}\varepsilon^2\eta^{-1}r_1(k) + \sum_{q\geq 2}\partial_\varphi^2\pi_q \otimes \eta^{-1}\varepsilon^2(D_3^2 + (\lambda_q - \lambda_1) - k^2)^{-1}\eta\Big).
\end{aligned}$$

It is clear that the last two terms are analytic and uniformly bounded in $B(0, r)$. For the first one, we note that the kernel of $\partial_3\eta^{-1}r_1(k)$, the kernel of r_1 being (2.6), is given by $(x_3, x_3') \mapsto -\frac{1}{2}\eta(x_3')\mathrm{sign}(x_3 - x_3')e^{ik|x_3 - x_3'|}$, and therefore $\partial_\varphi\pi_1 \otimes \eta^{-1}\varepsilon\partial_3\eta^{-1}r_1$ admits an analytic expansion which is uniformly bounded. The same arguments run for $\sum_{q\geq 2}\partial_\varphi\pi_q \otimes \eta^{-1}\varepsilon\partial_3(D_3^2 + (\lambda_q - \lambda_1) - k^2)^{-1}\eta$.

Finally, K_0 is of rank one for δ small enough, because $\partial_\varphi\psi_1 \neq 0$ (see [4, Proposition 2.2]). ☐

3. Meromorphic extension of the resolvent and study of the resonance

Lemma 3.1. *Let $\mathcal{D} \subset B(0, \sqrt{\lambda_2 - \lambda_1})$ be a compact neighborhood of zero. With the notation of Proposition 2.2, for δ sufficiently small, let us introduce the functions $\tilde{\Phi}_\delta = (\mathrm{Id} + \delta T(\delta, k))^{-1}\Phi_\delta$ and*

$$w_\delta(k) = \delta\langle\tilde{\Phi}_\delta|\psi_1 \otimes \eta\rangle. \tag{3.1}$$

Then:

(i) *There exists δ_0 such that for any $k \in \mathcal{D}$, $\delta \in (0, \delta_0)$,*

$$w_\delta(k) = i\mu\delta^2 + O(\delta^3) + \delta^2 k g_\delta(k), \tag{3.2}$$

where

$$\mu := \tfrac{1}{2}\sum_{q\geq 2}(\lambda_q - \lambda_1)\|\pi_q\partial_\varphi\psi_1\|^2\langle\varepsilon|(D_3^2 + \lambda_q - \lambda_1)^{-1}\varepsilon\rangle \tag{3.3}$$

is a positive constant, and g_δ is an analytic function in $\mathcal{D}$ satisfying

$$\sup_{\delta\in(0,\delta_0)}\sup_{k\in\mathcal{D}}|g_\delta(k)| < +\infty.$$

(ii) *When $\beta \in \mathbb{R} \cap -i\mathcal{D}$, there holds $w_\delta(i\beta) \in i\mathbb{R}$.*

Proof. We use the Taylor expansion and Proposition 2.2 to see that

$$(\mathrm{Id} + \delta T(\delta,k))^{-1} = \mathrm{Id} - \delta T(\delta,0) + \delta k G_\delta(k) + O(\delta^2), \tag{3.4}$$

where $G_\delta(k)$ is a holomorphic operator-valued function that is uniformly bounded for $k \in \mathcal{D}$ and δ small.

By definition of Φ_δ, we have

$$\begin{aligned}&\langle \Phi_\delta | \psi_1 \otimes \eta\rangle \\ &\qquad = -\tfrac{i}{2}\left(\langle \partial_\varphi \psi_1 | \psi_1\rangle_{L^2(\omega)}\,\langle \eta^{-1}\varepsilon' | \eta\rangle_{L^2(\mathbb{R})} + \delta \langle \partial_\varphi^2 \psi_1 | \psi_1\rangle_{L^2(\omega)}\,\langle \eta^{-1}\varepsilon^2 | \eta\rangle_{L^2(\mathbb{R})}\right).\end{aligned}$$

The first term is zero because ε tends to zero at infinity. Using integration by parts, since ψ_1 satisfies a Dirichlet boundary condition, we deduce that

$$\langle \Phi_\delta | \psi_1 \otimes \eta\rangle = \delta \frac{i}{2}\|\partial_\varphi \psi_1\|^2 \|\varepsilon\|^2.$$

Noticing that $\|\Phi_\delta\| = O(1)$, from (3.4) we get

$$w_\delta(k) = \delta^2 \frac{i}{2}\|\partial_\varphi \psi_1\|^2\|\varepsilon\|^2 - \delta^2 \langle T(\delta,0)\Phi_\delta | \psi_1 \otimes \eta\rangle + \delta^2 k g_\delta(k) + O(\delta^3), \tag{3.5}$$

where $g_\delta(k)$ is holomorphic and uniformly bounded for $k \in \mathcal{D}$ and δ small.

We now compute $\langle T(\delta,0)\Phi_\delta | \psi_1 \otimes \eta\rangle$. Recall that $T(\delta,k) = \delta^{-1}\eta^{-1}W(\delta)\eta^{-1}A_0(k)$. Next, note that since $\langle \partial_\varphi \psi_1 | \psi_1\rangle = 0$,

$$\pi_1 \partial_\varphi \psi_1 = 0,$$

and therefore, using the definition of Φ_δ in (2.8), we get

$$(\pi_1 \otimes r_1(0))\Phi_\delta = -\delta \frac{i}{2}\pi_1 \partial_\varphi^2 \psi_1 \otimes r_1(0)\eta^{-1}\varepsilon^2,$$

which in turn implies that

$$\langle (\delta^{-1}\eta^{-1}W(\delta)\eta^{-1})(\pi_1 \otimes r_1(0))\Phi_\delta | \psi_1 \otimes \eta\rangle = O(\delta).$$

In consequence, having in mind (2.8) again, we deduce that

$$\begin{aligned}&\langle T(\delta,0)\Phi_\delta, \psi_1 \otimes \eta\rangle + O(\delta) \qquad (3.6)\\ =&\langle \eta^{-1}\left(-2\varepsilon \partial_\varphi \partial_3 - \varepsilon' \partial_\varphi - \delta \varepsilon^2 \partial_\varphi^2\right)\eta^{-1}\Big(\sum_{q\geq 2}\pi_q \otimes \eta(D_3^2 + (\lambda_q - \lambda_1))^{-1}\eta\Big)\Phi_\delta | \psi_1 \otimes \eta\rangle\\ =&\frac{i}{2}\langle \eta^{-1}\big(2\varepsilon\partial_\varphi\partial_3 + \varepsilon'\partial_\varphi\big)\Big(\sum_{q\geq 2}\pi_q \otimes (D_3^2 + (\lambda_q - \lambda_1))^{-1}\Big)\partial_\varphi \psi_1 \otimes \varepsilon' | \psi_1 \otimes \eta\rangle.\end{aligned}$$

We compute the last expression using integration by parts, both in the φ and the x_3 variables:

$$\begin{aligned}&\langle \eta^{-1}(2\varepsilon\partial_\varphi\partial_3 + \varepsilon'\partial_\varphi)\left(\sum_{q\geq 2}\pi_q \otimes (D_3^2 + (\lambda_q - \lambda_1))^{-1}\right)\partial_\varphi \psi_1 \otimes \varepsilon' | \psi_1 \otimes \eta\rangle\\ &\quad = \sum_{q\geq 2}\langle \partial_\varphi \psi_1 | \pi_q \partial_\varphi \psi_1\rangle \times \langle \varepsilon' | (D_3^2 + \lambda_q - \lambda_1)^{-1}\varepsilon'\rangle.\end{aligned} \tag{3.7}$$

Now, we notice that

$$\begin{aligned}\langle \varepsilon' | (D_3^2 + \lambda_q - \lambda_1)^{-1}\varepsilon'\rangle &= \langle \varepsilon | (D_3^2 + \lambda_q - \lambda_1)^{-1}D_3^2 \varepsilon\rangle\\ &= \|\varepsilon\|^2 - (\lambda_q - \lambda_1)\langle \varepsilon | (D_3^2 + \lambda_q - \lambda_1)^{-1}\varepsilon\rangle.\end{aligned}$$

In addition, since $\pi_1\partial_\varphi\psi_1 = 0$ and $\sum_{q\geq 1}\pi_q = \mathrm{Id}$, we have that

$$\sum_{q\geq 2}\langle\partial_\varphi\psi_1|\pi_q\partial_\varphi\psi_1\rangle = \|\partial_\varphi\psi_1\|^2. \tag{3.8}$$

Then, from (3.6) and (3.7) we get

(3.9)

$$\begin{aligned}&\langle T(\delta,0)\Phi_\delta, \psi_1\otimes\eta\rangle\\ &= \tfrac{i}{2}\|\varepsilon\|^2\|\partial_\varphi\psi_1\|^2 - \tfrac{i}{2}\sum_{q\geq 2}(\lambda_q-\lambda_1)\langle\partial_\varphi\psi_1|\pi_q\partial_\varphi\psi_1\rangle\langle\varepsilon|(D_3^2+\lambda_q-\lambda_1)^{-1}\varepsilon\rangle + O(\delta).\end{aligned}$$

Putting together (3.5) and (3.9), we deduce (3.2). Moreover, μ is clearly nonnegative, and since $\partial_\varphi\psi_1 \neq 0$ (see [4, Proposition 2.2]), from (3.8) there exists $q \geq 2$ such that $\langle\partial_\varphi\psi_1|\pi_q\partial_\varphi\psi_1\rangle > 0$. Since $(D_3^2+\lambda_q-\lambda_1)^{-1}$ is a positive operator, we get $\mu > 0$.

Let us prove (ii). Let $\beta \in \mathbb{R}$ such that $i\beta \in \mathcal{D}$, i.e., $\beta \in -i\mathcal{D}$. Then $A_0(i\beta)$ has a real integral kernel; see (2.5). Therefore if $u \in L^2(\Omega)$ is real valued, so is $(\mathrm{Id}+\delta T(\delta,i\beta))^{-1}u$. In consequence, since Φ_δ has values in $i\mathbb{R}$, so is $\tilde\Phi_\delta = (\mathrm{Id}+\delta T(\delta,i\beta))^{-1}\Phi_\delta$, and we deduce that $w_\delta(i\beta)$ has values in $i\mathbb{R}$ as well. □

Theorem 3.2. *Let $\varepsilon : \mathbb{R}\to\mathbb{R}$ be a non-zero C^1-function satisfying* (1.1) *and let $\mathcal{D} \subset B(0,\sqrt{\lambda_2-\lambda_1})$ be a compact neighborhood of zero. Then, for δ sufficiently small, $k \mapsto R(k) = (H-\lambda_1-k^2)^{-1}$, initially defined in $\mathbb{C}^+$, admits a meromorphic operator-valued extension on $\mathcal{D}$ whose operator-values act from $\eta L^2(\Omega)$ into $\eta^{-1}L^2(\Omega)$. This function has exactly one pole $k(\delta)$ in $\mathcal{D}$, called a resonance of H, and it is of multiplicity one. Moreover, we have the asymptotic expansion*

$$k(\delta) = -i\mu\delta^2 + O(\delta^3),$$

with μ given by (3.3) *and* $\mathrm{Re}(k(\delta)) = 0$.

Proof. Consider the identity (2.7), and note that from Proposition 2.2 for $k \in \mathcal{D}\setminus\{0\}$ and δ sufficiently small we can write

$$\Big(\mathrm{Id}+\eta^{-1}W(\delta)R_0(k)\eta\Big) = \Big(\mathrm{Id}+\delta T(\delta,k)\Big)\Big(\mathrm{Id}+\frac{\delta}{k}(\mathrm{Id}+\delta T(\delta,k))^{-1}K_0\Big). \tag{3.10}$$

For $k \in \mathcal{D}\setminus\{0\}$ let us set

$$K := \frac{\delta}{k}(\mathrm{Id}+\delta T(\delta,k))^{-1}K_0 = \frac{\delta}{k}|\tilde\Phi_\delta\rangle\langle\psi_1\otimes\eta|,$$

which is a rank one operator. Then, we need to study the inverse of $(\mathrm{Id}+K)$.

Let us consider $\Pi_\delta^\perp$, the projection onto $(\mathrm{span}\,\{\psi_1\otimes\eta\})^\perp$ into the direction $\tilde\Phi_\delta$ and $\Pi_\delta = \mathrm{Id}-\Pi_\delta^\perp$, the projection onto span $\{\tilde\Phi_\delta\}$ into the direction normal to $(\psi_1\otimes\eta)$. We can easily see that

$$(\mathrm{Id}+K)\Pi_\delta^\perp = \Pi_\delta^\perp \qquad \text{and} \qquad (\mathrm{Id}+K)\Pi_\delta = (1+\frac{\delta}{k}\langle\tilde\Phi_\delta|\psi_1\otimes\eta\rangle)\Pi_\delta = \frac{k+w_\delta(k)}{k}\Pi_\delta.$$

Therefore, $\mathrm{Id}+K$ is invertible if and only if $k+w_\delta(k)\neq 0$, and

$$(\mathrm{Id}+K)^{-1} = \Pi_\delta^\perp + \frac{k}{k+w_\delta(k)}\Pi_\delta. \tag{3.11}$$

Let us consider the equation $k+w_\delta(k) = 0$. Using (3.2), for all $\kappa \in (0,\sqrt{\lambda_2-\lambda_1})$, for δ small enough, the equation has no solution for $k \in \mathcal{D}$ and $|k| \geq \kappa$. We then apply Rouché's theorem inside the ball $B(0,\kappa)$: consider the analytic functions

$h_\delta(k) = i\mu\delta^2 + k$ and $f_\delta(k) = w_\delta(k) + k$. The function h_δ has exactly one root, and on the circle $C(0,\kappa)$, using again (3.2), there holds $|h_\delta - f_\delta| \leq h_\delta$ for δ small enough. Thus, we deduce that the equation $k + w_\delta(k) = 0$ has exactly one solution $k(\delta)$ in $\mathcal{D}$, for each fixed δ small enough. In consequence, putting together (2.7), (2.4), (3.10), and (3.11) we obtain that for all $k \in \mathcal{D} \setminus \{0, k(\delta)\}$,

$$\eta R(k)\eta = \Big(\frac{1}{k}\pi_1 \otimes \alpha_0 + A_0(k)\Big)\Big(\Pi_\delta^\perp + \frac{k}{k + w_\delta(k)}\Pi_\delta\Big)\left(\mathrm{Id} + \delta T(\delta,k)\right)^{-1}.$$

By the definition of $\Pi_\delta^\perp$, we have that $(\pi_1 \otimes \alpha_0)\Pi_\delta^\perp = 0$. This relation is crucial in order to drop the degeneracy of the kernel near $k = 0$; indeed it provides

$$\eta R(k)\eta = \frac{1}{k + w_\delta(k)}(\pi_1 \otimes \alpha_0)\Pi_\delta(\mathrm{Id} + \delta T(\delta,k))^{-1}$$

$$+\frac{k}{k + w_\delta(k)}A_0(k)\Pi_\delta(\mathrm{Id} + \delta T(\delta,k))^{-1} + A_0(k)\Pi_\delta^\perp(\mathrm{Id} + \delta T(\delta,k))^{-1}.$$

Therefore, for δ sufficiently small, the weighted resolvent $k \mapsto \eta R(k)\eta$ admits a meromorphic extension to $\mathcal{D} \setminus \{k(\delta)\}$, where the pole $k(\delta)$ is given by the solution of $k + w_\delta(k) = 0$.

Using (3.2), the asymptotic expansion of $k(\delta)$ follows immediately. Further, the multiplicity of this resonance is the rank of the residue of $\eta R(k)\eta$, which coincides with the rank of $(\pi_1 \otimes \alpha_0)\Pi_\delta + k(\delta)A_0(k(\delta))\Pi_\delta$. It is one because Π_δ is of rank one with its range in $\mathrm{span}\{\tilde{\Phi}_\delta\}$ and

$$\Big((\pi_1 \otimes \alpha_0) + k(\delta)A_0(k(\delta))\Big)\tilde{\Phi}_\delta = \frac{i}{2}\langle\tilde{\Phi}_\delta|\psi_1 \otimes \eta\rangle(\psi_1 \otimes \eta) + O(\delta^2) = -\frac{\delta\mu}{2}(\psi_1 \otimes \eta) + O(\delta^2)$$

does not vanish for δ sufficiently small.

Finally let us prove that $k(\delta) \in i\mathbb{R}$. As a consequence of Lemma 3.1(ii), we have that the function s_δ, defined on $\mathbb{R} \cap B(0,\delta)$ by $s_\delta(\beta) = i(i\beta + w_\delta(i\beta))$, is real valued. Moreover, using (3.2) for δ small, $s_\delta(0) < 0$ and $s_\delta(-\delta) > 0$. In consequence, this function admits a root $\beta(\delta)$ which is real. By uniqueness, $k(\delta) = i\beta(\delta)$. □

4. Upper thresholds

We now extend our analysis to the upper thresholds. We will show that if λ_{q_0} is an eigenvalue of multiplicity $m_0 \geq 1$ of $(-\Delta_\omega)$, then m_0 is a bound for the number of resonances around λ_{q_0}.

Let $(\psi_{q_0,j})_{j=1,\ldots,m_0}$ be a normalized basis of $\ker(-\Delta_\omega - \lambda_{q_0})$. In analogy with (3.3), for $1 \leq j,l \leq m_0$ define

$$\begin{aligned}\mu_{j,l} &= \langle\partial_\varphi\psi_{q_0,j}|\pi_{q_0}\partial_\varphi\psi_{q_0,l}\rangle\,||\varepsilon||^2\\ &+\frac{1}{2}\sum_{q\neq q_0}(\lambda_q - \lambda_{q_0})\langle\partial_\varphi\psi_{q_0,j}|\pi_q\partial_\varphi\psi_{q_0,l}\rangle\langle(D_3^2 + \lambda_q - \lambda_{q_0})^{-1}\varepsilon|\varepsilon\rangle,\end{aligned} \tag{4.1}$$

and let Υ_{q_0} be the matrix $(\mu_{j,l})$.

Introduce $r_0 := \min(\sqrt{|\lambda_{q_0} - \lambda_{q_0-1}|}, \sqrt{|\lambda_{q_0+1} - \lambda_{q_0}|})$ and the upper right part of the complex plane $\mathbb{C}^{++} := \{k \in \mathbb{C}^+;\ \mathrm{Re}\,k > 0\}$.

Theorem 4.1. *Suppose that λ_{q_0} is an eigenvalue of multiplicity $m_0 \geq 1$ of $(-\Delta_\omega)$, that $\varepsilon : \mathbb{R} \to \mathbb{R}$ is a non-zero C^1-function satisfying* (1.1) *with $\alpha > 2r_0$, and that $\mathcal{D} \subset B(0,r_0)$ is a compact neighborhood of zero. Then, for all δ sufficiently small, the operator-valued function $k \mapsto (H(\delta) - \lambda_{q_0} - k^2)^{-1}$, initially defined in*

$\mathbb{C}^{++}$, admits a meromorphic extension on $\mathcal{D}$, from $\eta L^2(\Omega)$ into $\eta^{-1}L^2(\Omega)$. This extension has at most m_0 poles, counted with multiplicity. These poles are among the zeros $(k_l(\delta))_{1\leq l\leq m_0}$ of some determinant which satisfy

$$k_l(\delta) = -i\nu_{q_0,l}\,\delta^2 + o(\delta^2), \quad \delta \downarrow 0,$$

where $(\nu_{q_0,l})_{1\leq l\leq m_0}$ are the eigenvalues of the matrix Υ_{q_0}.

Proof. Some points in this proof are close to what has been done for the first threshold. We will keep the same notation and explain how to modify the arguments of the previous sections. In analogy with Section 2 set

$$\Phi_{j,\delta} := -\frac{i}{2}((\partial_\varphi\psi_{q_0,j}\otimes\eta^{-1}\varepsilon') + \delta(\partial_\varphi^2\psi_{q_0,j}\otimes\eta^{-1}\varepsilon^2)) \text{ and } K_0 := \sum_{j=1}^{m_0} |\Phi_{j,\delta}\rangle\langle\psi_{q_0,j}\otimes\eta|,$$

where K_0 is of rank m_0 for δ small enough, because due to the non-radiality of ω, following the proof of [4, Proposition 2.2]), we check that $\{\partial_\varphi\psi_{q_0,j}\}_{0\leq j\leq m_0}$ are linearly independent. Then, the analoguous of Proposition 2.2 still holds. Here, since λ_{q_0} is in the interior of the essential spectrum, the resolvent $(H(\delta)-z)^{-1}$ is initially defined for $\operatorname{Im} z > 0$ near $z = \lambda_{q_0}$, and the extension of the weighted resolvent is done with respect to $k = \sqrt{z-\lambda_{q_0}}$ from $\mathbb{C}^{++}$ to a neighborhood of $k = 0$.

Also, as in the proof of Theorem 3.2, we have for $k \in \mathbb{C}^{++}$, with $R_0(k) := (H_0 - \lambda_{q_0} - k^2)^{-1}$ (and similar notation for $R(k)$),

$$\eta R(k)\eta = \eta R_0(k)\eta(\mathrm{Id}+K)^{-1}(\mathrm{Id}+\delta T(\delta,k))^{-1}, \tag{4.2}$$

where

$$K := \frac{\delta}{k}(\mathrm{Id}+\delta T(\delta,k))^{-1}K_0 = \frac{\delta}{k}\sum_{j=1}^{m_0}|\tilde{\Phi}_{j,\delta}\rangle\langle\psi_{q_0,j}\otimes\eta|$$

is now of rank m_0 for δ small enough with obvious notation for $\tilde{\Phi}_{j,\delta}$.

Next, let $\Pi_\delta^\perp$ be the projection over $\Big(\ker(-\Delta_\omega - \lambda_{q_0})\otimes\operatorname{span}\{\eta\}\Big)^\perp$ in the direction of $\operatorname{span}\{\tilde{\Phi}_{1,\delta},...,\tilde{\Phi}_{m_0,\delta}\}$ and $\Pi_\delta := \mathrm{Id} - \Pi_\delta^\perp$. Then, the matrix of $(\mathrm{Id}+K)\Pi_\delta$ in the basis $\{\tilde{\Phi}_{j,\delta}\}_1^{m_0}$ is given, for $k \neq 0$, by

$$\frac{1}{k}\begin{bmatrix} k+\omega_{1,1,\delta}(k) & \dots & \omega_{m_0,1,\delta}(k) \\ \vdots & \ddots & \vdots \\ \omega_{1,m_0,\delta}(k) & \dots & k+\omega_{m_0,m_0,\delta}(k)\end{bmatrix} := \frac{1}{k}M_\delta(k), \tag{4.3}$$

where we have set $w_{j,l,\delta}(k) = \delta\langle\tilde{\Phi}_{j,\delta}|\psi_{q_0,l}\otimes\eta\rangle$. Assume that $M_\delta(k)$ is invertible. Then by (4.2)

$$\begin{aligned}
&\eta R(k)\eta\ (\mathrm{Id}+\delta T(\delta,k)) \\
&= \left(\frac{i}{2k}\sum_j |\psi_{q_0,j}\otimes\eta\rangle\langle\psi_{q_0,j}\otimes\eta| + A_0(k)\right)\left(\Pi_\delta^\perp + kM_\delta(k)^{-1}\Pi_\delta\right) \\
&= \left(\frac{i}{2}\sum_j |\psi_{q_0,j}\otimes\eta\rangle\langle\psi_{q_0,j}\otimes\eta|M_\delta(k)^{-1}\Pi_\delta + A_0(k)\left(\Pi_\delta^\perp + kM_\delta(k)^{-1}\Pi_\delta\right)\right).
\end{aligned}$$

In consequence, since the $w_{l,k,\delta}$ are holomorphic, $\eta R\eta$ admits a meromorphic extension to $\mathcal{D}$, and the poles of this extension are the poles of

$$\Big(\frac{i}{2}\sum_j |\psi_{q_0,j}\otimes\eta\rangle\langle\psi_{q_0,j}\otimes\eta| + kA_0(k)\Big)M_\delta(k)^{-1}\Pi_\delta. \tag{4.4}$$

Evidently, the poles are included in the set of zeros of the determinant of $M_\delta(k)$.

Define

$$\Delta(k,\delta) := \det(M_\delta(k)).$$

We can check as in Lemma 3.1 that

$$w_{j,l,\delta}(k) = i\mu_{j,l}\delta^2 + O(\delta^3) + \delta^2 k g_{j,l}(k,\delta), \tag{4.5}$$

where the $\mu_{j,l}$ are given by (4.1). Then

$$\Delta(k,\delta) = \delta^{2m_0}\det(k\delta^{-2} + i\mu_{j,l} + O(\delta) + kg_{j,l}(k,\delta)),$$

and the zeros of $\Delta(\cdot,\delta)$ are the complex numbers of the form $k = u\delta^2$, with u being a zero of

$$\tilde\Delta(u,\delta) := \det(u + i\mu_{j,l} + O(\delta) + \delta^2 u g_{j,l}(\delta^2 u,\delta)).$$

Since

$$\tilde\Delta(u,\delta) = \tilde\Delta(u,0) + \delta h(u,\delta) = \det(u + i\mu_{j,l}) + \delta h(u,\delta), \tag{4.6}$$

where h is an analytic function in u and δ, taking the ball $B(0,C)$ with C larger than the modulus of the larger eigenvalue of Υ_{q_0} and applying Rouché's theorem, we conclude that all the zeros of $\tilde\Delta(\cdot,\delta)$ are inside this ball for δ sufficiently small. Moreover, if we denote by $\nu_{q_0,l}$ the eigenvalues of Υ_{q_0}, (4.6) yields $u_{q_0,l}(\delta) = -i(\nu_{q_0,l} + o(1))$. This immediately implies that all the zeros of $\Delta(\cdot,\delta)$ in $\mathcal{D}$, denoted by k_l, are inside the ball $B(0,C\delta^2)$ and satisfy

$$k_l(\delta) = -i\delta^2(\nu_{q_0,l} + o(1)).$$

□

Remark 4.2. In the last theorem, if $m_0 = 1$, we are able to obtain extra information. For instance, as in Theorem 3.2, for the unique zero of the determinant, $k_1(\delta)$, we have that $k_1(\delta) = -i\mu_{q_0}\delta^2 + O(\delta^3)$ with

$$\mu_{q_0} := \mu_{1,1} = \frac{1}{2}\sum_{q\neq q_0}(\lambda_q - \lambda_{q_0})\langle\partial_\varphi\psi_{q_0}|\pi_q\partial_\varphi\psi_{q_0}\rangle\langle\varepsilon|(D_3^2 + \lambda_q - \lambda_{q_0})^{-1}\varepsilon\rangle.$$

Then, as in the proof of Theorem 3.2, k_{q_0} is a pole of multiplicity one when $\mu_{q_0} \neq 0$. It is also important to notice that, for $q < q_0$, the operator $(D_3^2 + \lambda_q - \lambda_{q_0})^{-1}$ has to be understood as the limit of $(D_3^2 + \lambda_q - \lambda_{q_0} - k^2)^{-1}$, acting in weighted spaces, when $k \to 0$. It is not a self-adjoint operator anymore; therefore μ_{q_0} is not necessarily real. Actually, in general, it has a non-zero imaginary part coming from the first terms when $q < q_0$. Indeed, thanks to (2.3), for $q < q_0$, the imaginary part of $2(\lambda_{q_0} - \lambda_q)^{1/2}\,\langle\varepsilon|(D_3^2 + \lambda_q - \lambda_{q_0})^{-1}\varepsilon\rangle$ is given by

$$-\Big(\int_{\mathbb{R}}\cos(\sqrt{\lambda_{q_0}-\lambda_q}\,x_3)\,\varepsilon(x_3)\,dx_3\Big)^2 - \Big(\int_{\mathbb{R}}\sin(\sqrt{\lambda_{q_0}-\lambda_q}\,x_3)\,\varepsilon(x_3)\,dx_3\Big)^2$$
$$= -\sqrt{2\pi}|\widehat{\varepsilon}(\sqrt{\lambda_{q_0}-\lambda_q})|^2,$$

where $\widehat{\varepsilon}$ is the Fourier transform of ε. Then, the imaginary part of μ_{q_0} is

$$\operatorname{Im}(\mu_{q_0}) = -\frac{\sqrt{2\pi(\lambda_{q_0}-\lambda_q)}}{4} \sum_{q<q_0} \|\pi_q \partial_\varphi \psi_{q_0}\|^2 \, |\widehat{\varepsilon}(\sqrt{\lambda_{q_0}-\lambda_q})|^2.$$

This identity allows us to give sufficient conditions on the eigenfunctions of $-\Delta_\omega$ and on $\hat{\varepsilon}$, so that $\mu_{q_0} \neq 0$, giving rise to a unique resonance of multiplicity one, with $\operatorname{Re} k_1(\delta) < 0$.

For $m_0 > 1$ the resonances (i.e., the poles of (4.4)) would be the poles of $M_\delta(k)^{-1}$ when the operator $\sum_j |\psi_{q_0,j} \otimes \eta\rangle\langle\psi_{q_0,j} \otimes \eta|$ is invertible on the range of Π_δ. This property will be satisfied as soon as the matrix Υ_{q_0} is invertible, but it does not hold in general.

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