PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 11, November 2018, Pages 4937–4950
https://doi.org/10.1090/proc/14142
Article electronically published on August 8, 2018

EXAMPLES OF ITÔ CÀDLÀG ROUGH PATHS

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(Communicated by Zhen-Qing Chen)

Abstract. Based on a dyadic approximation of Itô integrals, we show the existence of Itô càdlàg rough paths above general semimartingales, suitable Gaussian processes, and nonnegative typical price paths. Furthermore, the Lyons–Victoir extension theorem for càdlàg paths is presented, stating that every càdlàg path of finite p-variation can be lifted to a rough path.

1. Introduction

Very recently, the notion of càdlàg rough paths was introduced by Friz and Shekhar [FS17] (see also [CF17, Che17]) extending the well-known theory of continuous rough paths initiated by Lyons [Lyo98]. These new developments significantly generalize an earlier work by Williams [Wil01]. While [Wil01] already provides a pathwise meaning to stochastic differential equations driven by certain Lévy processes, [FS17, CF17] develop a more complete picture about càdlàg rough paths, including rough path integration, differential equations driven by càdlàg rough paths, and the continuity of the corresponding solution maps. We refer to [LCL07, FV10b, FH14] for detailed introductions to classical rough path theory.

A càdlàg rough path is analogously defined to a continuous rough path using finite p-variation as required regularity (see Definitions 2.1 and 2.3), but (of course) dropping the assumption of continuity. Note that the notion of p-variation still works in the context of càdlàg paths without any modifications. Loosely speaking, for $p \in [2,3)$ a càdlàg rough path is a pair $(X, \mathbb{X})$ given by a càdlàg path $X\colon [0,T] \to \mathbb{R}^d$ of finite p-variation and its "iterated integral"

$$\mathbb{X}_{s,t} = \text{``} \int_s^t (X_{r-} - X_s) \otimes \mathrm{d}X_r \text{''}, \quad s,t \in [0,T], \tag{1.1}$$

which satisfies Chen's relation and is of finite $p/2$-variation in the rough path sense. While the "iterated integral" can be easily defined for smooth paths X, as for example via Young integration [You36], it is a nontrivial question whether any paths of finite p-variation can be lifted (or enhanced) to a rough path. In the setting of continuous rough paths this question was answered affirmative by the Lyons–Victoir extension theorem [LV07]. In Section 2 we prove the analogous

Received by the editors September 19, 2017, and, in revised form, February 12, 2018.
2010 *Mathematics Subject Classification.* Primary 60H99, 60G17; Secondary 91G99.
Key words and phrases. Càdlàg rough paths, Gaussian processes, Lyons–Victoir extension theorem, semimartingales, typical price paths.
The second author gratefully acknowledges financial support of the Swiss National Foundation under Grant No. 200021_163014 and was affiliated with ETH Zürich when this project was commenced.

result in the context of càdlàg rough paths stating that every càdlàg path of finite p-variation for arbitrary noninteger $p \geq 1$ can be lifted to a rough path.

The theory of càdlàg rough paths provides a novel perspective to many questions in stochastic analysis involving stochastic processes with jumps, which play a very important role in probability theory. For a long list of successful applications of continuous rough path theory we refer to the book [FH14]. However, for applications of rough path theory in probability theory the Lyons–Victoir extension theorem is not sufficient. Instead, it is of upmost importance to be able to lift stochastic processes to random rough paths via some type of stochastic integration.

In Section 3 we focus on stochastic processes with sample paths of finite p-variation for $p \in (2,3)$, which is the most frequently used setting in probability theory, and construct the corresponding random rough paths using Itô(-type) integration. More precisely, we define for a stochastic process X the "iterated integral" $\mathbb{X}$ (cf. (1.1)) as a limit of approximating left-point Riemann sums, which corresponds to classical Itô integration if X is a semimartingale. The main difficulty is to show that $\mathbb{X}$ is of finite $p/2$-variation in the rough path sense. For this purpose we provide a deterministic criterion to verify the $p/2$-variation of $\mathbb{X}$ based on a dyadic approximation of the path and its iterated integral; see Theorem 3.1. As an application of Theorem 3.1, we provide the existence of Itô càdlàg rough paths above general semimartingales (possibly perturbed by paths of finite q-variation), certain Gaussian processes, and typical nonnegative prices paths. Let us remark that related constructions of random càdlàg rough paths above stochastic processes are given in [FS17] and [CF17], on which we comment in more detail in the specific subsections.

1.1. **Organization of the paper.** In Section 2 the basic definitions and Lyons–Victoir extension theorem are presented. Section 3 provides the constructions of Itô càdlàg rough paths.

2. Càdlàg rough path and Lyons–Victoir extension theorem

In this section we briefly recall the definitions of càdlàg rough path theory as very recently introduced in [FS17, CF17] and present the Lyons–Victoir extension theorem in the càdlàg setting; see Proposition 2.4.

Let $D([0,T];E)$ be the space of càdlàg (right-continuous with left-limits) paths from $[0,T]$ into a metric space (E,d). A partition $\mathcal{P}$ of the interval $[0,T]$ is a set of essentially disjoint intervals covering $[0,T]$, i.e., $\mathcal{P} = \{[t_i, t_{i+1}] : 0 = t_0 < t_1 < \cdots < t_n = T,\, n \in \mathbb{N}\}$. A path $X \in D([0,T];E)$ is of finite p-variation for $p \in (0,\infty)$ if

$$\|X\|_{p\text{-var}} := \left(\sup_{\mathcal{P}} \sum_{[s,t] \in \mathcal{P}} d(X_s, X_t)^p \right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all partitions $\mathcal{P}$ of the interval $[0,T]$ and the sum denotes the summation over all intervals $[s,t] \in \mathcal{P}$. The space of all càdlàg paths of finite p-variation is denoted by $D^{p\text{-var}}([0,T];E)$. For a two-parameter function $\mathbb{X} \colon \Delta_T \to \mathbb{R}^{d \times d}$ we define

$$\|\mathbb{X}\|_{p/2\text{-var}} := \left(\sup_{\mathcal{P}} \sum_{[s,t] \in \mathcal{P}} |\mathbb{X}_{s,t}|^{\frac{p}{2}} \right)^{\frac{2}{p}}, \quad p \in (0,\infty), \tag{2.1}$$

where $\Delta_T := \{(s,t) \in [0,T] : s \le t\}$ and $d \in \mathbb{N}$. Furthermore, we use the shortcut $X_{s,t} := X_t - X_s$ for $X \in D([0,T];\mathbb{R}^d)$.

For $p \in [2,3)$ the fundamental definition of a càdlàg rough path was introduced in [FS17, Definition 12] and reads as follows.

Definition 2.1. For $p \in [2,3)$, a pair $\mathbf{X} = (X, \mathbb{X})$ is called a *càdlàg rough path* over $\mathbb{R}^d$ (in symbols $\mathbf{X} \in \mathcal{W}^p([0,T];\mathbb{R}^d)$) if $X\colon [0,T] \to \mathbb{R}^d$ and $\mathbb{X}\colon \Delta_T \to \mathbb{R}^{d\times d}$ satisfy:

(1) Chen's relation holds: $\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = X_{s,u} \otimes X_{u,t}$ for $0 \le s \le u \le t \le T$.
(2) The map $[0,T] \ni t \mapsto X_{0,t} + \mathbb{X}_{0,t} \in \mathbb{R}^d \times \mathbb{R}^{d\times d}$ is càdlàg.
(3) $\mathbf{X} = (X, \mathbb{X})$ is of finite p-variation in the rough path sense, i.e., $\|X\|_{p\text{-var}} + \|\mathbb{X}\|_{p/2\text{-var}} < \infty$.

An important subclass of rough paths are the so-called weakly geometric rough paths: For $N \ge 1$ let $G^N(\mathbb{R}^d) \subset T^N(\mathbb{R}^d) := \sum_{k=0}^N (\mathbb{R}^d)^{\otimes k}$ be the step-N free nilpotent Lie group over $\mathbb{R}^d$, embedded into the truncated tensor algebra $(T^N(\mathbb{R}^d), +, \otimes)$ which is equipped with the Carnot–Carathéodory norm $\|\cdot\|$ and the induced (left-invariant) metric d. For more details we refer to [FV10b, Chapter 7]. A rough path $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{W}^p([0,T];\mathbb{R}^d)$ for $p \in [2,3)$ is said to be a weakly geometric rough path if $1 + X_{0,t} + \mathbb{X}_{0,t}$ takes values in $G^2(\mathbb{R}^d)$.

Note that while the constructions of rough paths carried out in Section 3 lead in general to nongeometric rough paths, it is always possible to recover a weakly geometric one.

Remark 2.2. If $N = 2$ and $p \in [2,3)$, one can easily verify that if $\mathbf{X} = (X, \mathbb{X})$ is a càdlàg rough path, then there exists a càdlàg function $F\colon [0,T] \to \mathbb{R}^{d\times d}$ of finite $p/2$-variation such that $1 + X_{0,t} + \mathbb{X}_{0,t} + F_t$ is a weakly geometric rough path; cf. [FH14, Exercise 2.14].

The notion of weakly geometric rough paths naturally extends to arbitrary low regularity $p \in [1,\infty)$; see [CF17, Definition 2.2].

Definition 2.3. Let $1 \le p < N+1$, and let $N \in \mathbb{N}$. Any $\mathbf{X} \in D^{p\text{-var}}([0,T];G^N(\mathbb{R}^d))$ is called a *weakly geometric càdlàg rough path* over $\mathbb{R}^d$.

The next proposition is the Lyons–Victoir extension theorem (see, in particular, [LV07, Corollary 19]) in the context of càdlàg rough paths.

Proposition 2.4. *Let $p \in [1,\infty)\setminus\{2,3,\dots\}$, and let $N \in \mathbb{N}$ be such that $p < N+1$. For every càdlàg path $X\colon [0,T] \to \mathbb{R}^d$ of finite p-variation there exists a (in general nonunique) weakly geometric càdlàg rough path $\mathbf{X} \in D^{p\text{-var}}([0,T];G^N(\mathbb{R}^d))$ such that $\pi_1(\mathbf{X}) = X$, where $\pi_1\colon G^N(\mathbb{R}^d) \to \mathbb{R}^d$ is the canonical projection onto the first component.*

Proof. Let X be a càdlàg $\mathbb{R}^d$-valued path of finite p-variation. By a slight modification of [CG98, Theorem 3.1], there exists a nondecreasing function $\varphi\colon [0,T] \to [0,\varphi(T)]$ with $\varphi(T) < \infty$ and a $1/p$-Hölder continuous function $g\colon [0,\varphi(T)] \to \mathbb{R}^d$ such that $X = g \circ \varphi$. Since $\varphi(t)$ is nondecreasing, the set $\mathcal{N}$ of discontinuity points of φ is at most countable. Let us define a function ϕ such that $\phi(t) = \varphi(t)$ for $t \in ([0,T] \setminus \mathcal{N}) \bigcup \{T\}$ and $\phi(t) = \varphi(t+) := \lim_{s\downarrow t, s\notin\mathcal{N}} \varphi(s)$ if $t \in \mathcal{N}$. It is easy to verify that ϕ is nondecreasing, càdlàg, and $\phi(T) = \varphi(T)$. Moreover, since X is right-continuous and g is continuous, we have $g \circ \phi = X$.

By [LV07, Corollary 19] there exists a weakly geometric $1/p$-Hölder continuous rough path $\tilde{g}$ such that $\pi_1(\tilde{g}) = g$. Now we define $\tilde{\mathbf{X}} := \tilde{g} \circ \phi$. Since ϕ is càdlàg and

$\tilde{g}$ is continuous, $\tilde{\mathbf{X}}$ is also càdlàg. Furthermore, using [CG98, Theorem 3.1] again we conclude that $\tilde{\mathbf{X}}$ has finite p-variation and thus $\tilde{\mathbf{X}} \in D^{p\text{-var}}([0,T];G^{[p]}(\mathbb{R}^d))$ with $[p] := \max\{n \in \mathbb{N} : n \leq p\}$. Finally, it is obvious that $\pi_1(\tilde{\mathbf{X}}) = \pi_1(\tilde{g}) \circ \phi = g \circ \phi = X$ and the extension of $\tilde{\mathbf{X}}$ to a weakly geometric càdlàg rough path $\mathbf{X} \in D^{p\text{-var}}([0,T];G^N(\mathbb{R}^d))$ for every $N \in \mathbb{N}$ with $p < N+1$ is possible due to [FS17, Theorem 20]. $\square$

Further conventions: The space $\mathbb{R}^d$ (resp., $\mathbb{R}^{d\times d}$) is equipped with the Euclidean norm $|\cdot|$. For $X \in D([0,T];\mathbb{R}^d)$ the supremum norm is given by $\|X\|_\infty := \sup_{t\in[0,T]} |X_t|$ and X_- denotes the left-continuous version of X, i.e., $X_-(t) := X_{t-} := \lim_{s\to t,\, s<t} X_s$ for $t \in (0,T]$ and $X_-(0) := X_{0-} := X_0$. We write $A_\vartheta \lesssim B_\vartheta$, meaning that $A_\vartheta \leq C B_\vartheta$ for some constant $C > 0$ independent of a generic parameter ϑ, and $A_\vartheta \lesssim_\vartheta B_\vartheta$, meaning that $A_\vartheta \leq C(\vartheta) B_\vartheta$ for some constant $C(\vartheta) > 0$ depending on ϑ. The indicator function of a set $A \subset \mathbb{R}$ or $A \subset D([0,T];\mathbb{R}^d)$ is denote by $\mathbf{1}_A$ and $x \wedge y := \min\{x,y\}$ for $x, y \in \mathbb{R}$.

3. Construction of Itô rough paths

In order to lift stochastic processes using Itô type integration, we first prove a deterministic criterion to check the $p/2$-variation of the corresponding lift. The construction of random rough paths above (stochastic) processes is presented in the following subsections.

For $X \in D([0,T];\mathbb{R}^d)$ or for (later) any càdlàg process X, we define the dyadic (stopping) times $(\tau_k^n)_{n,k\in\mathbb{N}}$ by

$$\tau_0^n := 0 \quad \text{and} \quad \tau_{k+1}^n := \inf\{t \geq \tau_k^n \,:\, |X_t - X_{\tau_k^n}| \geq 2^{-n}\}.$$

Furthermore, for $t \in [0,T]$ and $n \in \mathbb{N}$ we introduce the dyadic approximation

$$X_t^n := \sum_{k=0}^{\infty} X_{\tau_k^n} \mathbf{1}_{(\tau_k^n, \tau_{k+1}^n]}(t) \quad \text{and} \quad \int_0^t X_s^n \otimes \mathrm{d}X_s := \sum_{k=0}^{\infty} X_{\tau_k^n} \otimes X_{\tau_k^n \wedge t, \tau_{k+1}^n \wedge t}. \tag{3.1}$$

Note that the integral $\int_0^t X_s^n \otimes \mathrm{d}X_s$ is well defined and $\|X^n - X_-\|_\infty \leq 2^{-n}$ for every $n \in \mathbb{N}$.

Theorem 3.1. *Suppose that $X \in D^{p\text{-var}}([0,T];\mathbb{R}^d)$ for every $p > 2$ and there exist a function $\int_0^\cdot X_- \otimes \mathrm{d}X \in D([0,T];\mathbb{R}^{d\times d})$ and a dense subset D_T containing T in $[0,T]$ satisfying that for every $t \in D_T$ and for every $\varepsilon \in (0,1)$, there exist an $N = N(t,\varepsilon) \in \mathbb{N}$ and a constant $c = c(p,\varepsilon)$ such that*

$$\left| \int_0^t X_s^n \otimes \mathrm{d}X_s - \int_0^t X_- \otimes \mathrm{d}X \right| \leq c 2^{-n(1-\varepsilon)} \quad \textit{for all } n \geq N. \tag{3.2}$$

Setting for $(s,t) \in \Delta_T$

$$\mathbb{X}_{s,t} := \int_s^t X_{r-} \otimes \mathrm{d}X_r - X_s \otimes X_{s,t} := \int_0^t X_{r-} \otimes \mathrm{d}X_r - \int_0^s X_{r-} \otimes \mathrm{d}X_r - X_s \otimes X_{s,t},$$

then $(X, \mathbb{X}) \in \mathcal{W}^p([0,T];\mathbb{R}^d)$ for every $p \in (2,3)$.

To prove Theorem 3.1, we adapted some arguments used in the proof of [PP16, Theorem 4.12], in which the existence of rough paths above typical continuous price paths is shown; cf. Subsection 3.3. As a preliminary step, we need a version of Young's maximal inequality (cf. [You36] or [LCL07, Theorem 1.16]) specific to the integral $\int X^n \otimes \mathrm{d}X$.

Recall that a function $c\colon \Delta_T \to [0,\infty)$ is called right-continuous superadditive if

$$c(s,u) + c(u,t) \le c(s,t) \quad \text{for} \quad 0 \le s \le u \le t \le T,$$

and $c(s,t)$ is right-continuous in t for fixed s. Note that $X \in D^{p\text{-var}}([0,T];\mathbb{R}^d)$ if and only if there exists a right-continuous superadditive function c s.t. $|X_{s,t}|^p \le c(s,t)$ for all $(s,t) \in \Delta_T$.

Lemma 3.2. *Let $X \in D^{p\text{-var}}([0,T];\mathbb{R}^d)$ for every $p > 2$. Then it holds that*

$$\Big| \int_0^t X_r^n \otimes \mathrm{d}X_r - \int_0^s X_r^n \otimes \mathrm{d}X_r - X_s \otimes X_{s,t} \Big| \lesssim \max\{2^{-n} c(s,t)^{1/q}, 2^{n(q-2)} c(s,t) + c(s,t)^{2/q}\},$$

for $q \in (2,3)$ and every superadditive function $c\colon \Delta_T \to [0,\infty)$ (which may depend on q) such that $|X_{s,t}|^q \le c(s,t)$ for all $(s,t) \in \Delta_T$.

The proof follows the classical arguments used to derive Young's maximal inequality.

Proof. Let $X \in D^{p\text{-var}}([0,T];\mathbb{R}^d)$, and let X^n be its dyadic approximation as defined in (3.1).

1. If there exists no k such that $\tau_k^n \in [s,t]$, then

$$\Big| \int_0^t X_r^n \otimes \mathrm{d}X_r - \int_0^s X_r^n \otimes \mathrm{d}X_r - X_s \otimes X_{s,t} \Big| \lesssim 2^{-n} c(s,t)^{1/q}$$

due to the estimate $|X_{s,t}| \le c(s,t)^{1/q}$.

2. If there exists a k such that $\tau_k^n \in [s,t]$, we may assume that $s = \tau_{k_0}^n$ for some k_0. Otherwise, we just add $c(s,t)^{2/q}$ to the right-hand side. Let $\tau_{k_0}^n, \dots, \tau_{k_0+N-1}^n$ be those $(\tau_k^n)_k$ which are in $[s,t)$. W.l.o.g. we may further suppose that $N \ge 2$. Abusing notation, we write $\tau_{k_0+N}^n = t$. The idea is now to successively delete points $(\tau_{k_0+\ell}^n)$ from $\tau_{k_0}^n, \dots, \tau_{k_0+N-1}^n$. Due to the superadditivity of c, there exist $\ell \in \{1, \dots, N-1\}$ such that

$$c(\tau_{k_0+\ell-1}^n, \tau_{k_0+\ell+1}^n) \le \frac{2}{N-1} c(s,t),$$

and thus

$$\begin{aligned}
&|X_{\tau_{k_0+\ell-1}^n} \otimes X_{\tau_{k_0+\ell-1}^n, \tau_{k_0+\ell}^n} + X_{\tau_{k_0+\ell}^n} \otimes X_{\tau_{k_0+\ell}^n, \tau_{k_0+\ell+1}^n} - X_{\tau_{k_0+\ell-1}^n} \otimes X_{\tau_{k_0+\ell-1}^n, \tau_{k_0+\ell+1}^n}| \\
&\qquad = |X_{\tau_{k_0+\ell-1}^n, \tau_{k_0+\ell}^n} \otimes X_{\tau_{k_0+\ell}^n, \tau_{k_0+\ell+1}^n}| \le c(\tau_{k_0+\ell-1}^n, \tau_{k_0+\ell+1}^n)^{2/q} \\
&\qquad \le \Big(\frac{2}{N-1} c(s,t) \Big)^{2/q}.
\end{aligned}$$

Successively deleting in this manner all the points except $\tau_{k_0}^n = s$ and $\tau_{k_0+N}^n = t$ from the partition generated by $\tau_{k_0}^n, \dots, \tau_{k_0+N}^n$ leads to the estimate

$$\begin{aligned}
&\Big| \int_0^t X_r^n \otimes \mathrm{d}X_r - \int_0^s X_r^n \otimes \mathrm{d}X_r - X_s \otimes X_{s,t} \Big| \\
&\qquad \le \sum_{k=2}^{N} \Big(\frac{2}{k-1} c(s,t) \Big)^{2/q} \lesssim N^{1-2/q} c(s,t)^{2/q} \\
&\qquad \lesssim (\#\{k : \tau_k^n \in [s,t]\})^{1-2/q} c(s,t)^{2/q} + c(s,t)^{2/q},
\end{aligned}$$

since $N \leq \#\{k : \tau_k^n \in [s,t]\}$.

Hence, 1 and 2 in combination with $\#\{k : \tau_k^n \in [s,t]\} \lesssim 2^{nq} c(s,t)$, imply the assertion. $\square$

With the auxiliary Lemma 3.2 at hand we come to the proof of Theorem 3.1.

Proof of Theorem 3.1. It is straightforward to check that $(X, \mathbb{X})$ satisfies conditions (1) and (2) of Definition 2.1 and that $\|X\|_{p\text{-var}} < \infty$. Therefore, it remains to show the $p/2$-variation (in the sense of (2.1)) of $\mathbb{X}$ for every $p > 2$.

Let c be a right-continuous superadditive function with $|X_{s,t}|^q \leq c(s,t)$. Then for all $(s,t) \in \Delta_T \cap D_T^2$, using (3.2) and Lemma 3.2, for every $\varepsilon > 0$ and $q \in (2,3)$ we get a constant $c = c(p,q,\varepsilon)$ such that

$$\begin{aligned} |\mathbb{X}_{s,t}| &\leq c\Big(2^{-n(1-\varepsilon)} + \Big|\int_0^t X_r^n \otimes \mathrm{d}X_r - \int_0^s X_r^n \otimes \mathrm{d}X_r - X_s \otimes X_{s,t}\Big|\Big) \\ &\leq c\Big(2^{-n(1-\varepsilon)} + \max\{2^{-n} c(s,t)^{1/q}, 2^{-n(2-q)} c(s,t) + c(s,t)^{2/q}\}\Big), \end{aligned} \tag{3.3}$$

for all $n \geq N$, where $N \in \mathbb{N}$ may depend on s, t, and ε.

In the case that $c(s,t) \leq 1$, we set $\alpha := p/2$ for $p \in (2,3)$ and choose $n \geq N$ such that $2^{-n} \leq c(s,t)^{1/(\alpha(1-\varepsilon))}$. Taking this n in (3.3), we obtain

$$\begin{aligned} |\mathbb{X}_{s,t}|^\alpha &\leq c\Big(c(s,t) + \max\Big\{c(s,t)^{1/(1-\varepsilon)} c(s,t)^{\alpha/q}, c(s,t)^{(2-q)/(1-\varepsilon)+\alpha} + c(s,t)^{2\alpha/q}\Big\}\Big) \\ &= c\Big(c(s,t) + \max\Big\{c(s,t)^{\frac{q+\alpha(1-\varepsilon)}{q(1-\varepsilon)}}, c(s,t)^{\frac{2-q+\alpha(1-\varepsilon)}{1-\varepsilon}} + c(s,t)^{2\alpha/q}\Big\}\Big) \end{aligned}$$

for some constant $c = c(\alpha, q, \varepsilon)$. Now we would like all the exponents in the maximum on the right-hand side to be larger than or equal to 1. For the first term, this is satisfied as long as $\varepsilon < 1$. For the third term, we need $\alpha \geq q/2$. For the second term, we need $\alpha \geq (q-1-\varepsilon)/(1-\varepsilon)$. Since $\varepsilon > 0$ can be chosen arbitrarily close to 0, it suffices if $\alpha > q-1$. This means, choosing a $q_0 > 2$ close to 2 enough such that $p/2 = \alpha > \max\{q_0/2, q_0 - 1\}$, we obtain that $|\mathbb{X}_{s,t}|^{p/2} \leq c \cdot c(s,t)$ for some constant $c = c(p, q_0)$.

For the remaining case $c(s,t) > 1$, we simply estimate

$$|\mathbb{X}_{s,t}|^{p/2} \leq c\Big(\Big\|\int_0^\cdot X_{r-} \otimes \mathrm{d}X_r\Big\|_\infty^{p/2} + \|X\|_\infty^p\Big) \leq c\Big(\Big\|\int_0^\cdot X_{r-} \otimes \mathrm{d}X_r\Big\|_\infty^{p/2} + \|X\|_\infty^p\Big) c(s,t).$$

Therefore, $|\mathbb{X}_{s,t}|^{p/2} \leq c \cdot c(s,t)$ for some constant $c = c(p)$ and for every $(s,t) \in \Delta_T \cap D_T^2$. Moreover, for an arbitrary $(s,t) \in \Delta_T$, picking any sequences $(s_k)_{k\in\mathbb{N}}$ and $(t_k)_{k\in\mathbb{N}}$ in D_T such that $s_k \downarrow s$ and $t_k \downarrow t$ as $k \to \infty$, we have

$$|\mathbb{X}_{s,t}|^{p/2} = \lim_{k\to\infty} |\mathbb{X}_{s_k,t_k}|^{p/2} \leq c(p) \limsup_{k\to\infty} c(s_k, t_k) \leq c(p) \lim_{k\to\infty} c(s, t_k) = c(p) c(s,t),$$

since $c(s,t)$ is right-continuous and superadditive. This ensures that $\|\mathbb{X}\|_{p\text{-var}} < \infty$. $\square$

Remark 3.3. All arguments in the proofs of Theorem 3.1 and of Lemma 3.2 extend immediately from $\mathbb{R}^d$ to (infinite-dimensional) Banach spaces. However, while the theory of continuous rough paths works for Banach spaces (cf. [Lyo98, LCL07]), the current results about càdlàg rough paths are developed in finite-dimensional settings (cf. [FS17, CF17]). For this reason we also focus only on $\mathbb{R}^d$-valued paths and stochastic processes.

3.1. **Semimartingales.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $(\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions. For a $\mathbb{R}^d$-valued semimartingale X we consider

(3.4)
$$\mathbb{X}_{s,t} := \int_s^t (X_{r-} - X_s) \otimes \mathrm{d}X_r = \int_0^t X_{r-} \otimes \mathrm{d}X_r - \int_0^s X_{r-} \otimes \mathrm{d}X_r - X_s \otimes X_{s,t}, \quad (s,t) \in \Delta_T,$$

where the integration $\int X_{r-} \otimes \mathrm{d}X$ is defined as an Itô integral. We refer to [Pro05] and [JS03] for more details on stochastic integration.

Proposition 3.4. *Let X be a $\mathbb{R}^d$-valued semimartingale. If $\mathbb{X}$ is defined as in (3.4) via Itô integration, then $(X, \mathbb{X}) \in \mathcal{W}^p([0,T]; \mathbb{R}^d)$ for every $p \in (2,3)$ $\mathbb{P}$-almost surely.*

Proof. First note that every semimartingale possesses càdlàg sample paths of finite p-variation for any $p > 2$ (see, e.g., [Pro05, Chapter II.1] and [Lép76]) and $\int X_{r-} \otimes \mathrm{d}X$ has càdlàg sample paths (see, e.g., [JS03, Theorem I.4.31]). Therefore, in order to deduce Proposition 3.4 from Theorem 3.1, it suffices to verify that the condition (3.2) holds $\mathbb{P}$-almost surely for $\int X_- \otimes \mathrm{d}X$ and its dyadic approximation $\int X^n \otimes \mathrm{d}X$ defined via (3.1).

1. Let us assume that $X = M$ is a square integrable martingale and M^n its approximation defined as in (3.1). By the Burkholder–Davis–Gundy inequality we observe

$$C(M,n) := \mathbb{E}\left[\left\| \int_0^{\cdot} M^n \otimes \mathrm{d}M - \int_0^{\cdot} M_- \otimes \mathrm{d}M \right\|_\infty^2 \right] \lesssim 2^{-2n}, \quad n \in \mathbb{N}, \tag{3.5}$$

where the constant depends on the quadratic variation of M. Combining Chebyshev's inequality with (3.5), we get

$$\mathbb{P}\left(\left\| \int_0^{\cdot} M^n \otimes \mathrm{d}M - \int_0^{\cdot} M_- \otimes \mathrm{d}M \right\|_\infty \geq 2^{-n(1-\epsilon)} \right) \leq 2^{2n(1-\epsilon)} C(M,n) \lesssim 2^{-2n\epsilon}.$$

Since the right-hand side is summable in n, the Borel–Cantelli lemma gives

$$\left\| \int_0^{\cdot} M^n \otimes \mathrm{d}M - \int_0^{\cdot} M_- \otimes \mathrm{d}M \right\|_\infty \lesssim_{\omega,\varepsilon} 2^{-n(1-\varepsilon)} \quad \mathbb{P}\text{-a.s.}$$

2. Let $X = M$ be a locally square integrable martingale. Let $(\sigma_k)_{k\in\mathbb{N}}$ be a localizing sequence of stopping times for M such that $\sigma_k \leq \sigma_{k+1}$, $\lim_{k\to\infty} \mathbb{P}(\sigma_k = T) = 1$, and for every k, the stopped process M^{σ_k} is a square integrable martingale. Thanks to 1 applied to every M^{σ_k}, for every k there exists a $\Omega_k \subset \Omega$ with $\mathbb{P}(\Omega_k) = 1$ such that for all $\omega \in \Omega_k$, it holds that

$$\left\| \int_0^{\cdot\wedge\sigma_k} M^n \otimes \mathrm{d}M - \int_0^{\cdot\wedge\sigma_k} M_- \otimes \mathrm{d}M \right\|_\infty \lesssim_{\omega,k} 2^{-n(1-\varepsilon)}$$

for any n. It follows immediately that (3.2) holds for any $\omega \in \bigcup_{k\in\mathbb{N}}(\{\sigma_k = T\} \cap \Omega_k)$ and it holds that $\mathbb{P}(\bigcup_{k\in\mathbb{N}}(\{\sigma_k = T\} \cap \Omega_k)) = 1$.

3. By [Pro05, Theorem III.29], every semimartingale X can be decomposed as $X = X_0 + M + A$, where $X_0 \in \mathbb{R}^d$, M is a locally square integrable martingale, and A has finite variation. By 2 we obtain that $\left\| \int_0^{\cdot} X^n \otimes \mathrm{d}M - \int_0^{\cdot} X_- \otimes \mathrm{d}M \right\|_\infty \lesssim_{\omega,\varepsilon} 2^{-n(1-\varepsilon)}$ $\mathbb{P}$-a.s.; on the other hand, since $\|X^n - X_-\|_\infty \leq 2^{-n}$, we also have $\left\| \int_0^{\cdot} X^n \otimes \mathrm{d}A - \int_0^{\cdot} X_- \otimes \mathrm{d}A \right\|_\infty \lesssim_\omega 2^{-n}$. ∎

Remark 3.5.[1] Very recently, Chevyrev and Friz proved that every semimartingale can be lifted via the "Marcus lift" to a weakly geometric càdlàg rough path based on a new enhanced Burkholder–Davis–Gundy inequality; see [CF17, Section 4]. Their result allows for deducing the existence of Itô rough paths due to [FS17, Proposition 16]. However, let us emphasize that our approach directly provides the existence of an Itô rough path only relying on classical Itô integration and fairly elementary analysis (cf. Theorem 3.1). Moreover, it is independent of the results from [CF17, FS17].

Two natural generalizations of semimartingales are semimartingales perturbed by paths of finite q-variation for $q \in [1,2)$ and Dirichlet processes. While these stochastic processes are beyond the scope of classical Itô integration, one can still construct corresponding random rough paths as a limit of approximating Riemann sums.

For $Y \in D^{q\text{-var}}([0,T];\mathbb{R}^d)$ with $q \in [1,2)$, the Young integral

$$\int_0^\cdot Y_{r-} \otimes \mathrm{d}Y_r := \lim_{n\to\infty} \sum_{[s,t]\in\mathcal{P}^n} Y_{s-} \otimes Y_{s\wedge\cdot,t\wedge\cdot}$$

exists along suitable sequences of the partition $(\mathcal{P}^n)_{n\in\mathbb{N}}$ and belongs to $D^{q\text{-var}}([0,T];\mathbb{R}^{d\times d})$; see for instance [You36] or [FS17, Proposition 14]. In this case the Young integral can also be obtained via the dyadic approximation (Y^n) as defined in (3.1). Indeed, using the Young–Loeve inequality (see, e.g., [FS17, Theorem 2]) and a standard interpolation argument, one gets

$$\begin{aligned}\Big\| \int_0^\cdot Y_r^n \otimes \mathrm{d}Y_r - \int_0^\cdot Y_{r-} \otimes \mathrm{d}Y_r \Big\|_\infty &\lesssim \|Y^n - Y_-\|_{q'\text{-var}} \|Y\|_{q'\text{-var}} \\ &\lesssim \|Y^n - Y_-\|_{q\text{-var}}^{q/q'} \|Y - Y^n\|_\infty^{1-q/q'} \|Y\|_{q'\text{-var}} \lesssim \|Y\|_{q\text{-var}}^{q/q'+1} 2^{-n(1-q/q')}\end{aligned}$$

for $1 \le q < q' < 2$, and thus $\lim_{n\to\infty} \| \int_0^\cdot Y_r^n \otimes \mathrm{d}Y_r - \int_0^\cdot Y_{r-} \otimes \mathrm{d}Y_r \|_\infty = 0$.

As a consequence of Proposition 3.4 and the previous discussion, it follows that semimartingales perturbed by paths of finite q-variation admit a natural rough path lift in the spirit of Itô integration. A similar result for the canonical Marcus lift was presented in [CF17, Section 5.1].

Corollary 3.6. *Let $Z = X + Y$ be semimartingale perturbed by paths of finite q-variation with $q \in [1,2)$, i.e., X is a semimartingale and Y is a stochastic process with sample paths of finite q-variation for $q \in [1,2)$. Then, there exists a càdlàg rough path $(Z,\mathbb{Z}) \in \mathcal{W}^p([0,T];\mathbb{R}^d)$ for every $p \in (2,3)$ $\mathbb{P}$-almost surely, where $\mathbb{Z}$ can be constructed as a limit of approximating left-point Riemann sums.*

In the case $Z = X + Y$ for a stochastic process Y with continuous sample paths of finite q-variation with $q \in [1,2)$, the process Z belongs to the class of so-called càdlàg Dirichlet processes; cf. [Str88] and [CMS03]. Furthermore, let us remark that if $Y \in C^{0,2\text{-var}}([0,T];\mathbb{R}^d)$ admits a rough path lift, then it has to coincide with the Young integral; cf. [FH14, Exercise 2.12]. Here $C^{0,2\text{-var}}([0,T];\mathbb{R}^d)$ denotes the closure of smooth paths on $[0,T]$ w.r.t. $|\cdot|_{2\text{-var}}$.

[1] After completion of the present work, it was pointed out in [FZ17] that the Itô lift of semimartingales can also be constructed using an enhanced version of the Burkholder–Davis–Gundy inequality.

3.2. **Gaussian processes.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $(\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions, and let $X = (X^1, \ldots, X^d)\colon \Omega \times [0,T] \to \mathbb{R}^d$ be a d-dimensional Gaussian process. A natural candidate for the corresponding $\mathbb{X} = (\mathbb{X}^{i,j})_{i,j=1,\ldots,d}$ is

(3.6)
$$\mathbb{X}^{i,j}_{s,t} := \int_0^t X^i_{r-}\,\mathrm{d}X^j_r - \int_0^s X^i_{r-}\,\mathrm{d}X^j_r - X^i_s X^j_{s,t} \quad \text{and} \quad \mathbb{X}^{i,i}_{s,t} := \frac{1}{2}(X^i_{s,t})^2, \quad (s,t) \in \Delta_T,$$

where $i \neq j$ and where the integral is given as an L^2-limit of left-point Riemann–Stieltjes approximations. For more details on Gaussian processes in the context of rough path theory, we refer to [FV10b, Chapter 15].

Proposition 3.7. *Let $(X_t)_{t\in[0,T]}$ be a d-dimensional separable centered Gaussian process with independent components and càdlàg sample paths. If for every $q > 1$*

$$\sup_{\mathcal{P},\mathcal{P}'} \sum_{[s,t]\in\mathcal{P},[u,v]\in\mathcal{P}'} |\mathbb{E}[X_{s,t} \otimes X_{u,v}]|^q < \infty, \tag{3.7}$$

then $(X, \mathbb{X}) \in \mathcal{W}^p([0,T];\mathbb{R}^d)$ for every $p \in (2,3)$ $\mathbb{P}$-almost surely, where $\mathbb{X}$ is defined as in (3.6) *and $\mathbb{X}^{i,j}$ exists in the sense of an L^2-limit of Riemann–Stieltjes approximations for $i \neq j$.*

Proof. Proceeding as in [FS17, Section 10.3], the sample paths of X have finite p-variation for any $p > 2$ due to (3.7) and there exists a centered Gaussian process $\widetilde{X}$ with continuous sample paths such that $\widetilde{X} \circ F = X$, where $F^i(t) := \sup_{\mathcal{P}} \sum_{[u,v]\in\mathcal{P}} |X^i_u - X^i_v|^{2q}_{L^2}$ for every $i = 1, \ldots, d$.

By [FV10a, Theorem 35 (iv)] the integral $\int_0^\cdot \widetilde{X}^i_r\,\mathrm{d}\widetilde{X}^j_r$ exists as the L^2-limit of Riemann–Stieltjes approximation and has continuous sample paths. Furthermore, using the Young–Towghi maximal inequality (see [FV11, Theorem 3]) it can be verified that

(3.8)
$$\int_0^\cdot \widetilde{X}^i_r\,\mathrm{d}\widetilde{X}^j_r \circ F(t) = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} X^i_u (X^j_{v\wedge\cdot} - X^j_{u\wedge\cdot}) = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} X^i_{u-} (X^j_{v\wedge\cdot} - X^j_{u\wedge\cdot}),$$

for $i,j = 1, \ldots, d$ with $i \neq j$, where the limits are taken in L^2 and in Refinement Riemann–Stieltjes sense (cf. [FS17, Definition 1]). We denote by $\int_0^\cdot X^i_{r-}\,\mathrm{d}X^j_r$ the integral from (3.8), which has càdlàg sample paths.

It remains to check condition (3.2) for $\int_0^\cdot X^i_{r-}\,\mathrm{d}X^j_r$, which then implies the proposition by Theorem 3.1. With an abuse of notation, we now write X for X^i, and $\widetilde{X}$ for X^j. Let X^n be given as in (3.1) such that $\|X^n - X_-\|_\infty \leq 2^{-n}$. We define for $Y^n := X^n - X_-$, and for $s, u \in [0,T]$ we set $R^n\binom{s}{u} := \mathbb{E}[Y^n_{0,s} Y^n_{0,u}]$ and $\widetilde{R}\binom{s}{u} := \mathbb{E}[\widetilde{X}_{0,s}\widetilde{X}_{0,u}]$. Thanks to (3.7), $\widetilde{R}$ has finite q-variation for any $q > 1$. We claim that R^n has finite p-variation for any $p > 2$. Indeed, for every rectangle $[s,t] \times [u,v] \subset [0,T]^2$, we have $|\mathbb{E}[Y^n_{s,t} Y^n_{u,v}]|^p \leq \|Y^n_{s,t}\|^p_{L^2} \|Y^n_{u,v}\|^p_{L^2}$. Using [BOW16, Proposition 1.7] and the definition of X^n we obtain that $\mathbb{E}[\|Y^n\|^p_{p\text{-var}}] \lesssim \mathbb{E}[\|X\|^p_{p\text{-var}}] < \infty$. Then, by Jensen's inequality we deduce that

$$\mathbb{E}[\|Y^n\|^p_{p\text{-var}}] \geq \sup_{\mathcal{P}} \sum_{[s,t]\in\mathcal{P}} \mathbb{E}[|Y^n_{s,t}|^p] \geq \sup_{\mathcal{P}} \sum_{[s,t]\in\mathcal{P}} (\mathbb{E}[|Y^n_{s,t}|^2])^{p/2} = \sup_{\mathcal{P}} \sum_{[s,t]\in\mathcal{P}} \|Y^n_{s,t}\|^p_{L^2},$$

which means that Y^n has finite p-variation w.r.t. the L^2-distance. Let $|||Y^n|||_{p\text{-var}}$ denote the p-variation norm of Y^n in the L^2-distance; then $c_n(s,t) := |||Y^n|||^p_{p\text{-var};[s,t]}$ is superadditive and $c_n(0,T) \lesssim \mathbb{E}[\|X\|^p_{p\text{-var}}]$ for all n. Hence, for any partitions $\mathcal{P}$, $\mathcal{P}'$ of $[0,T]$ and for

$$R^n\binom{s,t}{u,v} := R^n\binom{s}{u} + R^n\binom{t}{v} - R^n\binom{s}{v} - R^n\binom{t}{u}, \quad u,v,s,t \in [0,T],$$

it holds that

$$\sum_{[s,t]\in\mathcal{P},[u,v]\in\mathcal{P}'} R^n\binom{s,t}{u,v}^p \leq \sum_{[s,t]\in\mathcal{P},[u,v]\in\mathcal{P}'} c_n(s,t)c_n(u,v) \leq c_n(0,T)^2 \lesssim \mathbb{E}[\|X\|^p_{p\text{-var}}]^2.$$

Now, for any $p > 2$, we can choose any $q > 1$ close enough to 1 such that $1/p + 1/q > 1$. Since Y^n and $\widetilde{X}$ are independent, applying the Young–Towghi maximal inequality to the discrete integrals $\mathbb{E}[(\sum_{t_i\in\mathcal{P}} Y^n_{t_i}\widetilde{X}_{t_i,t_{i+1}})^2]$ and then sending $|\mathcal{P}|$ to zero, by Fatou's lemma we obtain that

$$\mathbb{E}\bigg[\bigg(\int_0^t Y^n_{0,r}\,\mathrm{d}\widetilde{X}_r\bigg)^2\bigg] \lesssim V_p(R^n)V_q(\widetilde{R}), \quad t \in [0,T],$$

where V_p denotes p-variation on $[0,T]^2$ in the sense of [FV11, Definition 1], given by

$$V_p(R) := \sup_{\mathcal{P},\mathcal{P}'}\bigg(\sum_{[s,t]\in\mathcal{P},[u,v]\in\mathcal{P}'} R\binom{s,t}{u,v}^p\bigg)^{1/p}$$

for a function $R\colon [0,T]^2 \to \mathbb{R}$. By an interpolation argument we have for $p' > p$,

$$V_{p'}(R^n) \leq V_p(R^n)^{p/p'}\bigg(\sup_{s\neq t,u\neq v}\bigg|R^n\binom{s,t}{u,v}\bigg|\bigg)^{1-p/p'}.$$

Hence, noting that $|R^n\binom{s,t}{u,v}| = |\mathbb{E}[Y^n_{s,t}Y^n_{u,v}]| \lesssim 2^{-2n}$ due to $\|Y^n\|_\infty \leq 2^{-n}$, the above inequality applied for p' and q with $1/p' + 1/q > 1$ gives

$$\mathbb{E}\bigg[\bigg(\int_0^t Y^n_{0,r}\mathrm{d}\widetilde{X}_r\bigg)^2\bigg] \lesssim V_p(R^n)^{p/p'}V_q(\widetilde{R})2^{-2n(1-p/p')}.$$

In particular, for a given $p > 2$ and $\varepsilon > 0$, we choose $p' = p/\varepsilon$, and for a corresponding parameter q close enough to 1 such that $1/p' + 1/q > 1$, we get

$$\mathbb{E}\bigg[\bigg(\int_0^t Y^n_{0,r}\mathrm{d}\widetilde{X}_r\bigg)^2\bigg] \lesssim_\varepsilon 2^{-2n(1-\varepsilon)}.$$

Then by Chebyshev's inequality, for each n and each $t \in [0,T]$ we have (note that $Y^n_0 = 0$)

$$\mathbb{P}\bigg(\bigg|\int_0^t Y^n_r\mathrm{d}\widetilde{X}_r\bigg| \geq 2^{-n(1-2\varepsilon)}\bigg) \leq 2^{2n(1-2\varepsilon)}\mathbb{E}\bigg[\bigg(\int_0^t Y^n_{0,r}\mathrm{d}\widetilde{X}_r\bigg)^2\bigg] \lesssim_\varepsilon 2^{-2n\varepsilon}.$$

Since the right-hand side of the above inequality is summable over $n \in \mathbb{N}$, by the Borel–Cantelli lemma, we conclude that for every $t \in [0,T]$ there exists a $\Omega_t \subset \Omega$ with $\mathbb{P}[\Omega_t] = 1$ such that for every $\omega \in \Omega_t$, when n is large enough (n may depend on ω and t),

$$\bigg|\int_0^t (X^n - X_{r-})\mathrm{d}\widetilde{X}_r\bigg| \leq 2^{-n(1-\varepsilon)}$$

holds. Let D_T be any countable dense subset in $[0,T]$ containing T and $\widetilde{\Omega} := \bigcap_{t\in D_T} \Omega_t$. Therefore, condition (3.2) is satisfied for every $\omega \in \widetilde{\Omega}$, which finishes the proof. $\square$

Remark 3.8. The Gaussian rough path as constructed in Proposition 3.7 is in fact a weakly geometric càdlàg rough path which coincides with the one given in [FS17, Theorem 60]. However, while the proof of [FS17, Theorem 60] is entirely based on time-change arguments and on corresponding well-known results for continuous Gaussian rough paths, the above proof gives a direct verification of the required rough path regularity via Theorem 3.1.

3.3. **Typical price paths.** In recent years, initiated by Vovk, a model-free, hedging-based approach to mathematical finance emerged that uses arbitrage considerations to investigate which sample path properties are satisfied by "typical price paths"; see for instance [Vov08, TKT09, PP16]. In particular, Vovk's framework allows for setting up a model-free Itô integration; see [PP16, ŁPP18, Vov16]. Based on this integration, we show in the present subsection that "typical price paths" can be lifted to càdlàg rough paths.

Let $\Omega_+ := D([0,T];\mathbb{R}^d_+)$ be the space of all nonnegative càdlàg functions $\omega\colon [0,T] \to \mathbb{R}^d_+$. The space Ω_+ can be interpreted as all possible price trajectories on a financial market. For each $t \in [0,T]$, $\mathcal{F}^\circ_t$ is defined to be the smallest σ-algebra on Ω_+ that makes all functions $\omega \mapsto \omega(s)$, $s \in [0,t]$, measurable and $\mathcal{F}_t$ is defined to be the universal completion of $\mathcal{F}^\circ_t$. Stopping times $\tau\colon \Omega_+ \to [0,T] \cup \{\infty\}$ w.r.t. the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and the corresponding σ-algebras $\mathcal{F}_\tau$ are defined as usual. The coordinate process on Ω_+ is denoted by $S = (S^1,\dots,S^d)$, i.e., $S_t(\omega) := \omega(t)$ and $S^i_t(\omega) := \omega^i(t)$ for $\omega = (\omega^1,\dots,\omega^d) \in \Omega_+$, $t \in [0,T]$ and $i = 1,\dots,d$.

A process $H\colon \Omega_+ \times [0,T] \to \mathbb{R}^d$ is a *simple* (*trading*) *strategy* if there exist a sequence of stopping times $0 = \sigma_0 < \sigma_1 < \sigma_2 < \cdots$ such that for every $\omega \in \Omega_+$ there exist an $N(\omega) \in \mathbb{N}$, such that $\sigma_n(\omega) = \sigma_{n+1}(\omega)$ for all $n \ge N(\omega)$, and a sequence of $\mathcal{F}_{\sigma_n}$-measurable bounded functions $h_n\colon \Omega_+ \to \mathbb{R}^d$, such that $H_t(\omega) = \sum_{n=0}^\infty h_n(\omega)\mathbf{1}_{(\sigma_n(\omega),\sigma_{n+1}(\omega)]}(t)$ for $t \in [0,T]$. Therefore, for a simple strategy H the corresponding integral process

$$(H\cdot S)_t(\omega) := \sum_{n=0}^{\infty} h_n(\omega) S_{\sigma_n\wedge t,\sigma_{n+1}\wedge t}(\omega)$$

is well defined for all $(t,\omega) \in [0,T]\times\Omega_+$. For $\lambda > 0$ we write $\mathcal{H}_\lambda$ for the set of all simple strategies H such that $(H\cdot S)_t(\omega) \ge -\lambda$ for all $(t,\omega) \in [0,T]\times\Omega_+$.

Definition 3.9. *Vovk's outer measure* $\overline{P}$ of a set $A \subset \Omega_+$ is defined as the minimal superhedging price for $\mathbf{1}_A$, that is,

$$\overline{P}(A) := \inf\Big\{\lambda > 0 \,:\, \exists (H^n)_{n\in\mathbb{N}} \subset \mathcal{H}_\lambda \\ \text{s.t. } \forall \omega\in\Omega_+ \,:\, \liminf_{n\to\infty}(\lambda + (H^n\cdot S)_T(\omega)) \ge \mathbf{1}_A(\omega) \Big\}.$$

A set $\mathcal{A} \subset \Omega_+$ is called a *null set* if it has outer measure zero. A property (P) holds for *typical price paths* if the set $\mathcal{A}$ where (P) is violated is a null set.

Note that $\overline{P}$ is indeed an outer measure, which dominates all local martingale measures on the space Ω_+; see [ŁPP18, Lemma 2.3 and Proposition 2.5]. For more details about Vovk's outer measure we refer for example to [ŁPP18, Section 2].

Proposition 3.10. *Typical price paths belonging to Ω_+ can be enhanced to càdlàg rough paths $(S, A) \in \mathcal{W}^p([0,T];\mathbb{R}^d)$ for every $p > 2$ where*

$$A_{s,t} := \int_0^t S_{r-} \otimes \mathrm{d}S_r - \int_0^s S_{r-} \otimes \mathrm{d}S_r - S_s \otimes S_{s,t}, \quad (s,t) \in \Delta_T,$$

and $\int S_- \otimes \mathrm{d}S$ denotes the model-free Itô integral from [ŁPP18, Theorem 4.2].

Proof. It follows from [Vov11, Theorem 1] that typical price paths belonging to Ω_+ are of finite p-variation for every $p > 2$. Hence, it remains to check condition (3.2) of Theorem 3.1 to prove the assertion.

Let S^n be the dyadic approximation of S as defined in (3.1) for $n \in \mathbb{N}$ and let us recall that [ŁPP18, Corollary 4.9] extends to the estimate

$$\overline{P}\bigg(\bigg\{\bigg\|\int_0^\cdot (S^n - S_-)\otimes \mathrm{d}S\bigg\|_\infty \geq a_n\bigg\} \cap \{|[S]_T| \leq b\} \cap \{\|S\|_\infty \leq b\}\bigg) \lesssim 6(\sqrt{b}+2+2b)\frac{c_n}{a_n},$$

where $c_n := \|S^n - S\|_\infty \lesssim 2^{-n}$, $|[S]_T| := \big(\sum_{i,j=1}^d [S^i, S^j]_T^2\big)^{1/2}$, and $[S^i, S^j]$ denotes the quadratic co-variation as defined in [ŁPP18, Corollary 3.11]. Due to the countable subadditivity of $\overline{P}$, it is enough to consider a fixed $b > 0$. Setting $a_n := 2^{-(1-\varepsilon)n}$ for $\varepsilon \in (0,1)$ and applying the Borel–Cantelli lemma for $\overline{P}$ (see [ŁPP18, Lemma A.1]), we get $\overline{P}(B_b) = 0$ with

$$B_b := \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} A_{b,n}$$

and

$$A_{b,n} := \bigg\{\bigg\|\int_0^\cdot (S^n - S_-) \otimes \mathrm{d}S\bigg\|_\infty \geq a_n\bigg\} \cap \{|[S]_T| \leq b\} \cap \{\|S\|_\infty \leq b\}.$$

In particular, for typical price paths (belonging to Ω_+) we have

$$\bigg\|\int_0^\cdot (S^n - S_-) \otimes \mathrm{d}S\bigg\|_\infty \lesssim_\omega 2^{-(1-\varepsilon)n}$$

for all $n \in \mathbb{N}$, and thus typical price paths satisfy condition (3.2). □

Let us briefly comment on various aspects of Proposition 3.10.

Remark 3.11.

(1) Proposition 3.10 implies the (robust) existence of Itô càdlàg rough paths in the sense that the set of all nonnegative càdlàg paths which do not possess an Itô rough path has measure zero with respect to all local martingale measures on Ω_+. This justifies taking the existence of Itô rough paths above price paths as an underlying assumption in model-free financial mathematics.
(2) The nonexistence of Itô càdlàg rough paths above nonnegative price paths leads to a pathwise arbitrage of the first kind; cf., [ŁPP18, Proposition 2.6].
(3) In the case of continuous (price) paths the assertion of Proposition 3.10 was obtained in [PP16, Theorem 4.12].
(4) Proposition 3.10 can be generalized in a straightforward manner from Ω_+ to the more general sample spaces considered in [ŁPP18].

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