PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 11, November 2018, Pages 4837–4842
https://doi.org/10.1090/proc/14155
Article electronically published on July 23, 2018

A NOTE ON LIOUVILLE TYPE EQUATIONS ON GRAPHS

HUABIN GE, BOBO HUA, AND WENFENG JIANG

(Communicated by Guofang Wei)

ABSTRACT. In this note, we study the Liouville equation $\Delta u = -e^u$ on a graph G satisfying a certain isoperimetric inequality. Following the idea of W. Ding, we prove that there exists a uniform lower bound for the energy, $\sum_G e^u$, of any solution u to the equation. In particular, for the 2-dimensional lattice graph $\mathbb{Z}^2$, the lower bound is given by 4.

1. INTRODUCTION

The Liouville equation

$$\Delta u + e^u = 0 \tag{1.1}$$

on 2-dimensional manifolds has been extensively studied in the literature. From the point of view of the theory of partial differential equations, it is critical, i.e., on the borderline of Sobolev embedding theorems in the 2-dimensional case, which is closely related to the so-called Moser-Trudinger inequalities; see e.g., [1, 12, 14] for references.

Let u be a solution to the Liouville equation on the plane with finite energy, i.e.,

$$\begin{cases} \Delta u + e^u = 0, \\ \int_{\mathbb{R}^2} e^u < \infty. \end{cases} \tag{1.2}$$

An interesting argument initiated by Weiyue Ding, see [2], shows that

$$\int_{\mathbb{R}^2} e^u \geq 8\pi.$$

The key ingredient of the proof is the following isoperimetric inequality: for any bounded domain Ω of finite perimeter in $\mathbb{R}^2$,

$$\text{Length}(\partial\Omega)^2 \geq 4\pi \cdot \text{Area}(\Omega), \tag{1.3}$$

where Length$(\partial\Omega)$ (Area(Ω), resp.) denotes the length of the boundary of Ω (the area of Ω, resp.). The estimate is sharp since one can construct a family of explicit solutions,

$$F_{x_0,\lambda}(x) := \ln\left[\frac{32\lambda^2}{(4 + \lambda^2|x - x_0|^2)^2}\right], \quad \lambda > 0, x_0 \in \mathbb{R}^2, \tag{1.4}$$

Received by the editors November 10, 2017, and, in revised form, March 7, 2018.

2010 *Mathematics Subject Classification.* Primary 35R02; Secondary 58J05.

The research was supported by the National Natural Science Foundation of China (NSFC) under grants No. 11501027 (the first author) and No. 11401106 (the second author).

whose energy attains the above lower bound. Based on a delicate argument using moving plane methods, Chen and Li [2] further proved that all solutions to (1.2) are exactly given by (1.4).

As is well known, one of the difficulties for the analysis on graphs lies in the lack of chain rules for discrete Laplace operators. While linear equations have been studied extensively on graphs, people recently began to consider nonlinear problems on graphs, such as semilinear equations. For semilinear equations with the nonlinearity of power type, one refers to, e.g., [6, 7, 9, 10]. A class of semilinear equations with the exponential nonlinearity, so-called Kazdan-Warner equations, has been studied by [3–5, 8] on graphs. The exponential nonlinearity usually causes additional difficulties for the analysis in the discrete setting. In this paper, we study the Liouville type equations on graphs, analogous to (1.1), which are special cases of Kazdan-Warner equations. Following W. Ding's idea, we prove a uniform lower bound of the energy for the solutions to the Liouville equations on graphs satisfying isoperimetric inequalities analogous to (1.3); see Theorem 2.1. As a corollary, for the 2-dimensional lattice graph which is a discrete analog of $\mathbb{R}^2$, we obtain an explicit lower bound for the energy of solutions to the Liouville equation; see Corollary 2.3. This could be regarded as a preliminary step to understanding the Liouville type equations on infinite graphs.

The paper is organized as follows: in the next section, we introduce the basic setting and state our main results. Section 3 is devoted to the proof of Theorem 2.1.

2. Basic setting and main results

Let (V, E) be a simple, undirected, and locally finite graph, where V denotes the set of vertices and E denotes the set of edges. Two vertices x and y are called neighbors, denoted by $x \sim y$, if there is an edge connecting them, i.e., $\{x, y\} \in E$. We assign weights on vertices and edges as follows:

$$\mu : V \to (0, \infty), \quad V \ni x \mapsto \mu_x,$$

and

$$w : E \to (0, \infty), \quad E \ni \{x, y\} \mapsto w_{xy} = w_{yx},$$

and call the quadruple $G = (V, E, \mu, w)$ a *weighted graph*. For discrete measure spaces (V, μ) and (E, w), we write $\mu(A) := \sum_{x \in A} \mu_x$ and $w(B) := \sum_{e \in B} w_e$ for any subsets $A \subset V, B \subset E$. For simplicity, for a function u on V we write

$$\int_V u = \sum_{x \in V} u(x)\mu_x,$$

whenever it makes sense.

The Laplacian on $G = (V, E, \mu, w)$ is defined as, for any function u on V and $x \in V$,

$$\Delta u(x) = \frac{1}{\mu_x} \sum_{y \in V : y \sim x} w_{xy}(u(y) - u(x)).$$

For any vertex x, its weighted degree is given by

$$\mathrm{Deg}(x) := \frac{\sum_{y : y \sim x} w_{xy}}{\mu_x}.$$

The Laplacian is a bounded operator on $\ell^2(V,\mu)$, i.e., the Hilbert space of ℓ^2 summable functions on V w.r.t. the measure μ, if and only if

$$(BLap) \qquad \mathrm{Deg}(G) := \sup_{x\in V} \mathrm{Deg}(x) < \infty.$$

In this paper, we always assume $(BLap)$ holds.

For any finite subset Ω in V, we denote by

$$\partial\Omega := \{\{x,y\} \in E : x \in \Omega, y \in V \setminus \Omega, \text{ or vice versa}\}$$

the (edge) boundary of Ω. We say that a weighted graph $G = (V, E, \mu, w)$ satisfies the 2-dimensional isoperimetric inequality, denoted by IS_2, if

$$(\mathrm{IS}_2) \qquad C_{IS} := \inf \frac{(w(\partial\Omega))^2}{\mu(\Omega)} > 0,$$

where the infimum is taken over all finite $\Omega \subset V$; see [13].

In this note, we study the discrete Liouville equation

$$\Delta u + e^u = 0 \tag{2.1}$$

on a weighted graph G. Following W. Ding, see Lemma 1.1 in [2], we obtain our main result, a discrete analog of the energy estimate for the solutions to the Liouville equation under the assumption of the isoperimetric inequality.

Theorem 2.1. *Let G be a weighted graph satisfying $(BLap)$ and $\inf_{x\in V} \mu_x > 0$. Suppose that (IS_2) holds, then for any solutions u of (2.1),*

$$\int_V e^u \geq \frac{C_{IS}}{\mathrm{Deg}(G)}.$$

Remark 2.2. One may generalize the result to the following equation:

$$\Delta u + F(u) = 0, \tag{2.2}$$

for some nonnegative function F on $\mathbb{R}$ satisfying $F' \geq 0$ and $F'' \geq 0$.

We denote by $\mathbb{Z}^2$ the standard lattice graph with the set of vertices $\{(x,y) \in \mathbb{R}^2 : x, y \in \mathbb{Z}\}$, the set of edges

$$\{\{(x_1,y_1),(x_2,y_2)\} : |x_1 - x_2| + |y_1 - y_2| = 1\},$$

and weights $\mu \equiv 4$ and $w \equiv 1$. It is known that it satisfies (IS_2) with $C_{IS} = 4$; see Theorem 6.30 in [11]. Then by the above theorem we have the following corollary.

Corollary 2.3. *For any solution u of (2.1) on the lattice $\mathbb{Z}^2$, we have*

$$\int_{\mathbb{Z}^2} e^u \geq 4.$$

This suggests the following interesting problems for further investigation.

Problem 1. What is the sharp constant in Corollary 2.3, i.e.,

$$C := \inf \int_{\mathbb{Z}^2} e^u,$$

where the infimum is taken over all solutions to (2.1) on $\mathbb{Z}^2$?

Problem 2. Is there any solution u to (2.1) on $\mathbb{Z}^2$ with finite energy, i.e., $\int_{\mathbb{Z}^2} e^u < \infty$?

3. Proof of Theorem 2.1

For any $\sigma \in \mathbb{R}$, set

$$\Omega_\sigma = \{x \in V | u(x) \geq \sigma\}.$$

It is no restriction to assume that Ω_σ is finite for any σ; otherwise by $\inf_{x\in V} \mu_x > 0$,

$$\int_V e^u = +\infty.$$

By (2.1),

$$\begin{aligned}\int_{\Omega_\sigma} e^u = \int_{\Omega_\sigma} -\Delta u &= \sum_{x\in\Omega_\sigma} \sum_{y\in V: y\sim x} w_{xy}(u(x) - u(y)) \\ &= \sum_{x\in\Omega_\sigma} \sum_{y\in\Omega_\sigma: y\sim x} w_{xy}(u(x) - u(y)) + \sum_{x\in\Omega_\sigma} \sum_{y\notin\Omega_\sigma: y\sim x} w_{xy}(u(x) - u(y)).\end{aligned}$$

We denote the first summand by A. Then

$$\begin{aligned}A &= \sum_{x,y\in\Omega_\sigma: x\sim y} w_{xy}(u(x) - u(y)) \\ &= - \sum_{x,y\in\Omega_\sigma: x\sim y} w_{xy}(u(y) - u(x)) \\ &= - \sum_{y\in\Omega_\sigma} \sum_{x\in\Omega_\sigma: x\sim y} w_{xy}(u(y) - u(x)) = -A.\end{aligned}$$

This yields that $A = 0$, and we get

$$\int_{\Omega_\sigma} e^u = \sum_{e=\{x,y\}\in E, u(x)<\sigma\leq u(y)} w_{xy}(u(y) - u(x)). \tag{3.1}$$

For any $\sigma \in \mathbb{R}$, let

$$G(\sigma) = \sum_{e=\{x,y\}\in E, u(x)<\sigma\leq u(y)} w_{xy}/\big(u(y) - u(x)\big).$$

For any subset $K \subset \mathbb{R}$, we denote by $\mathbb{1}_K$ the characteristic function on K; i.e., $\mathbb{1}_K(\sigma) = 1$ if $\sigma \in K$, and $\mathbb{1}_K(\sigma) = 0$ otherwise. We have

$$\begin{aligned}\int_{-\infty}^{+\infty} e^\sigma G(\sigma) d\sigma &= \int_{-\infty}^{+\infty} e^\sigma \sum_{e=\{x,y\}\in E, u(y)>u(x)} w_{xy}\big(u(y) - u(x)\big)^{-1} \mathbb{1}_{(u(x),u(y)]}(\sigma) d\sigma \\ &= \sum_{e=\{x,y\}\in E, u(y)>u(x)} w_{xy}\big(u(y) - u(x)\big)^{-1} \int_{-\infty}^{+\infty} e^\sigma \mathbb{1}_{(u(x),u(y)]}(\sigma) d\sigma \\ &= \sum_{e=\{x,y\}\in E, u(y)>u(x)} w_{xy} \frac{e^{u(y)} - e^{u(x)}}{u(y) - u(x)} \\ &\leq \sum_{e=\{x,y\}\in E, u(y)>u(x)} w_{xy} e^{u(y)} \\ &\leq \operatorname{Deg}(G) \sum_{y\in V} e^{u(y)} \mu_y,\end{aligned}$$

where we have used the elementary inequality $\frac{e^b-e^a}{b-a} \le e^b$ for any $a < b$ and the definition of $\mathrm{Deg}(G)$ in $(BLap)$. Hence by the above inequality,

$$\int_{-\infty}^{+\infty} e^\sigma G(\sigma) \int_{\Omega_\sigma} e^u d\sigma \le \int_V e^u \int_{-\infty}^{+\infty} e^\sigma G(\sigma) d\sigma \le \mathrm{Deg}(G) \left(\int_V e^u \right)^2. \tag{3.2}$$

On the other hand, by (3.1) and the Cauchy-Schwarz inequality,

$$\begin{aligned} & G(\sigma) \int_{\Omega_\sigma} e^u \\ = & \left(\sum_{e=\{x,y\}\in E, u(x)<\sigma\le u(y)} \frac{w_{xy}}{u(y)-u(x)} \right) \left(\sum_{e=\{x,y\}\in E, u(x)<\sigma\le u(y)} w_{xy}(u(y)-u(x)) \right) \\ \ge & \left(\sum_{e=\{x,y\}\in E, u(x)<\sigma\le u(y)} w_{xy} \right)^2 = (w(\partial\Omega_\sigma))^2 \\ \ge & C_{IS} \cdot \mu(\Omega_\sigma), \end{aligned}$$

where the last inequality follows from the isoperimetric inequality. This yields that

$$\int_{-\infty}^{+\infty} e^\sigma G(\sigma) \int_{\Omega_\sigma} e^u \ge C_{IS} \int_{-\infty}^{+\infty} \mu(\Omega_\sigma) e^\sigma = C_{IS} \int_V e^u.$$

We prove the theorem by combining the above inequality with (3.2).

Acknowledgments

We thank the anonymous referee for his/her valuable comments and suggestions.

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Department of Mathematics, Beijing Jiaotong University, Beijing 100044, People's Republic of China

Current address: School of Mathematics, Renmin University of China, Beijing, 100872, People's Republic of China

Email address: hbge@bjtu.edu.cn

School of Mathematical Sciences, LMNS, Fudan University, Shanghai 200433, People's Republic of China

Email address: bobohua@fudan.edu.cn

School of Mathematics (Zhuhai), Sun Yat-Sen University, Zhuhai, People's Republic of China

Email address: wen_feng1912@outlook.com