

THE MEAN CURVATURE FLOW BY PARALLEL HYPERSURFACES

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ABSTRACT. It is shown that a hypersurface of a space form is the initial data for a solution to the mean curvature flow by parallel hypersurfaces if and only if it is isoparametric. By solving an ordinary differential equation, explicit solutions are given for all isoparametric hypersurfaces of space forms. In particular, for such hypersurfaces of the sphere, the exact collapsing time into a focal submanifold is given in terms of its dimension, the principal curvatures, and their multiplicities.

1. INTRODUCTION

The Mean Curvature Flow (MCF) is a gradient-type flow for the volume functional. Under the MCF, a closed hypersurface in \mathbb{R}^{n+1} locally evolves in the direction where the volume element decreases the fastest and it eventually becomes extinct. Along the flow singularities may occur, and one is interested in studying such singularities. There is extensive literature on the subject starting with the early work in material science dating from the 1920s. We refer the reader to the excellent survey by Colding, Minicozzi, and Pedersen [7] and the references within.

In recent years, self-similar solutions to the MCF have been studied. These are solutions given by the composition of isometries and homotheties. In the Euclidean space, the simplest examples of self-similar solutions are the spheres and the cylinders, which are self-contracting hypersurfaces. Other examples of self-contractile hypersurfaces in the Euclidean space can be found in [2, 10], and translation hypersurfaces are found in [6, 8, 13, 14]. There are very few results on the mean curvature flow in non-Euclidean spaces. In [9], Hungerbühler and Smoczyk considered a particular case of self-similar solutions evolving by the MCF by a group of isometries of the ambient space, which are known as solitons, and presented several examples of these hypersurfaces on Riemannian manifolds. In [11], Liu and Terng studied the MCF on isoparametric submanifolds with higher codimension of the Euclidean space and of the sphere, where they proved that the flow preserves the condition of being isoparametric and develops singularities in finite time, converging to a smooth submanifold of lower dimension.

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In this paper, we prove that any immersed hypersurface M^n of a space form evolves through the MCF by parallel hypersurfaces if and only if M is an isoparametric hypersurface. The MCF of such a hypersurface is obtained by solving an ordinary differential equation. We solve this equation for all isoparametric hypersurfaces of space forms. In particular, we provide explicit solutions to the MCF of isoparametric hypersurfaces of the sphere and of the hyperbolic space, including the exact collapsing time and the converging submanifold in terms of its dimension, the principal curvatures, and their multiplicities. Our results are stated in the following section and the proofs are given in Section 3.

2. MEAN CURVATURE FLOW OF ISOPARAMETRIC HYPERSURFACES.
MAIN RESULTS

In what follows, $\mathbb{M}^{n+1}(\bar{\kappa})$ will be a space form of constant sectional curvature $\bar{\kappa} \in \{-1, 0, 1\}$, i.e., \mathbb{R}^{n+1} if $\bar{\kappa} = 0$, $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ if $\bar{\kappa} = 1$, and $\mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ if $\bar{\kappa} = -1$, where \mathbb{L}^{n+2} is the Lorentzian space. We consider $F : M^n \rightarrow \mathbb{M}^{n+1}(\bar{\kappa})$ a hypersurface immersed in the space form $\mathbb{M}^{n+1}(\bar{\kappa})$, with the induced metric $g(v, w) = \langle dF(v), dF(w) \rangle$, for all vector fields v, w tangent to M . If $F(M)$ is oriented and N is a unit normal vector field, the *second fundamental form* of $F(M)$ is given by $h(v, w) = -\langle dN(v), dF(w) \rangle$. Let e_1, \dots, e_n be orthonormal vector fields which are principal directions, and let $\kappa_1, \dots, \kappa_n$, be the *principal curvatures* of $F(M)$, i.e., $g(e_i, e_j) = \delta_{ij}$ and $h(e_i, e_j) = \kappa_i \delta_{ij}$, for $1 \leq i, j \leq n$. We will denote the *mean curvature* by $H = \sum_{i=1}^n \kappa_i$. When the principal curvatures κ_i of $F(M)$ do not depend on x , for all $i = 1, \dots, n$, we say that $F(M)$ is an *isoparametric* hypersurface. From now on, we consider connected hypersurfaces.

Let $F : M^n \rightarrow \mathbb{M}^{n+1}(\bar{\kappa})$ be an oriented hypersurface with a unit normal vector field N . A one-parameter family of hypersurfaces $\hat{F} : M^n \times I \rightarrow \mathbb{M}^{n+1}(\bar{\kappa})$, $I \subset \mathbb{R}$, is a solution to the *mean curvature flow* (MCF) with initial condition F if

$$(2.1) \quad \begin{cases} \frac{\partial}{\partial t} \hat{F}(x, t) = \hat{H}(x, t) \hat{N}(x, t), \\ \hat{F}(x, 0) = F(x), \end{cases}$$

where $\hat{H}^t(\cdot) = \hat{H}(\cdot, t) = \sum_{i=1}^n \hat{k}_i^t$ is the mean curvature and $\hat{N}^t(\cdot) = \hat{N}(\cdot, t)$ is a unit normal vector field of $\hat{F}^t(M)$. When F is a minimal hypersurface, i.e., $H = 0$, then the family $\hat{F}^t(x) = F(x)$ gives a trivial solution to the MCF.

In this paper, we consider a special type of solution to the MCF by imposing that the hypersurfaces \hat{F}^t be parallel. We first introduce the following notation:

$$(2.2) \quad c(\xi) = \begin{cases} 1, & \text{if } \bar{\kappa} = 0, \\ \cos(\xi), & \text{if } \bar{\kappa} = 1, \\ \cosh(\xi), & \text{if } \bar{\kappa} = -1 \end{cases} \quad \text{and} \quad s(\xi) = \begin{cases} \xi, & \text{if } \bar{\kappa} = 0, \\ \sin(\xi), & \text{if } \bar{\kappa} = 1, \\ \sinh(\xi), & \text{if } \bar{\kappa} = -1. \end{cases}$$

Definition 2.1. Let $\hat{F} : M^n \times I \rightarrow \mathbb{M}^{n+1}(\bar{\kappa})$ be a solution to the mean curvature flow in $\mathbb{M}^{n+1}(\bar{\kappa})$ with initial condition $F : M^n \rightarrow \mathbb{M}^{n+1}(\bar{\kappa})$. We say \hat{F} is a *solution to the mean curvature flow by parallel hypersurfaces* if there is a function $\xi : I \rightarrow \mathbb{R}$ such that $\xi(0) = 0$ and

$$(2.3) \quad \hat{F}^t(x) = c(\xi(t))F(x) + s(\xi(t))N(x),$$

for all $t \in I$, where $c : \mathbb{R} \rightarrow \mathbb{R}$ and $s : \mathbb{R} \rightarrow \mathbb{R}$ are the functions defined in (2.2).

We now state our main results.

Theorem 2.2. *Let $F : M^n \rightarrow \mathbb{M}^{n+1}(\bar{\kappa})$ be a hypersurface in a space form $\mathbb{M}^{n+1}(\bar{\kappa})$. Then $F(M)$ is the initial data of a solution to the MCF by parallel hypersurfaces if and only if $F(M)$ is an isoparametric hypersurface.*

As a consequence of the proof of this theorem, given in Section 3, one obtains the MCF of the isoparametric hypersurfaces of space forms by solving an ordinary differential equation. Namely, we prove the following.

Corollary 2.3. *Let $F : M^n \rightarrow \mathbb{M}^{n+1}(\bar{\kappa})$ be an isoparametric hypersurface with unit normal vector field N and principal curvatures κ_i . Then the solution to the MCF with initial data F is given by (2.3), where s and c are the functions defined in (2.2) and $\xi(t)$ is the solution of*

$$\xi'(t) = \sum_{i=1}^n \frac{\bar{\kappa}_i s(\xi(t)) + \kappa_i c(\xi(t))}{c(\xi(t)) - \kappa_i s(\xi(t))}, \quad \xi(0) = 0.$$

As an application of Corollary 2.3, we explicitly obtain the MCF by parallel hypersurfaces of the isoparametric hypersurfaces of \mathbb{R}^{n+1} and of \mathbb{H}^{n+1} in Propositions 2.4-2.7. The MCFs for a nonminimal hypersurface of \mathbb{S}^{n+1} with g distinct curvatures are given in Propositions 2.8-2.12.

For the sake of completeness we include the result for isoparametric hypersurfaces of the Euclidean space, without proof, since it is well known.

Proposition 2.4. *Let $F : \mathbb{S}^m \times \mathbb{R}^{n-m} \rightarrow \mathbb{R}^{n+1}$, $m \neq 0$, be the immersion of a cylinder (or sphere if $m = n$) in the Euclidean space, with m principal curvatures equal to $\kappa \neq 0$ and $n - m$ null principal curvatures. Then, the solution to the MCF with initial condition $F(M)$ is*

$$(2.4) \quad \widehat{F}^t(x) = F(x) + \frac{1 - \sqrt{1 - 2m\kappa^2 t}}{\kappa} N(x),$$

for all $t \in (-\infty, t^*)$, where $t^* = \frac{1}{2m\kappa^2}$. Moreover, if $m = n$ the solution collapses into a point at t^* . If $m \neq n$, then the solution collapses into an $(n - m)$ -dimensional plane of \mathbb{R}^{n+1} , at t^* .

Proposition 2.5. *Let $F : \mathbb{R}^n \rightarrow \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ be the immersion of a horosphere in the hyperbolic space, with unit normal vector field N and all principal curvatures $\kappa = \pm 1$. Then, the solution to the MCF with initial data F is*

$$(2.5) \quad \widehat{F}^t(x) = \cosh(nt)F(x) + \kappa \sinh(nt)N(x),$$

for all $t \in \mathbb{R}$. Moreover, $\widehat{F}^t(\mathbb{R}^n)$ is a horosphere for all $t \in \mathbb{R}$.

The totally umbilic hypersurfaces of the hyperbolic space, different from the horospheres, are treated in the following result.

Proposition 2.6. *Let $F : M^n \rightarrow \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ be the immersion of a totally umbilic hypersurface in the hyperbolic space, with unit normal vector field N and all principal curvatures equal to κ where $\kappa \notin \{0, \pm 1\}$. Then, the solution to the MCF with initial condition $F(M)$ is given by*

$$(2.6) \quad \widehat{F}^t(x) = \frac{\kappa^2 e^{-nt} - \sqrt{1 - \kappa^2 + \kappa^2 e^{-2nt}}}{\kappa^2 - 1} F(x) + \frac{\kappa e^{-nt} - \kappa \sqrt{1 - \kappa^2 + \kappa^2 e^{-2nt}}}{\kappa^2 - 1} N(x).$$

- (1) If $0 < |\kappa| < 1$, then \widehat{F}^t is defined for $t \in \mathbb{R}$, and it converges to a totally geodesic n -dimensional manifold when $t \rightarrow +\infty$.

- (2) If $|\kappa| > 1$, then \widehat{F}^t is defined for $t \in (-\infty, t^*)$, where $t^* = \frac{1}{2n} \ln \left(\frac{\kappa^2}{\kappa^2 - 1} \right)$, and it collapses to a point at t^* .

For the hyperbolic cylinder, we have the following solution to the mean curvature flow.

Proposition 2.7. *Let $F : \mathbb{S}^{m_1} \times \mathbb{H}^{m_2} \rightarrow \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ be the immersion of a cylinder in the hyperbolic space, with m_1 principal curvatures equal to $\kappa_1 > 1$ and m_2 principal curvatures equal to κ_2 , such that $\kappa_1\kappa_2 = 1$. Then the solution to the MCF with initial condition F is given by*

$$(2.7) \quad \widehat{F}^t(x) = \cosh(\xi(t))F(x) + \sinh(\xi(t))N(x),$$

where

$$(2.8) \quad \cosh(2\xi(t)) = \frac{a\ell(t) - 2\sqrt{q(t)}}{a^2 - 4}, \quad \sinh(2\xi(t)) = \frac{2\ell(t) - a\sqrt{q(t)}}{a^2 - 4},$$

$$(2.9)$$

$$q(t) = \ell^2(t) - a^2 + 4; \quad \ell(t) = (a-b)e^{-2nt} + b, \quad a = \kappa_1 + \kappa_2, \quad b = -\frac{m_1 - m_2}{n}(\kappa_1 - \kappa_2).$$

\widehat{F}^t is defined for all $t \in (-\infty, t^*)$ where $t^* = \frac{1}{2n} \ln \frac{m_1\kappa_1^2 + m_2}{m_1(\kappa_1^2 - 1)}$ and it collapses into an m_2 -dimensional focal submanifold at t^* .

We will now consider the isoparametric hypersurfaces of the sphere. Münzner [12] showed that the number g of distinct principal curvatures, for an isoparametric hypersurface $M^n \subset \mathbb{S}^{n+1}$, is restricted to be 1, 2, 3, 4, or 6. Moreover, he showed that connected isoparametric hypersurfaces of the sphere can be extended to compact ones. Cartan [3] classified these hypersurfaces when $g \leq 3$. If $g = 1$, then M^n is a sphere obtained as the intersection of \mathbb{S}^{n+1} with a hyperplane of \mathbb{R}^{n+2} . If $g = 2$, then M^n must be the standard product of spheres $\mathbb{S}_{r_1}^l \times \mathbb{S}_{r_2}^{n-l} \subset \mathbb{S}^{n+1}$, where $r_1^2 + r_2^2 = 1$. When $g = 3$, Cartan proved that there are only four distinct isoparametric hypersurfaces of \mathbb{S}^{n+1} with three distinct principal curvatures. Their dimensions are $n = 3m$, where $m = 1, 2, 4$, or 8 and all the principal curvatures have the same multiplicity m . The classification of the isoparametric hypersurfaces of the sphere with $g = 4$ or 6 is still not complete. However, when $g = 4$, Münzner [12] (see also Cecil–Chi–Jensen [5]) proved that the principal curvatures $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ can be ordered so that their corresponding multiplicities satisfy $m_1 = m_3$ and $m_2 = m_4$. When $g = 6$, Münzner [12] showed that all the principal curvatures must have the same multiplicities m , and Abresch [1] showed that $m = 1$ or $m = 2$.

One can determine the principal curvatures of $M^n \subset \mathbb{S}^{n+1}$ up to a constant. In fact, for all $g = 2, 3, 4, 6$, let $a \in \mathbb{R} - 1 < a < 1$, and $a = \cos(gs)$, i.e., $0 < s < \pi/g$. We consider

$$(2.10) \quad \kappa_j = \cot \left(s + \frac{j-1}{g} \pi \right), \quad j = 1, \dots, g.$$

The hypersurfaces for different values of the constant a are parallel in \mathbb{S}^{n+1} .

In the next propositions, we consider the solution \widehat{F}^t to the MCF, by parallel hypersurfaces, with initial data an isoparametric hypersurface F of the sphere. Therefore,

$$(2.11) \quad \widehat{F}^t(x) = \cos(\xi(t))F(x) + \sin(\xi(t))N(x).$$

The function $\xi(t)$ will be determined in the following results, according to the number g of distinct principal curvatures. We start considering the case $g = 1$, i.e., the umbilical hypersurfaces of \mathbb{S}^{n+1} .

Proposition 2.8. *Let $F : M^n \rightarrow \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ be the immersion of a totally umbilic hypersurface in \mathbb{S}^{n+1} , with unit normal vector field N and all principal curvatures equal to $\kappa \neq 0$. Then the solution to the MCF with F as initial data is given by*

$$(2.12) \quad \widehat{F}^t(x) = \frac{\kappa^2 e^{nt} + \sqrt{q(t)}}{\kappa^2 + 1} F(x) + \frac{\kappa e^{nt} - \kappa \sqrt{q(t)}}{\kappa^2 + 1} N(x), \quad \text{where } q(t) = \kappa^2 + 1 - \kappa^2 e^{2nt}.$$

\widehat{F}^t is defined for all $t \in (-\infty, t^*)$ where $t^* = \frac{1}{2n} \ln\left(\frac{\kappa^2 + 1}{\kappa^2}\right)$, and it collapses to a point at t^* .

Proposition 2.9. *Let $F : \mathbb{S}_{r_1}^l \times \mathbb{S}_{r_2}^{n-l} \rightarrow \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ be an isoparametric hypersurface in \mathbb{S}^{n+1} , with two distinct principal curvatures κ_1 and κ_2 with multiplicities l and $n - l$, respectively. Then $\kappa_1 \kappa_2 = -1$, and assuming the immersion is not minimal, we may consider $\kappa_1 > \sqrt{(n-l)/l} > 1$. The solution to the MCF with initial data F is \widehat{F}^t given by (2.11), where*

$$(2.13) \quad \cos(2\xi(t)) = \frac{a q(t) + 2\sqrt{a^2 + 4 - q^2(t)}}{a^2 + 4}, \quad \sin(2\xi(t)) = \frac{2q(t) - a\sqrt{a^2 + 4 - q^2(t)}}{a^2 + 4},$$

and

$$(2.14) \quad a = \kappa_1 + \kappa_2, \quad b = -\frac{n-2l}{n}(\kappa_1 - \kappa_2), \quad q(t) = (a + b)e^{2nt} - b.$$

\widehat{F}^t is defined for all $t \in [0, t^*)$, where $t^* = \frac{1}{2n} \ln\left(\frac{l(\kappa_1^2 + 1)}{l(\kappa_1^2 + 1) - n}\right)$, and it collapses into an $(n - l)$ -dimensional focal submanifold of F at t^* .

Proposition 2.10. *Let $F : M^n \rightarrow \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ be a nonminimal isoparametric hypersurface in \mathbb{S}^{n+1} , with unit normal vector field N and three distinct principal curvatures $\kappa_1, \kappa_2, \kappa_3$. Then all the principal curvatures have the same multiplicity m , where $m = 1, 2, 4$, or 8 , i.e., $n = 3m$. Moreover, we may consider $\kappa_1 > \sqrt{3}/3$ and κ_2 and κ_3 given by (2.10). The solution to the MCF with initial data F is \widehat{F}^t given by (2.11), where*

$$(2.15) \quad \begin{aligned} \cos(3\xi(t)) &= \frac{a^2 e^{9mt} + 3\sqrt{q(t)}}{a^2 + 9}, & \sin(3\xi(t)) &= \frac{a(3e^{9mt} - \sqrt{q(t)})}{a^2 + 9}, \\ a = \kappa_1 + \kappa_2 + \kappa_3 &= \frac{3\kappa_1(\kappa_1^2 - 3)}{3\kappa_1^2 - 1}, & q(t) &= a^2 + 9 - a^2 e^{18mt}. \end{aligned}$$

\widehat{F}^t is defined for all $t \in [0, t^*)$, where $t^* = \frac{1}{18m} \ln\left(1 + \frac{9}{a^2}\right)$, and it collapses into a $2m$ -dimensional focal submanifold of $F(M)$ at t^* .

Proposition 2.11. *Let $F : M^n \rightarrow \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ be a nonminimal isoparametric hypersurface of \mathbb{S}^{n+1} , with unit normal vector field N and four distinct principal curvatures κ_j , with multiplicities m_j , $j = 1, 2, 3, 4$. Then we may consider*

$$(2.16) \quad \kappa_1 > 1, \quad \kappa_2 = \frac{\kappa_1 - 1}{\kappa_1 + 1}, \quad \kappa_3 = \frac{-1}{\kappa_1}, \quad \kappa_4 = \frac{-(\kappa_1 + 1)}{\kappa_1 - 1},$$

where the multiplicities m_j satisfy $m_1 = m_3$ and $m_2 = m_4$, $n = 2(m_1 + m_2)$. The solution to the MCF with initial data F is \widehat{F}^t given by (2.11), where

$$(2.17) \quad \cos(4\xi(t)) = \frac{aq(t) + 4\sqrt{a^2 + 16 - q^2(t)}}{a^2 + 16}, \quad \sin(4\xi(t)) = \frac{4q(t) - a\sqrt{a^2 + 16 - q^2(t)}}{a^2 + 16},$$

$$(2.18) \quad a = \sum_{j=1}^4 \kappa_j = \frac{\kappa_1^4 - 6\kappa_1^2 + 1}{\kappa_1(\kappa_1^2 - 1)}, \quad b = \frac{2(m_1 - m_2)(\kappa_1^2 + 1)^2}{n\kappa_1(\kappa_1^2 - 1)}, \quad q(t) = (a + b)e^{4nt} - b.$$

Moreover, \widehat{F}^t is defined for all $t \in [0, t^*)$, where $t^* = \frac{1}{4n} \ln\left(\frac{b + \sqrt{a^2 + 16}}{a + b}\right)$, and it collapses into an $(m_1 + 2m_2)$ -dimensional focal submanifold of $F(M)$.

Proposition 2.12. *Let $F : M^n \rightarrow \mathbb{S}^{n+1} \subset R^{n+2}$ be a nonminimal isoparametric hypersurface in \mathbb{S}^{n+1} , with unit normal vector field N and six distinct principal curvatures κ_j , $j = 1, \dots, 6$. Then $n = 6m$, where $m = 1, 2$, and we may consider $\kappa_1 > \sqrt{3}$ and κ_j given by (2.10), for $2 \leq j \leq 6$. The solution to the MCF with initial data F is \widehat{F}^t given by (2.11), where*

$$(2.19) \quad \cos(6\xi(t)) = \frac{a^2 e^{36mt} + 6\sqrt{q(t)}}{a^2 + 36}, \quad \sin(6\xi(t)) = \frac{a(6e^{36mt} - \sqrt{q(t)})}{a^2 + 36},$$

where

$$(2.20) \quad a = \sum_{j=1}^6 \kappa_j = \frac{\kappa_1^6 - 15\kappa_1^4 + 15\kappa_1^2 - 1}{\kappa_1(\kappa_1^2 - 3)(3\kappa_1^2 - 1)} \quad \text{and} \quad q(t) = a^2 + 36 - a^2 e^{72mt},$$

which is defined for all $t \in [0, t^*)$, where $t^* = \frac{1}{72m} \ln\left(1 + \frac{36}{a^2}\right)$. Moreover, the solution collapses into a 5m-dimensional focal submanifold of $F(M)$ at t^* .

3. PROOF OF THE MAIN RESULTS

In order to prove our main results, we first state some well-known properties of parallel hypersurfaces. In fact, our next lemma can be easily proved as a consequence of (2.3) and the fact that the functions $c(\xi)$ and $s(\xi)$, defined by (2.2), satisfy the properties

$$(3.1) \quad c'(\xi) = -\bar{\kappa}s(\xi), \quad s'(\xi) = c(\xi), \quad c^2(\xi) + \bar{\kappa}s^2(\xi) = 1.$$

Lemma 3.1. *Let $F : M^n \rightarrow \mathbb{M}^{n+1}(\bar{\kappa})$ be an oriented hypersurface with unit normal vector field N , and let $\widetilde{F}^\xi : M^n \rightarrow \mathbb{M}^{n+1}(\bar{\kappa})$ be a parallel hypersurface given by $\widetilde{F}^\xi(x) = c(\xi)F(x) + s(\xi)N(x)$. Then, the unit vector field \widetilde{N}^ξ normal to \widetilde{F}^ξ and the corresponding principal curvatures $\widetilde{\kappa}_i^\xi$ are given by*

$$(3.2) \quad \widetilde{N}^\xi(x) = -\bar{\kappa}s(\xi)F(x) + c(\xi)N(x), \quad \widetilde{\kappa}_i^\xi(x) = \frac{\bar{\kappa}s(\xi) + \kappa_i(x)c(\xi)}{c(\xi) - \kappa_i(x)s(\xi)},$$

where $\kappa_i(x)$ is the i th principal curvature of $F(M)$ on $x \in M$, for all $i = 1, \dots, n$. Moreover, if $\{e_1, \dots, e_n\} \subset T_x M$ is an orthonormal basis of eigenvectors of the second fundamental form of $F(M)$, then

$$(3.3) \quad \widetilde{g}_x^\xi(e_i, e_j) = [c(\xi) - \kappa_i(x)s(\xi)]^2 \delta_{ij}.$$

Proof of Theorem 2.2. Suppose that \widehat{F}^t is given by (2.3). Then, as a consequence of (3.1),

$$(3.4) \quad \frac{\partial}{\partial t} \widehat{F}(x, t) = \xi'(t) (-\bar{\kappa}s(\xi(t))F(x) + c(\xi(t))N(x)).$$

If \widehat{F}^t is a solution to MCF, then (2.1) reduces to

$$\xi'(t) (-\bar{\kappa}s(\xi(t))F(x) + c(\xi(t))N(x)) = \widehat{H}^t(x)\widehat{N}^t(x),$$

and from (3.2) we conclude that

$$(3.5) \quad \xi'(t) = \widehat{H}^t(x).$$

Therefore, for each t fixed, the mean curvature \widehat{H}^t is constant. Then, it follows from a classical result of Cartan [3] (see also Theorem 3.6 in [4]) that $F(M)$ is an isoparametric hypersurface.

Reciprocally, if $F(M)$ is an isoparametric hypersurface, let $\xi(t)$ be the unique solution of the ordinary differential equation

$$(3.6) \quad \xi'(t) = \sum_{i=1}^n \widehat{\kappa}_i^t = \sum_{i=1}^n \frac{\bar{\kappa}s(\xi(t)) + \kappa_i c(\xi(t))}{c(\xi(t)) - \kappa_i s(\xi(t))},$$

such that $\xi(0) = 0$, where κ_j is the j th principal curvature of $F(M)$.

Considering $\widehat{F}^t(x)$ given by (2.3), then (3.4) holds. Moreover, it follows from Lemma 3.1 that

$$\widehat{N}^t(x) = -\bar{\kappa}s(\xi(t))F(x) + c(\xi(t))N(x)$$

and

$$\widehat{H}^t(x) = \sum_{i=1}^n \widehat{\kappa}_i^t = \sum_{i=1}^n \frac{\bar{\kappa}s(\xi(t)) + \kappa_i c(\xi(t))}{c(\xi(t)) - \kappa_i s(\xi(t))} = \xi'(t).$$

Therefore,

$$\frac{\partial}{\partial t} \widehat{F}^t(x) = \widehat{H}^t(x)\widehat{N}^t(x).$$

Moreover, since $\xi(0) = 0$, we have $\widehat{F}^0(x) = F(x)$, i.e., \widehat{F}^t is a solution to the MCF. □

The proof of Corollary 2.3 is an immediate consequence of the proof given above.

From now on, we will apply Corollary 2.3 to obtain the mean curvature flow whose initial data is a nonminimal isoparametric hypersurface of a space form, i.e., we will prove Propositions 2.4–2.12.

Proof of Proposition 2.5. From Corollary 2.3 and the fact that $\kappa = \pm 1$, we have $\widehat{F}^t(x) = \cosh(\xi(t))F(x) + \sinh(\xi(t))N(x)$, where $\xi'(t) = \kappa n$. Integrating this equation, since $\xi(0) = 0$, we have $\xi(t) = \kappa n t$, for all $t \in \mathbb{R}$. Therefore, (2.5) follows. Moreover, from (3.2), we have $\widehat{\kappa}_i^t = \kappa \ \forall i$, which concludes the proof. □

Proof of Proposition 2.6. From Corollary 2.3, the solution to the MCF by parallel hypersurfaces is given by

$$\widehat{F}^t(x) = \cosh(\xi(t))F(x) + \sinh(\xi(t))N(x),$$

where

$$\xi'(t) = -n \frac{\sinh(\xi(t)) - \kappa \cosh(\xi(t))}{\cosh(\xi(t)) - \kappa \sinh(\xi(t))}.$$

Integrating the equation for ξ , with $\xi(0) = 0$, we have $\sinh(\xi(t)) = \kappa \cosh(\xi(t)) - \kappa e^{-nt}$. The square of this equation reduces to

$$(\kappa^2 - 1) \cosh^2(\xi(t)) - 2\kappa^2 e^{-nt} \cosh(\xi(t)) + \kappa^2 e^{-2nt} + 1 = 0.$$

Therefore,

$$\cosh(\xi(t)) = \frac{\kappa^2 e^{-nt} - \sqrt{q(t)}}{\kappa^2 - 1}, \quad \sinh(\xi(t)) = \frac{\kappa e^{-nt} - \kappa \sqrt{q(t)}}{\kappa^2 - 1},$$

where

$$q(t) = 1 - \kappa^2 + \kappa^2 e^{-2nt},$$

which proves (2.6). Let $\{E_1, \dots, E_n\}$ be a local orthonormal frame of principal directions on $F(M)$. It follows from Lemma 3.1 that the first fundamental form and the principal curvatures of \widehat{F}^t are given, respectively, by $\widehat{g}^t(E_i, E_j) = q(t)\delta_{ij}$ and $\widehat{\kappa}^t = \kappa e^{-nt} / \sqrt{q(t)}$.

If $0 < |\kappa| < 1$, then $q(t) > 0$ and the solution is defined for all $t \in \mathbb{R}$. Moreover, since $\lim_{t \rightarrow +\infty} \widehat{\kappa}^t = 0$, $\widehat{F}^t(\mathbb{H}^n)$ converges to a totally geodesic submanifold. If $|\kappa| > 1$, then $q(t^*) = 0$, where $t^* = \frac{1}{2n} \ln\left(\frac{\kappa^2}{\kappa^2 - 1}\right)$. Hence $\widehat{F}^t(\mathbb{S}^n)$ is defined for $t \in (-\infty, t^*)$, and it collapses to a point at t^* . \square

Proof of Proposition 2.7. It follows from Corollary 2.3 that the solution \widehat{F}^t is given by (2.7) with $\xi(t)$ satisfying

$$\xi'(t) = -n \frac{2 \sinh(2\xi(t)) - a \cosh(2\xi(t)) + b}{2 \cosh(2\xi(t)) - a \sinh(2\xi(t))},$$

where a and b are given by (2.9). Integrating this equation with $\xi(0) = 0$, we get $2 \sinh(2\xi(t)) = a \cosh(2\xi(t)) - b + (b - a)e^{-2nt}$. The square of this equation reduces to

$$(a^2 - 4) \cosh^2(2\xi(t)) - 2a\ell(t) \cosh(2\xi(t)) + \ell^2(t) + 4 = 0,$$

where $\ell(t)$ is given by (2.9). Therefore, we obtain $\cosh(2\xi(t))$ and $\sinh(2\xi(t))$ as in (2.8). Without loss of generality, we are considering $\kappa_1 > \kappa_2 > 0$.

Moreover, \widehat{F}^t is defined for all t , for which $q(t) = \ell^2(t) - a^2 + 4 > 0$. Since $-a^2 + 4 = -(\kappa_1 - \kappa_2)^2$ and $b - a = -2(m_1\kappa_1 + m_2\kappa_2)/n$, we conclude that $t < t^*$, where $t^* = 1/n \ln \frac{m_1\kappa_1 + m_2\kappa_2}{m_1(\kappa_1 - \kappa_2)}$.

Let $\{E_1, \dots, E_{m_1}, E_{m_1+1}, \dots, E_{m_1+m_2} = E_n\}$ be an orthonormal frame of principal directions such that E_1, \dots, E_{m_1} are tangent to \mathbb{S}^{m_1} and E_{m_1+1}, \dots, E_n are tangent to \mathbb{H}^{m_2} . It follows from (3.3) that $\widehat{g}^{t^*}(E_i, E_j) = 0$ for $1 \leq i \leq m_1$ and $\widehat{g}^{t^*}(E_i, E_j) = \kappa_2(\kappa_1 - \kappa_2)\delta_{ij}$ for $m_1 + 1 \leq i, j \leq n$. Thus, the solution \widehat{F}^t collapses into an m_2 -dimensional focal submanifold at t^* , since $\coth(\xi(t^*)) = \kappa_1$. \square

We will now prove Propositions 2.8–2.12 that consider the MCF of isoparametric hypersurfaces of the sphere.

Proof of Proposition 2.8. It follows from Corollary 2.3 that the solution to the MCF is given by (2.11), where $\xi(t)$ satisfies

$$\xi'(t) = n \frac{\sin(\xi(t)) + \kappa \cos(\xi(t))}{\cos(\xi(t)) - \kappa \sin(\xi(t))}, \quad \text{with} \quad \xi(0) = 0.$$

Integrating this equation, we have $\sin(\xi(t)) = \kappa(e^{nt} - \cos(\xi(t)))$. The square of this equation reduces to

$$(\kappa^2 + 1) \cos^2(\xi(t)) - 2\kappa^2 e^{nt} \cos(\xi(t)) + \kappa^2 e^{2nt} - 1 = 0.$$

Therefore, we obtain (2.12) and the solution is defined for all $t \in (-\infty, t^*)$ where $t^* = \frac{1}{2n} \ln\left(\frac{\kappa^2 + 1}{\kappa^2}\right)$. Let $\{E_1, \dots, E_n\}$ be a local orthonormal frame of principal directions of $F(M)$. From Lemma 3.1, we have $\hat{g}^t(E_i, E_j) = q^2(t)\delta_{ij}$. Since $q(t^*) = 0$, we have $\hat{g}^{t^*}(E_i, E_j) = 0$, for all i, j . Therefore, $\hat{F}^t(M)$ collapses into a point at t^* . \square

Proof of Proposition 2.9. Since we are considering that F is not a minimal hypersurface, we may assume that the mean curvature $H > 0$. Since (2.10) implies that $\kappa_2 = -1/\kappa_1$, hence, without loss of generality, we may consider $\kappa_1 > \sqrt{(n-l)/l}$ and $n-l > l$, i.e., $\kappa_1 > 1$. It follows from Corollary 2.3 and a straightforward computation that the solution of the mean curvature flow is given by (2.11), where $\xi(t)$ must satisfy

$$\xi'(t) = n \frac{2 \sin(2\xi(t)) + a \cos(2\xi(t)) + b}{2 \cos(2\xi(t)) - a \sin(2\xi(t))}, \quad \xi(0) = 0,$$

where a and b are given by (2.14). Integrating, we get

$$(3.7) \quad 2 \sin(2\xi(t)) + a \cos(2\xi(t)) + b = (a + b)e^{2nt}.$$

This equation implies that $\cos(2\xi(t))$ must satisfy the algebraic equation

$$(a^2 + 4) \cos^2(2\xi(t)) - 2aq(t) \cos(2\xi(t)) + q^2(t) - 4 = 0,$$

where $q(t)$ is given by (2.14). Solving for $\cos(2\xi(t))$ with $\xi(0) = 0$ and using (3.7), we get (2.13).

Let t^* be such that $a^2 + 4 - q^2(t^*) = 0$. Then, since $a + b = 2H/n > 0$, we get

$$t^* = \frac{1}{2n} \ln\left(\frac{b + \sqrt{a^2 + 4}}{a + b}\right) = \frac{1}{2n} \ln\left(\frac{l(\kappa_1^2 + 1)}{l(\kappa_1^2 + 1) - n}\right),$$

and it follows from (2.13) that $\cot(2\xi(t^*)) = a/2$ and $\cot(\xi(t^*)) = \kappa_1$. Let $\{E_1, \dots, E_l\}$ and $\{E_{l+1}, \dots, E_n\}$ be an orthonormal frame of principal vector fields corresponding to κ_1 and κ_2 , respectively. It follows from Lemma 3.1 that

$$\hat{g}^{t^*}(E_i, E_j) = \sin(\xi(t^*))^2 (\cot(\xi(t^*)) - \kappa_1)^2 \delta_{ij} = 0$$

for

$$1 \leq i, j \leq l,$$

and

$$\hat{g}^{t^*}(E_i, E_j) = \sin(\xi(t^*))^2 (\cot(\xi(t^*)) + 1/\kappa_1)^2 \delta_{ij} \neq 0,$$

for $l + 1 \leq i, j \leq n$. Therefore, \hat{F}^t is defined for $t \in [0, t^*)$, and when t tends to t^* , the mean curvature flow collapses into an $(n - l)$ -dimensional focal submanifold of $F(M)$. \square

Proof of Proposition 2.10. When $g = 3$ the three distinct principal curvatures are determined by (2.10). Since $H > 0$ we may consider $\kappa_1 > \sqrt{3}/3$. Then $\kappa_2 = (\kappa_1\sqrt{3}-3)/(3\kappa_1+\sqrt{3})$ and $\kappa_3 = (\kappa_1\sqrt{3}+3)/(-3\kappa_1+\sqrt{3})$. Moreover, Cartan showed that all the principal curvatures have the same multiplicity m , where $m = 1, 2, 4$, or 8 , and hence $n = 3m$. It follows from Corollary 2.3 and a straightforward computation that the solution of the mean curvature flow is given by (2.11), where $\xi(t)$ must satisfy

$$\xi'(t) = 3m \frac{3 \sin(3\xi(t)) + a \cos(3\xi(t))}{3 \cos(3\xi(t)) + a \sin(3\xi(t))},$$

where $a = \sum_{j=1}^3 \kappa_j$. Integrating this equation, and using the fact that $\xi(0) = 0$, we get

$$(3.8) \quad 3 \sin(3\xi(t)) + a \cos(3\xi(t)) = ae^{9mt}.$$

Considering the square of this equation, we obtain

$$(a^2 + 9) \cos^2(3\xi(t)) - 2a^2 e^{9mt} \cos(3\xi(t)) + a^2 e^{18mt} - 9 = 0.$$

Solving this equation for $\cos(3\xi(t))$ with $\xi(0) = 0$ and using (3.8) we obtain $\cos(3\xi(t))$ and $\sin(3\xi(t))$ given by (2.15). Let t^* be such that $q(t^*) = 0$. Then

$$t^* = \frac{1}{18m} \ln \left(1 + \frac{9}{a^2} \right) = \frac{1}{18m} \ln \left(\frac{(\kappa_1^2 + 2)^3}{\kappa_1^2(\kappa_1^2 - 3)^2} \right).$$

and it follows from (2.15) that $\cot(3\xi(t^*)) = a/3$. Moreover, $\cot(\xi(t^*)) = \kappa_1$. When t tends to t^* the hypersurface \hat{F}^t collapses into a $2m$ -dimensional focal submanifold of $F(M)$. In fact, considering E_{j1}, \dots, E_{jm} , $j = 1, 2, 3$, an orthonormal vector field of principal direction corresponding to the principal curvature κ_j , we conclude that $\hat{g}^{t^*}(E_{1\ell}, E_{1r}) = \sin(\xi(t^*))(\cot(\xi(t^*)) - \kappa_1) = 0$ for all $1 \leq \ell, r \leq m$, while $\hat{g}^{t^*}(E_{j\ell}, E_{j\ell}) \neq 0$ for $j \neq 1$. □

Proof of Proposition 2.11. Since $g = 4$, it follows from (2.10) that we may consider the principal curvature given by (2.16). Moreover, Münzner proved that the corresponding multiplicities satisfy $m_1 = m_3$ and $m_2 = m_4$, hence $n = 2(m_1 + m_2)$. It follows from Corollary 2.3 and a straightforward computation that the solution of the MCF \hat{F}^t is given by (2.11), where $\xi(t)$ must satisfy

$$\xi'(t) = n \frac{4 \sin(4\xi(t)) + a \cos(4\xi(t)) + b}{4 \cos(4\xi(t)) - a \sin(4\xi(t))}, \quad \xi(0) = 0,$$

where a and b are the constants given by (2.18). Integrating, we get

$$(3.9) \quad 4 \sin(4\xi(t)) + a \cos(4\xi(t)) + b = (a + b)e^{4nt}.$$

It follows from this equation that $\cos(4\xi(t))$ must satisfy the algebraic equation

$$(a^2 + 16) \cos^2(4\xi(t)) - 2a((a + b)e^{4nt} - b) \cos(4\xi(t)) + ((a + b)e^{4nt} - b)^2 - 16 = 0.$$

Solving this equation for $\cos(4\xi)$ and considering $\xi(0) = 0$, the expression $q(t)$ defined by (2.18) and (3.9), we conclude that $\cos(4\xi(t))$ and $\sin(4\xi(t))$ are given by (2.17).

Let t^* be such that $a^2 + 16 - q^2(t^*) = 0$. Then

$$t^* = \frac{1}{4n} \ln \left(\frac{b + \sqrt{a^2 + 16}}{a + b} \right) = \frac{1}{4n} \ln \left(\frac{m_1(\kappa_1^2 + 1)^2}{m_1(\kappa_1^2 + 1)^2 - 2n\kappa_1^2} \right).$$

It follows from (2.17) that $\cot(4\xi(t^*)) = \frac{a}{4}$ and $\cot(\xi(t^*)) = \kappa_1$.

When t tends to t^* , the hypersurface collapses into an $(n - m_1)$ -dimensional submanifold of \mathbb{S}^{n+1} , which is a focal submanifold of $F(M)$. In fact, if we consider $\{E_1, \dots, E_{m_1}\}$, $\{E_{m_1+1}, \dots, E_{m_1+m_2}\}$, $\{E_{m_1+m_2+1}, \dots, E_{2m_1+m_2}\}$, and $\{E_{2m_1+m_2+1}, \dots, E_n\}$ an orthonormal frame of principal directions of FM , corresponding to $\kappa_1, \kappa_2, \kappa_3, \kappa_4$, respectively, then $\lim_{t \rightarrow t^*} \widehat{g}^t(E_\nu, E_j) = 0$ for all ν, j such that $1 \leq \nu, j \leq m_1$ and $\widehat{g}^{t^*}(E_\ell, E_r) = 0$ for $\ell \neq r$ and $\ell > m_1$ or $r > m_1$, while $\widehat{g}^{t^*}(E_\ell, E_\ell) \neq 0$, for $m_1 + 1 \leq \ell \leq n$. \square

Proof of Proposition 2.12. When $g = 6$ it follows from results of Münzner and Abresch that all the principal curvatures have the same multiplicity m and that $m = 1, 2$. Moreover, since F is not a minimal hypersurface we may consider that $\kappa_1 > \sqrt{3}$ and κ_j are given by (2.10) for $2 \leq j \leq 6$. It follows from Corollary 2.3 and a straightforward computation that the solution is given by (2.11), where $\xi(t)$ must satisfy

$$\xi'(t) = \frac{6m(a \cos(6\xi(t)) + 6 \sin(6\xi(t)))}{6 \cos(6\xi(t)) - a \sin(6\xi(t))}, \quad \xi'(0) = 0.$$

Integrating this equation we get $a \cos(6\xi(t)) + 6 \sin(6\xi(t)) = ae^{36mt}$. It follows from this equation that $\cos(6\xi(t))$ must satisfy the algebraic equation

$$(a^2 + 36) \cos^2(6\xi(t)) - 2a^2 e^{36mt} \cos(6\xi(t)) + a^2 e^{72mt} - 36 = 0.$$

Solving this equation for $\cos(6\xi(t))$ and considering $\xi(0) = 0$, we conclude that $\cos(6\xi(t))$ and $\sin(6\xi(t))$ are given by (2.19). Let t^* be such that $q(t^*) = 0$. Then $t^* = \frac{1}{72m} \ln\left(1 + \frac{36}{a^2}\right)$, and it follows from (2.19) that $\cot(6\xi(t^*)) = a/6$. Moreover, $\cot(\xi(t^*)) = \kappa_1$.

Then \widehat{F}^t is defined for $t \in [0, t^*)$, and when t tends to t^* the hypersurface it collapses into a $5m$ -dimensional submanifold of \mathbb{S}^{6m+1} that is a focal submanifold of $F(M)$. In fact, considering E_{j1}, E_{jm} , $j = 1, \dots, 6$, an orthonormal vector field of principal directions corresponding to the principal curvature κ_j , we conclude that $\widehat{g}^{t^*}(E_{1\ell}, E_{1r}) = 0$, for all $1 \leq \ell, r \leq m$, while $\widehat{g}^{t^*}(E_{j\ell}, E_{j\ell}) \neq 0$, for $j \neq 1$. \square

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