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A CLASS OF COMPLETELY MIXED MONOTONIC FUNCTIONS INVOLVING THE GAMMA FUNCTION WITH APPLICATIONS

ZHEN-HANG YANG AND JING-FENG TIAN

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Abstract. In this paper, we introduce the notion of completely mixed monotonicity of a function of several variables, very few of which have appeared. We give a necessary and sufficient condition for a function constructed by ratios of gamma functions to be completely mixed monotonic. From this, some new inequalities for gamma, psi, and polygamma functions are derived.

1. Introduction

Recall that a function f is called completely monotonic (for short, CM) on an interval I if f has derivatives of all orders on I and satisfies

$$(-1)^k (f(x))^{(k)} \geq 0$$

for all $k \geq 0$ on I (see [1], [2]). A function f is called logarithmically completely monotonic (for short, LCM) on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies

$$(-1)^k (\ln f(x))^{(k)} \geq 0$$

for all $k \in \mathbb{N}$ on I (see [3], [4]). The celebrated Bernstein–Widder's Theorem [2, p. 161, Theorem 12b] states that a necessary and sufficient condition that $f(x)$ should be completely monotonic for $0 < x < \infty$ is that

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t),$$

where $\alpha(t)$ is nondecreasing and the integral converges for $0 < x < \infty$.

For convenience, we denote the sets of the completely and logarithmically completely monotonic functions on I by $\mathcal{C}[I]$ and $\mathcal{L}[I]$, respectively.

We now introduce the notion of a completely monotonic function in several variables.

Definition 1 ([5, Theorem 4.2.2])**.** A function $f(x_1, x_2, \ldots, x_k)$ is called completely monotonic in a cone $X = X_1 \times X_2 \times \cdots \times X_k$ if it is C^∞ and satisfies

$$(-1)^{n_1+n_2+\cdots+n_k} \frac{\partial^{n_1+n_2+\cdots+n_k} f}{\partial x_1^{n_1} \partial x_2^{n_2} \cdots \partial x_k^{n_k}} \geq 0$$

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The second author is the corresponding author.

for all combinations $n_1, n_2, \ldots, n_k \geq 0$. This class of completely monotonic functions is denoted by $\mathcal{C}[X]$.

Remark 1. To avoid confusion with univariate functions, we prefer to say that $f(x_1, x_2, \ldots, x_k)$ is completely mixed monotonic on X.

Remark 2. If a function $f(x_1, x_2, \ldots, x_k)$ is completely mixed monotonic in a cone X, then f is completely monotonic in every x_j on X_j, $j = 1, 2, \ldots, k$.

The classical Euler's gamma function Γ is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \tag{1.1}$$

for $x > 0$, and its logarithmic derivative $\psi(x) = \Gamma'(x)/\Gamma(x)$ is known as the psi or digamma function, while ψ', ψ'', ... are called polygamma functions.

In 1986, Ismail, Lorch and Muldoon [6] showed that the function

$$x \mapsto x^{b-a} \frac{\Gamma(x+a)}{\Gamma(x+b)}$$

for $a > b \geq 0$ is logarithmically completely monotonic on $(0, \infty)$ if and only if $a + b \geq 1$. In the same year, Bustoz and Ismail [7] further presented some complete monotonicity results involving ratios of gamma functions. Since then, many papers on this topic have been published; see for example, [4], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20].

However, there are very few articles on complete mixed monotonicity in various journals. The aim of this paper is to investigate the complete mixed monotonicity of the ratio $\mathcal{G}_{p,q}(x, y)/\mathcal{G}_{r,s}(x, y)$, where

$$\mathcal{G}_{p,q}(x, y) = \begin{cases} \left[\dfrac{\Gamma(x+p)\Gamma(y+q)}{\Gamma(y+p)\Gamma(x+q)}\right]^{1/((p-q)(x-y))} & \text{if } (p-q)(x-y) \neq 0, \\ \exp\left[\dfrac{\psi(x+p) - \psi(y+p)}{x-y}\right] & \text{if } p = q,\ x \neq y, \\ \exp\left[\dfrac{\psi(x+p) - \psi(x+q)}{p-q}\right] & \text{if } p \neq q,\ x = y, \\ \exp\left[\psi'(x+p)\right] & \text{if } p = q,\ x = y \end{cases} \tag{1.2}$$

for $x, y > -\min(p, q)$.

The rest of this paper is organized as follows. In Section 2, we list some properties of an important auxiliary function $\eta_{u,v} : \mathbb{R} \longrightarrow \mathbb{R}$ defined for $u, v \in \mathbb{R}$ by

$$\eta_{u,v}(t) = \begin{cases} \dfrac{e^{-ut} - e^{-vt}}{v-u} & \text{if } u \neq v, \\ te^{-ut} & \text{if } u = v. \end{cases} \tag{1.3}$$

These properties, especially Properties 3-4, are crucial to the proof of our results. In Sections 3, the necessary and sufficient conditions for $\ln(\mathcal{G}_{p,q}(x, y)/\mathcal{G}_{r,s}(x, y))$ to be completely mixed monotonic are stated and proved. In the last section, as applications, some new inequalities involving gamma, psi, and polygamma functions are established.

2. Properties of the auxiliary function $\eta_{u,v}(t)$

The auxiliary function $\eta_{u,v}(t)$ has the following properties, which were listed in [21, Section 2].

Property 1. We have

$$\eta_{u,v}(t) = t\int_0^1 \exp(-utx - vt(1-x))\,dx, \tag{2.1}$$

$$\eta_{u,v}(t) = \begin{cases} e^{-(u+v)t/2}\dfrac{\sinh[(u-v)t/2]}{(u-v)/2} & \text{if } u \neq v, \\ te^{-ut} & \text{if } u = v. \end{cases} \tag{2.2}$$

Property 2. Let $u, v, t \in \mathbb{R}$. Then $\eta_{u,v}(t)$ satisfies that
(i) $\eta_{u,v}(t) > (<)\, 0$ for $t > (< 0)$;
(ii) $\eta_{u,v}(t) = \eta_{v,u}(t)$;
(iii) $e^{-\rho t}\eta_{u-\rho,v-\rho}(t) = \eta_{u,v}(t)$ for any $\rho \in \mathbb{R}$.

Property 3. Let $u, v, r, s \in \mathbb{R}$, and let $\eta_{u,v}$ be defined on $(0,\infty)$ by (2.1). Then the comparison inequality $\eta_{u,v}(t) \geq \eta_{r,s}(t)$ holds for all $t > 0$ if and only if

$$u + v \leq r + s \quad \text{and} \quad \min(u,v) \leq \min(r,s). \tag{2.3}$$

The auxiliary function $\eta_{u,v}(t)$ also has a new property.

Property 4. The function $(u,v) \mapsto \eta_{u,v}(t)$ is strictly completely mixed monotonic on $\mathbb{R}^2$ for $t \in (0,\infty)$.

Proof. Using integral representation (2.1), we immediately get

$$(-1)^n \frac{\partial^n \eta_{u,v}}{\partial u^k \partial v^{n-k}} = t^{n+1}\int_0^1 x^k (1-x)^{n-k} \exp(-utx - vt(1-x))\,dx > 0 \tag{2.4}$$

for $t \in (0,\infty)$, which proves the completely mixed monotonicity of $\eta_{u,v}(t)$ with respect to (u,v) on $\mathbb{R}^2$. □

3. Main results

With the aid of those properties of $\eta_{u,v}(t)$ presented in the previous section, the aim of this section is to investigate the complete mixed monotonicity of $(p,q,x,y) \mapsto \mathcal{G}_{p,q}(x,y)$ and the necessary and sufficient conditions for $(x,y) \mapsto \ln(\mathcal{G}_{p,q}(x,y)/\mathcal{G}_{r,s}(x,y))$ to be completely mixed monotonic.

Theorem 1. *Let $\mathcal{G}_{p,q}(x,y)$ be defined by (1.2). Then $(p,q,x,y) \mapsto \ln(\mathcal{G}_{p,q}(x,y))$ is completely mixed monotonic on Ω, where*

$$\Omega = \{x, y > -\min(p,q),\, p,q,x,y \in \mathbb{R}\}.$$

Proof. To prove the desired assertion, we need an integral representation of $\ln \mathcal{G}_{p,q}(x,y)$:

$$\ln \mathcal{G}_{p,q}(x,y) = \int_0^\infty \eta_{p,q}(t)\,\eta_{x,y}(t)\,\frac{dt}{t(1-e^{-t})}, \tag{3.1}$$

where $\eta_{p,q}(t)$ is defined by (1.3). In fact, by using the integral representation of $\ln\Gamma(z)$ [22, p. 258, (6.1.50)]

$$\ln\Gamma(z) = \int_0^\infty \left[(z-1)e^{-t} - \frac{e^{-t}-e^{-zt}}{1-e^{-t}}\right]\frac{dt}{t} \quad (\mathrm{Re}(z) > 0), \tag{3.2}$$

we get that for $(x-y)(u-v)\neq 0$,

$$\ln \mathcal{G}_{p,q}(x,y)=\frac{\ln\Gamma(x+p)+\ln\Gamma(y+q)-\ln\Gamma(y+p)-\ln\Gamma(x+q)}{(p-q)(x-y)}$$

$$\begin{aligned}
&= \int_0^\infty \frac{e^{-t(x+p)}+e^{-t(y+q)}-e^{-t(y+p)}-e^{-t(x+q)}}{(p-q)(x-y)}\frac{dt}{t(1-e^{-t})}\\
&= \int_0^\infty \frac{e^{-tp}-e^{-tq}}{q-p}\frac{e^{-tx}-e^{-ty}}{y-x}\frac{dt}{t(1-e^{-t})}=\int_0^\infty \eta_{p,q}(t)\,\eta_{x,y}(t)\frac{dt}{t(1-e^{-t})},
\end{aligned}$$

which is also true for $(p-q)(x-y)=0$.

Now by (2.4) we have

$$\begin{aligned}
&(-1)^{n_1+n_2+n_3+n_4}\frac{\partial^{n_1+n_2+n_3+n_4}[\ln \mathcal{G}_{p,q}(x,y)]}{\partial p^{n_1}\partial q^{n_2}\partial x^{n_3}\partial y^{n_4}}\\
&= \int_0^\infty \left[(-1)^{n_1+n_2}\frac{\partial^{n_1+n_2}\eta_{p,q}(t)}{\partial p^{n_1}\partial q^{n_2}}\right]\left[(-1)^{n_3+n_4}\frac{\partial^{n_3+n_4}\eta_{x,y}(t)}{\partial x^{n_3}\partial y^{n_4}}\right]\frac{dt}{t(1-e^{-t})}>0,
\end{aligned}$$

which completes the proof. □

Remark 3. Theorem 1 yields $x,y\mapsto \ln\mathcal{G}_{p,q}(x,y)\in\mathcal{C}[(-\min(p,q),\infty)]$, that is to say, $\ln\mathcal{G}_{p,q}(x,y)$ is a CM function both in x and y on $(-\min(p,q),\infty)$. This was first proved in [10, Theorem 3] by Qi.

Theorem 2. *For fixed* $p,q,r,s\in\mathbb{R}$, $\rho=\min(p,q,r,s)$, *let* $\mathcal{G}_{p,q}(x,y)$ *be defined on* $(-\min(p,q),\infty)^2$ *by* (1.2). *Then* $\ln(\mathcal{G}_{p,q}/\mathcal{G}_{r,s})\in\mathcal{C}\left[(-\rho,\infty)^2\right]$ *if and only if*

$$p+q\leq r+s \quad \text{and} \quad \min(p,q)\leq\min(r,s). \tag{3.3}$$

Proof. From the integral representation (3.1), we obtain

$$\ln\frac{\mathcal{G}_{p,q}(x,y)}{\mathcal{G}_{r,s}(x,y)}=\int_0^\infty[\eta_{p,q}(t)-\eta_{r,s}(t)]\frac{\eta_{x,y}(t)}{t(1-e^{-t})}dt. \tag{3.4}$$

We first prove the sufficiency. By the integral representation (3.1) we easily find that

$$(-1)^n\frac{\partial^n\ln(\mathcal{G}_{p,q}/\mathcal{G}_{r,s})}{\partial x^k\partial y^{n-k}}=\int_0^\infty\frac{\eta_{p,q}(t)-\eta_{r,s}(t)}{t(1-e^{-t})}\left[(-1)^n\frac{\partial^n\eta_{x,y}(t)}{\partial x^k\partial y^{n-k}}\right]dt. \tag{3.5}$$

The application of Property 3 yields $\eta_{p,q}(t)-\eta_{r,s}(t)\geq 0$ if $p+q\leq r+s$ and $\min(p,q)\leq\min(r,s)$, while $\eta_{x,y}(t)$ is completely mixed monotonic in (x,y) on $(-\rho,\infty)^2$ due to Property 4. This proves the sufficiency.

The necessity can follow from the inequality $\ln(\mathcal{G}_{p,q}(x,x)/\mathcal{G}_{r,s}(x,x))\geq 0$, that is, $\mathcal{G}_{p,q}(x,x)\geq\mathcal{G}_{r,s}(x,x)$, for all $x>-\rho$. If $(p-q)(r-s)\neq 0$, then this is equivalent to

$$\frac{\psi(x+p)-\psi(x+q)}{p-q}\geq\frac{\psi(x+r)-\psi(x+s)}{r-s},$$

which can be changed into

$$\psi_1^{-1}\left(\frac{\int_q^p\psi_1(x+t)\,dt}{p-q}\right)-x\leq\psi_1^{-1}\left(\frac{\int_s^r\psi_1(x+t)\,dt}{r-s}\right)-x, \tag{3.6}$$

where $\psi_n=(-1)^{n-1}\psi^{(n)}$ is decreasing on $(0,\infty)$.

It has been shown in [20, Theorem 1.3] that

$$\lim_{x\to-\min(p,q)} A_{\psi_n}(x;p,q) = \lim_{x\to-\min(p,q)} \left[\psi_n^{-1}\left(\frac{\int_q^p \psi_n(x+t)\,dt}{p-q}\right)-x\right] = \min(p,q),$$

$$\lim_{x\to\infty} A_{\psi_n}(x;p,q) = \lim_{x\to\infty} \left[\psi_n^{-1}\left(\frac{\int_q^p \psi_n(x+t)\,dt}{p-q}\right)-x\right] = \frac{p+q}{2}.$$

Then letting $x\to\infty$ in inequality (3.6) yields

$$\frac{p+q}{2}\le\frac{r+s}{2},$$

which is the first necessary condition. We claim that the second one is $\min(p,q)\le\min(r,s)$. If not, that is, $\min(p,q)>\min(r,s)$, this indicates $\rho=\min(p,q,r,s)$ $=\min(r,s)$. Taking $x\to-\rho=-\min(r,s)$ in inequality (3.6) leads to

$$\psi_1^{-1}\left(\frac{\int_q^p \psi_1(t-\min(r,s))\,dt}{p-q}\right)+\min(r,s)\le\min(r,s),$$

which is obviously a contradiction. Clearly, this is also true if $(p-q)(r-s)=0$. We thus prove the desired assertion. $\square$

For $p\neq q$ and $s=r+1,r$, the ratio $\mathcal{G}_{p,q}(x,y)/\mathcal{G}_{r,s}(x,y)$ can be written as

$$\frac{\mathcal{G}_{p,q}(x,y)}{\mathcal{G}_{r,r+1}(x,y)} = \begin{cases} \left(\dfrac{y+r}{x+r}\right)^{1/(x-y)}\left[\dfrac{\Gamma(x+p)\Gamma(y+q)}{\Gamma(x+q)\Gamma(y+p)}\right]^{1/((p-q)(x-y))} & \text{if } x\neq y,\\ \exp\left(\dfrac{\psi(x+p)-\psi(x+q)}{p-q}-\dfrac{1}{x+r}\right) & \text{if } x=y,\end{cases}$$

$$\frac{\mathcal{G}_{p,q}(x,y)}{\mathcal{G}_{r,r}(x,y)} = \begin{cases} \left[\dfrac{\Gamma(x+p)\Gamma(y+q)}{\Gamma(x+q)\Gamma(y+p)}\right]^{1/((p-q)(x-y))}\exp\left[-\dfrac{\psi(x+r)-\psi(y+r)}{x-y}\right] & \text{if } x\neq y,\\ \exp\left[\dfrac{\psi(x+p)-\psi(x+q)}{p-q}-\psi'(x+r)\right] & \text{if } x=y.\end{cases}$$

Then by Theorem 2, the following corollaries are immediate.

Corollary 1. *For $p,q,r\in\mathbb{R}$, $\rho=\min(p,q,r)$, let $\mathcal{G}_{p,q}(x,y)$ be defined on $(-\min(p,q),\infty)^2$ by (1.2). Then $\ln(\mathcal{G}_{p,q}/\mathcal{G}_{r,r+1})\in\mathcal{C}\left[(-\rho,\infty)^2\right]$ if and only if $r\ge\max[(p+q-1)/2,\min(p,q)]$, while $\ln(\mathcal{G}_{r,r+1}/\mathcal{G}_{p,q})\in\mathcal{C}\left[(-\rho,\infty)^2\right]$ if and only if $r\le\min[(p+q-1)/2,\min(p,q)]$.*

Corollary 2. *For $p,q,r\in\mathbb{R}$, $\rho=\min(p,q,r)$, let $\mathcal{G}_{p,q}(x,y)$ be defined on $(-\min(p,q),\infty)^2$ by (1.2). Then $\ln(\mathcal{G}_{p,q}/\mathcal{G}_{r,r})\in\mathcal{C}\left[(-\rho,\infty)^2\right]$ if and only if $r\ge(p+q)/2$, while $\ln(\mathcal{G}_{r,r}/\mathcal{G}_{p,q})\in\mathcal{C}\left[(-\rho,\infty)^2\right]$ if and only if $r\le\min(p,q)$.*

The integral representation (3.4) together with $\eta_{x,x}(t) = te^{-xt}$ and (iii) of Property 2 gives

$$\ln \frac{\mathcal{G}_{p,q}(x,x)}{\mathcal{G}_{r,s}(x,x)} = \int_0^\infty [\eta_{p,q}(t) - \eta_{r,s}(t)] \frac{e^{-xt}}{1-e^{-t}} dt = \int_0^\infty [\eta_{p-\rho,q-\rho}(t) - \eta_{r-\rho,s-\rho}(t)] \frac{e^{-(x+\rho)t}}{1-e^{-t}} dt, \tag{3.7}$$

which, by Bernstein–Widder's Theorem [2, p. 161, Theorem 12b] and Property 3, implies that the following.

Theorem 3. *For fixed $p, q, r, s \in \mathbb{R}$, $\rho = \min(p,q,r,s)$, let $\mathcal{G}_{p,q}(x,y)$ be defined on $(-\min(p,q), \infty)^2$ by* (1.2). *Then the function $x \mapsto \ln[\mathcal{G}_{p,q}(x,x)/\mathcal{G}_{r,s}(x,x)]$ is completely monotonic on $(-\rho, \infty)$ if and only if the conditions of* (3.3) *are satisfied.*

Denote by

$$W_{u,v}(x) = \begin{cases} \left(\dfrac{\Gamma(x+u)}{\Gamma(x+v)}\right)^{1/(u-v)} & \text{if } u \neq v, \\ \exp[\psi(x+u)] & \text{if } u = v \end{cases} \tag{3.8}$$

for $x > -\min(u,v)$. It is evident that

$$\ln \frac{\mathcal{G}_{p,q}(x,x)}{\mathcal{G}_{r,s}(x,x)} = \frac{d}{dx}\left[\ln \frac{W_{p,q}(x)}{W_{r,s}(x)}\right] \quad \text{and} \quad \lim_{x\to\infty} \ln \frac{W_{p,q}(x)}{W_{r,s}(x)} = 0,$$

of which the latter follows from $\Gamma(x+p)/\Gamma(x+q) \sim x^{p-q}$ as $x \to \infty$. Then by Theorem 3 we immediately get the following corollary, which was proved in [21, Theorem 3.1].

Corollary 3. *For fixed $p, q, r, s \in \mathbb{R}$, $\rho = \min(p,q,r,s)$, let $W_{u,v}(x)$ be defined on $(-\min(u,v), \infty)$ by* (3.8). *Then $\ln(W_{p,q}/W_{r,s}) \in \mathcal{C}[(-\rho,\infty)]$ if and only if*

$$p + q \geq r + s \quad \text{and} \quad \min(p,q) \geq \min(r,s).$$

Theorem 4. *For fixed $p, q, r, s \in \mathbb{R}$, $\rho = \min(p,q,r,s)$, let $\mathcal{G}_{p,q}(x,y)$ be defined on $(-\min(p,q), \infty)^2$ by* (1.2). *Then $\mathcal{G}_{p,q}/\mathcal{G}_{r,s}$ is log-convex on $(-\rho,\infty)^2$ if and only if the conditions of* (3.3) *are satisfied.*

Proof. We first prove the sufficiency. It suffices to prove

$$\frac{\partial^2 \ln(\mathcal{G}_{p,q}/\mathcal{G}_{r,s})}{\partial x^2} \frac{\partial^2 \ln(\mathcal{G}_{p,q}/\mathcal{G}_{r,s})}{\partial y^2} - \left[\frac{\partial^2 \ln(\mathcal{G}_{p,q}/\mathcal{G}_{r,s})}{\partial x \partial y}\right]^2 \geq 0 \text{ and } \frac{\partial^2 \ln(\mathcal{G}_{p,q}/\mathcal{G}_{r,s})}{\partial x^2} > 0$$

if the conditions of (3.3) are satisfied. First, Property 3 and identity (2.4) yield

$$P(t) = \frac{\eta_{p,q}(t) - \eta_{r,s}(t)}{t(1-e^{-t})} > 0 \text{ and } \frac{\partial^2 \eta_{x,y}(t)}{\partial x^2} = t^3 \int_0^1 \theta^2 e^{\phi(\theta)} d\theta > 0,$$

where $\phi(\theta) = -(\theta x + (1-\theta) y) t$, which implies

$$\frac{\partial^2 \ln(\mathcal{G}_{p,q}/\mathcal{G}_{r,s})}{\partial x^2} = \int_0^\infty P(t) \frac{\partial^2 \eta_{x,y}(t)}{\partial x^2} dt > 0.$$

Second, application of the Cauchy–Schwarz inequality yields

$$\frac{\partial^2 \ln(\mathcal{G}_{p,q}/\mathcal{G}_{r,s})}{\partial x^2}\frac{\partial^2 \ln(\mathcal{G}_{p,q}/\mathcal{G}_{r,s})}{\partial y^2} = \int_0^\infty P(t)\frac{\partial^2 \eta_{x,y}(t)}{\partial x^2}dt\int_0^\infty P(t)\frac{\partial^2 \eta_{x,y}(t)}{\partial y^2}$$

$$\geq \left(\int_0^\infty P(t)\sqrt{\frac{\partial^2 \eta_{x,y}(t)}{\partial x^2}\frac{\partial^2 \eta_{x,y}(t)}{\partial y^2}}dt\right)^2 \geq \left(\int_0^\infty P(t)\frac{\partial^2 \eta_{x,y}(t)}{\partial x\partial y}dt\right)^2,$$

where the last inequality holds due to

$$\frac{\partial^2 \eta_{x,y}(t)}{\partial x^2}\frac{\partial^2 \eta_{x,y}(t)}{\partial y^2} - \left(\frac{\partial^2 \eta_{x,y}(t)}{\partial x\partial y}\right)^2$$

$$= \left(t^3\int_0^1 \theta^2 e^{\phi(\theta)}d\theta\right)\left(t^3\int_0^1 (1-\theta)^2 e^{\phi(\theta)}d\theta\right) - \left(t^3\int_0^1 \theta(1-\theta)e^{\phi(\theta)}d\theta\right)^2 \geq 0,$$

which proves the sufficiency.

Conversely, if $\mathcal{G}_{p,q}/\mathcal{G}_{r,s}$ is log-convex on $(-\rho,\infty)^2$, then $\partial^2\ln(\mathcal{G}_{p,q}/\mathcal{G}_{r,s})/\partial x^2 > 0$ for all $x, y > -\rho$. From the identity (2.4) we clearly see that $\lim_{x\to\infty}\partial\eta_{x,y}(t)/\partial x = \lim_{x\to\infty}\eta_{x,y}(t) = 0$. It then follows that $\ln(\mathcal{G}_{p,q}(x,y)/\mathcal{G}_{r,s}(x,y)) > 0$ for all $x, y > -\rho$. From the proof of necessity of Theorem 2, we find that this implies $p+q \leq r+s$ and $\min(p,q) \leq \min(r,s)$, which proves the necessity.

This completes the proof. □

4. Some new inequalities for gamma, psi, and polygamma functions

Over the past decades, bounding for certain ratios of gamma functions has been researched by many researchers; see for example, Wendel [23], Gautschi [24], Kečkić and Vasić [25], Kershaw [26], Lorch [27], Laforgia [28], Alzer [29], Elezović et al. [30], Batir [31], Qi et al. [32], [34], [33], Merkle [35], and Yang and Tian [37].

More bounding for such ratios can be found in Qi's review article [16] and recent papers [36], [17], [18], [20] and references therein.

From completely (mixed) monotonicity of those functions presented in the third section, we can deduce many known inequalities for gamma, psi, and polygamma functions. In this section, we only list some new inequalities to illustrate applications of our results.

Corollary 4. *Let $p > q > 0$ with $p+q-1 > 0$. Then the double inequality*

$$\frac{\Gamma(p)}{\Gamma(q)}\left(1+\frac{x}{r_1}\right)^{p-q} < \frac{\Gamma(x+p)}{\Gamma(x+q)} < \frac{\Gamma(p)}{\Gamma(q)}\left(1+\frac{x}{r_2}\right)^{p-q} \tag{4.1}$$

holds for $x > 0$ if $r_1 \geq \max(q,(p+q-1)/2)$ and $r_2 \leq \min(q,(p+q-1)/2)$.

In particular, for $p = 1, q \in (0,1)$, and $r_1 = q$, $r_2 = q/2$, we have

$$\frac{1}{\Gamma(q)}\left(1+\frac{x}{q}\right)^{1-q} < \frac{\Gamma(x+1)}{\Gamma(x+q)} < \frac{1}{\Gamma(q)}\left(1+\frac{2x}{q}\right)^{1-q} \tag{4.2}$$

for $x > 0$.

Proof. Corollary 1 tells us that the function $\ln\left(\mathcal{G}_{p,q}/\mathcal{G}_{r,r+1}\right) \in \mathcal{C}\left[(-\rho,\infty)^2\right]$ if and only if $r \geq \max\left(\left(p+q-1\right)/2, \min\left(p,q\right)\right)$, where $\rho = \min\left(q,r\right)$, and so $x \mapsto \ln\left(\mathcal{G}_{p,q}\left(x,y\right)/\mathcal{G}_{r,r+1}\left(x,y\right)\right) \in \mathcal{C}\left[(-\rho,\infty)\right]$ for $y > -\rho$. Therefore, when $r \geq \max\left(\left(p+q-1\right)/2, q\right)$, we obtain that for $x > 0$,

$$0 = \lim_{x\to\infty} \ln \frac{\mathcal{G}_{p,q}\left(x,y\right)}{\mathcal{G}_{r,r+1}\left(x,y\right)} < \ln \frac{\mathcal{G}_{p,q}\left(x,y\right)}{\mathcal{G}_{r,r+1}\left(x,y\right)} < \frac{\mathcal{G}_{p,q}\left(0,y\right)}{\mathcal{G}_{r,r+1}\left(0,y\right)},$$

that is,

$$\begin{aligned} 1 &< \left(\frac{y+r}{x+r}\right)^{1/(x-y)} \left[\frac{\Gamma\left(x+p\right)\Gamma\left(y+q\right)}{\Gamma\left(x+q\right)\Gamma\left(y+p\right)}\right]^{1/((p-q)(x-y))} \\ &< \left(\frac{y+r}{r}\right)^{-1/y} \left[\frac{\Gamma\left(p\right)\Gamma\left(y+q\right)}{\Gamma\left(q\right)\Gamma\left(y+p\right)}\right]^{-1/((p-q)y)}. \end{aligned} \tag{4.3}$$

Letting $y = 0$ and $r = r_1$ in the left-hand side inequality in (4.3) yields

$$1 < \left(\frac{r_1}{x+r_1}\right)^{1/x} \left[\frac{\Gamma\left(x+p\right)\Gamma\left(q\right)}{\Gamma\left(x+q\right)\Gamma\left(p\right)}\right]^{1/((p-q)x)},$$

which is equivalent to the left-hand side inequality in (4.1).

Similarly, by the complete monotonicity of $x \mapsto \ln\left(\mathcal{G}_{p,q}\left(x,y\right)/\mathcal{G}_{r,r+1}\left(x,y\right)\right)$ on $(-\rho,\infty)$ for $r \leq \min\left(\left(p+q-1\right)/2, \min\left(p,q\right)\right)$, we can obtain the right-hand side inequality in (4.1).

Taking $p = 1, q \in (0,1)$, $r_1 = \max\left(q, q/2\right) = q$, and $r_2 = \min\left(q, q/2\right) = q/2$ in (4.1) gives (4.2), which completes the proof. $\square$

Corollary 5. *Let $t \in (0,1)$, and let $y \in (0,\infty)$. Then we have*

$$\left(\frac{2y}{t+2y}\right)^{2(2y+1)} e^{t(2+\psi(1+y))} < \frac{\Gamma\left(t+y\right)}{\Gamma\left(y\right)} < \left(\frac{y}{t+y}\right)^{y+1} e^{t(1+\psi(1+y))}, \tag{4.4}$$

$$\left(\frac{y+1}{y+t}\right)^{y+1} e^{-(1-t)(1+\psi(1+y))} < \frac{\Gamma\left(t+y\right)}{\Gamma\left(1+y\right)} < \left(\frac{2y+1}{2y+t}\right)^{2(2y+1)} e^{-(1-t)(2+\psi(1+y))}. \tag{4.5}$$

Proof. Letting $p = 1$, $q \in (0,1)$, and $x \to y$ in the first inequality in (4.3) gives

$$1 < \exp\left(\frac{\psi\left(1+y\right) - \psi\left(q+y\right)}{1-q} - \frac{1}{r_1+y}\right)$$

for $y > 0$ and $r_1 \geq \max\left(\min\left(p,q\right), \left(p+q-1\right)/2\right) = q$, which is reduced to

$$\frac{1}{r_1+y} < \frac{\psi\left(1+y\right) - \psi\left(q+y\right)}{1-q}.$$

This is reversed for $r_2 \leq \min\left(\min\left(p,q\right), \left(p+q-1\right)/2\right) = q/2$. Taking $r_1 = q$ and $r_2 = q/2$ we have

$$\frac{1}{q+y} < \frac{\psi\left(1+y\right) - \psi\left(q+y\right)}{1-q} < \frac{1}{q/2+y},$$

which can be changed into

$$\psi\left(1+y\right) - \frac{1-q}{q/2+y} < \psi\left(q+y\right) < \psi\left(1+y\right) - \frac{1-q}{q+y}.$$

Then integrating over $[0,t]$ ($t\in[0,1]$) for q yields

$$\begin{aligned}&(2+\psi(1+y))\,t-2(2y+1)\ln\left(\frac{t}{2y}+1\right)\\ <\ &\ln\frac{\Gamma(t+y)}{\Gamma(y)}<(1+\psi(1+y))\,t-(y+1)\ln\left(\frac{t}{y}+1\right),\end{aligned}$$

$$(3-\gamma)\,t-6\ln(t+2)<\ln\Gamma(t+1)<(2-\gamma)\,t-2\ln(t+1),$$

which implies (4.4).

Analogously, then integrating over $[t,1]$ ($t\in[0,1]$) for q yields

$$\begin{aligned}&(2+\psi(1+y))(1-t)+2(2y+1)\ln\frac{2y+t}{2y+1}\\ <\ &\ln\frac{\Gamma(1+y)}{\Gamma(t+y)}<(1-t)(1+\psi(1+y))+(y+1)\ln\frac{y+t}{y+1},\end{aligned}$$

which implies (4.5). ☐

Remark 4. Putting $y=1$ in inequalities (4.4) and (4.5) we derive

$$\frac{e^{(3-\gamma)t}}{(1+t/2)^6}<\Gamma(t+1)<\frac{e^{(2-\gamma)t}}{(1+t)^2},\tag{4.6}$$

$$4\frac{e^{-(2-\gamma)(1-t)}}{(t+1)^2}<\Gamma(t+1)<3^6\frac{e^{-(3-\gamma)(1-t)}}{(t+2)^6}\tag{4.7}$$

for $t\in(0,1)$. Note that

$$\frac{d}{dt}\left[\frac{e^{(3-\gamma)t}}{(1+t/2)^6}\right]=64(3-\gamma)\frac{e^{(3-\gamma)t}}{(t+2)^7}\left(t-\frac{2\gamma}{3-\gamma}\right),$$

which implies that the lower bound given in (4.6) has a minimum value at $t=2\gamma/(3-\gamma)\approx 0.476\,49$, that is,

$$\frac{e^{(3-\gamma)t}}{(1+t/2)^6}\geq\left[\frac{e^{(3-\gamma)t}}{(1+t/2)^6}\right]_{t=2\gamma/(3-\gamma)}=\frac{(3-\gamma)^6}{729}e^{2\gamma}\approx 0.880\,08.$$

Therefore, we obtain a constant lower bound for $\Gamma(t+1)$ on $(0,1)$:

$$0.88008\approx\frac{(3-\gamma)^6}{729}e^{2\gamma}\leq\frac{e^{(3-\gamma)t}}{(1+t/2)^6}<\Gamma(t+1).$$

This constant lower bound is superior to one given in [38, Corollary 3], since it is more concise and

$$\frac{(3-\gamma)^6}{729}e^{2\gamma}-\left(2\frac{\sqrt{\gamma(1-\gamma)}-(1-\gamma)}{2\gamma-1}\right)^{\gamma(1-\gamma)/(2\gamma-1)}\approx 4.183\,6\times10^{-5}>0.$$

Similarly, the lower bound given in (4.7) has a minimum value at $t=\gamma/(2-\gamma)\approx 0.405\,69$, which implies that for $t\in(0,1)$,

$$\Gamma(t+1)>4\frac{e^{-(2-\gamma)(1-t)}}{(t+1)^2}\geq\left[4\frac{e^{-(2-\gamma)(1-t)}}{(t+1)^2}\right]_{t=\gamma/(2-\gamma)}=(2-\gamma)^2e^{2\gamma-2}\approx 0.869\,07.$$

This constant lower bound is superior to another one, $e^{-\gamma/4}$, given in [38, Corollary 3].

Corollary 6. *If $r \geq 1$, then the double inequality*

$$\frac{1}{x-1}\ln\left(\frac{x+r}{1+r}\right) < \frac{\psi(x+1)-\psi(2)}{x-1} < \ln\frac{re}{1+r} + \frac{1}{x-1}\ln\left(\frac{x+r}{1+r}\right) \tag{4.8}$$

holds for $x>0$ with $x \neq 1$. It is reversed if $0<r\leq 1/2$. In particular, we have

$$\psi(2)+\ln\left(\frac{x+1}{2}\right) < (>)\, \psi(x+1) < (>)\, \psi(2)+\ln\left(\frac{2x+1}{3}\right) \tag{4.9}$$

for $0<x>(<)\,1$.

Proof. Taking $y=1$ and $p\to q=1$ in the double inequality (4.3) yields

$$1 < \left(\frac{1+r}{x+r}\right)^{1/(x-1)} \exp\left(\frac{\psi(x+1)-\psi(2)}{x-1}\right) < \frac{re}{1+r}$$

for $x>0$ with $x\neq 1$ and $r \geq \max(\min(p,q), (p+q-1)/2) = 1$, which is equivalent to (4.8). It is reversed if $r \leq \min(\min(p,q), (p+q-1)/2) = 1/2$.

Let $r=1$ and $r=1/2$ in (4.8) and its reverse, respectively. Then we obtain

$$\frac{1}{x-1}\ln\left(\frac{x+1}{2}\right) < \frac{\psi(x+1)-\psi(2)}{x-1} < \frac{1}{x-1}\ln\left(\frac{x+1/2}{1+1/2}\right)$$

for $x>0$ with $x\neq 1$, which proves (4.9). $\square$

As a direct consequence of Theorem 3, we have the following.

Corollary 7. *For $(p-q)(r-s)\neq 0$ and every nonnegative integer n, the inequality*

$$(-1)^n \frac{\psi^{(n)}(x+p)-\psi^{(n)}(x+q)}{p-q} > (<)\, (-1)^n \frac{\psi^{(n)}(x+r)-\psi^{(n)}(x+s)}{r-s} \tag{4.10}$$

holds for $x>(-\min(p,q,r,s),\infty)$ if and only if $p+q\leq(\geq)\, r+s$ and $\min(p,q)\leq (\geq)\min(r,s)$. In particular, the double inequality

$$\frac{n!}{(x+r_1)^{n+1}} < (-1)^n \frac{\psi^{(n)}(x+p)-\psi^{(n)}(x+q)}{p-q} < \frac{n!}{(x+r_2)^{n+1}} \tag{4.11}$$

holds for $x>-\min(p,q,r_1,r_2)$ if and only if $r_1 \geq \max(\min(p,q),(p+q-1)/2)$ and $r_2 \leq \min(\min(p,q),(p+q-1)/2)$, while the inequalities

$$(-1)^n\psi^{(n+1)}(x+r_1) < (-1)^n \frac{\psi^{(n)}(x+p)-\psi^{(n)}(x+q)}{p-q} < (-1)^n\psi^{(n+1)}(x+r_2) \tag{4.12}$$

hold for $x>-\min(p,q,r_1,r_2)$ if and only if $r_1\geq(p+q)/2$ and $r_2\leq\min(p,q)$.

Proof. The first assertion is an immediate consequence of Theorem 3. Letting $(r,s)=(r_i,r_i+1)$ $(i=1,2)$ in (4.10) and using the recurrence formula

$$\psi^{(n)}(x+1)-\psi^{(n)}(x) = (-1)^n\frac{n!}{x^{n+1}}$$

yield the inequalities (4.11), while inequalities (4.12) follow by putting $(r,s)\to(r_i,r_i)$ $(i=1,2)$ in (4.10). $\square$

Remark 5. Clearly, inequalities (4.11) slightly improve the ones given in [13, Theorem 3]. Inequalities (4.10) can be rewritten as

$$\psi_{n+1}(x+r_1) < \frac{\int_q^p \psi_{n+1}(x+t)\,dt}{p-q} < \psi_{n+1}(x+r_2),$$

where $\psi_n = (-1)^{n-1}\psi^{(n)}$, which was proved in [20, Corollary 1.4] for $r_1 = (p+q)/2$ and $r_2 = \min(p,q)$.

The following is a consequence of the log-convexity of $\mathcal{G}_{p,q}/\mathcal{G}_{r,s}$ in (x,y) on $(-\rho,\infty)^2$ given in Theorem 4.

Corollary 8. *Let $p,q,r,s,x,y \in \mathbb{R}$ with $(p-q)(r-s)(x-y) \neq 0$ and $\rho = \min(p,q,r,s)$.*

(i) *The inequality*

$$\begin{aligned}(4.13)\qquad &\frac{1}{p-q}\int_q^p \left[\frac{\int_y^x \psi'(u+v)dv}{x-y} - \psi'\left(u+\tfrac{x+y}{2}\right)\right]du \\ &\quad > (<)\, \frac{1}{r-s}\int_s^r \left[\frac{\int_y^x \psi'(u+v)dv}{x-y} - \psi'\left(u+\tfrac{x+y}{2}\right)\right]du\end{aligned}$$

holds for $x,y>-\rho$ if and only if $p+q \le (\ge)\, r+s$ and $\min(p,q) \le (\ge) \min(r,s)$.

(ii) *The double inequality*

$$\begin{aligned}(4.14)\qquad &\frac{1}{L(x+r_1,y+r_1)} - \frac{1}{A(x+r_1,y+r_1)} \\ &< \frac{1}{p-q}\int_q^p \left[\frac{\int_y^x \psi'(u+v)\,dv}{x-y} - \psi'\left(u+\frac{x+y}{2}\right)\right]du \\ &< \frac{1}{L(x+r_2,y+r_2)} - \frac{1}{A(x+r_2,y+r_2)}\end{aligned}$$

holds for $x > -\min(p,q,r_1,r_2)$ if and only if $r_1 \ge \max(\min(p,q),(p+q-1)/2)$ and $r_2 \le \min(\min(p,q),(p+q-1)/2)$, where $L(a,b) = (a-b)/(\ln a - \ln b)$ and $A(a,b) = (a+b)/2$ are the logarithmic and arithmetical means of distinct positive numbers a and b, respectively.

(iii) *The double inequality*

$$\begin{aligned}(4.15)\qquad &\frac{\psi(x+r_1)-\psi(y+r_1)}{x-y} - \psi'\left(\frac{x+y}{2}+r_1\right) \\ &< \frac{1}{p-q}\int_q^p \left[\frac{\int_y^x \psi'(u+v)\,dv}{x-y} - \psi'\left(u+\frac{x+y}{2}\right)\right]du \\ &< \frac{\psi(x+r_2)-\psi(y+r_2)}{x-y} - \psi'\left(\frac{x+y}{2}+r_2\right)\end{aligned}$$

holds for $x > -\min(p,q,r_1,r_2)$ if and only if $r_1 \ge (p+q)/2$ and $r_2 \le \min(p,q)$.

Proof.

(i) By Theorem 4 we obtain that for $x,y>-\rho$,

$$\frac{1}{2}\left[\ln\frac{\mathcal{G}_{p,q}(x,y)}{\mathcal{G}_{r,s}(x,y)} + \ln\frac{\mathcal{G}_{p,q}(y,x)}{\mathcal{G}_{r,s}(y,x)}\right] \ge (\le) \ln\frac{\mathcal{G}_{p,q}((x+y)/2,(x+y)/2)}{\mathcal{G}_{r,s}((x+y)/2,(x+y)/2)}$$

if $p+q \leq (\geq)\, r+s$ and $\min(p,q) \leq (\geq) \min(r,s)$. This together with $\mathcal{G}_{p,q}(x,y) = \mathcal{G}_{p,q}(y,x)$ and

$$\ln \mathcal{G}_{p,q}(x,y) = \frac{\int_q^p \int_y^x \psi'(u+v)\,du dv}{(p-q)(x-y)}$$

for $(p-q)(x-y) \neq 0$ yields (4.13), which proves the sufficiency.

Conversely, if inequality (4.13) holds for all $x, y > -\rho$, that is,

$$\ln \mathcal{G}_{p,q}(x,y) - \mathcal{G}_{p,q}\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \geq (\leq) \ln \mathcal{G}_{r,s}(x,y) - \ln \mathcal{G}_{r,s}\left(\frac{x+y}{2}, \frac{x+y}{2}\right).$$

Dividing by $(x-y)^2$ and letting $y \to x$ gives

$$\frac{1}{p-q}\int_q^p \frac{1}{24}\psi'''(u+x)\,du > (<) \frac{1}{r-s}\int_s^r \frac{1}{24}\psi'''(u+x)\,du,$$

which is written as

$$\psi_3^{-1}\left(\frac{\int_q^p \psi_3(u+x)\,du}{p-q}\right) - x < (>)\, \psi_3^{-1}\left(\frac{\int_s^r \psi_3(u+x)\,du}{r-s}\right) - x,$$

where $\psi_n = (-1)^{n-1}\psi^{(n)}$ is decreasing on $(0,\infty)$. Similar to proof of necessity of Theorem 2, it follows that $p+q \leq (\geq)\, r+s$ and $\min(p,q) \leq (\geq) \min(r,s)$, which proves the necessity.

(ii) Letting $(r,s) = (r_i, r_i+1)$ $(i = 1,2)$ in (4.13) and noting that

$$\ln \mathcal{G}_{r,r+1}(x,y) = \frac{1}{x-y}\ln\frac{x+r}{y+r} \quad \text{and} \quad \ln \mathcal{G}_{r,r+1}(x,x) = \frac{1}{x+r}$$

give the inequalities (4.14).

(iii) Taking $(r,s) \to (r_i, r_i)$ $(i = 1,2)$ in (4.13) and noting that

$$\ln \mathcal{G}_{r,r}(x,y) = \frac{\psi(x+r) - \psi(y+r)}{x-y} \quad \text{and} \quad \ln \mathcal{G}_{r,r}(x,x) = \psi'(x+r)$$

lead to (4.15).

This ends the proof. □

Remark 6. Let $p, q > 0$ with $p+q-1 > 0$. Then taking $(x,y) = (1,0)$ in the double inequality (4.14) and noting that

$$\int_0^1 \psi'(u+v)\,dv = \psi(u+1) - \psi(u) = \frac{1}{u},$$

we obtain

$$\ln\left(1+\frac{1}{r_1}\right) - \frac{1}{r_1+1/2} < \frac{1}{L(p,q)} - \frac{1}{p-q}\int_q^p \psi'\left(u+\frac{1}{2}\right)du < \ln\left(1+\frac{1}{r_2}\right) - \frac{1}{r_2+1/2}$$

if $r_1 \geq \max(\min(p,q), (p+q-1)/2)$ and $0 < r_2 \leq \min(\min(p,q), (p+q-1)/2)$.

Remark 7. Likewise, for $p > q > 0$, taking $(x,y) = (1,0)$ and $(r_1, r_2) = ((p+q)/2, q)$ in the double inequality (4.14) we conclude that

$$\frac{2}{p+q} - \psi'\left(\frac{p+q+1}{2}\right) < \frac{1}{L(p,q)} - \frac{1}{p-q}\int_q^p \psi'\left(u+\frac{1}{2}\right)du < \frac{1}{q} - \psi'\left(\frac{1}{2}+q\right).$$

Clearly, taking into account the left-hand side inequality above and that the Hermite–Hadamard inequality [39] due to ψ' is convex on $(0,\infty)$, we arrive at

$$0<\frac{\int_q^p \psi'(x+1/2)\,dx}{p-q}-\psi'\left(\frac{p+q+1}{2}\right)<\frac{1}{L(p,q)}-\frac{1}{A(p,q)}. \tag{4.16}$$

Remark 8. Letting $q=p+1$ in the double inequality (4.16) we have

$$\frac{2}{p+1/2}-\ln\left(1+\frac{1}{p}\right)<\psi'(p+1)<\frac{1}{p+1/2}$$

for $p>0$.

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College of Science and Technology, North China Electric Power University, Baoding, Hebei Province, 071051, People's Republic of China — and — Department of Science and Technology, State Grid Zhejiang Electric Power Company Research Institute, Hangzhou, Zhejiang, 310014, People's Republic of China

Email address: yzhkm@163.com

College of Science and Technology, North China Electric Power University, Baoding, Hebei Province, 071051, People's Republic of China

Email address: tianjf@ncepu.edu.cn