6. Pascal's Triangle

We now turn to a triangle that comes from polynomial equations in a very curious way.

In France, in the year 1635, the 13-year-old Blaise Pascal discovered a triangle of a new sort. In middle-school or high-school algebra you may have learned that

$$\begin{aligned} (x+1) &= x+1 \\ (x+1)^2 &= x^2+2x+1 \\ (x+1)^3 &= x^3+3x^2+3x+1 \end{aligned}$$

and so on...

If we lay out the coefficients of $(x+1)^n$, we get the following diagram, called Pascal's triangle:

$$\begin{array}{c} 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \end{array}$$

and so on...

In this form, we see that in each row after the first, the beginning and end number is 1, and each of the other numbers is the sum of the two numbers directly above it. Try adding up the numbers in each row. Will we always get a power of 2? One way to see why is to plug the value $x = 1$ into the equation $(x+1)^n$.

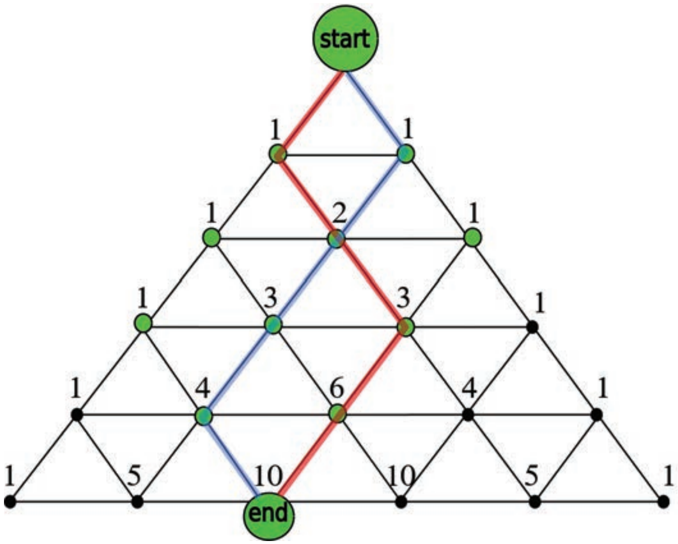
In the picture, the farmer of Pascal's gardens is still holding one of the pumpkins from the first row. I think he did not make the top shelf quite wide enough.

There are lots of different ways to think about how Pascal's triangle works. Some people may like the algebraic way shown on the previous page. It comes down to seeing the number of ways that a power of x appears in the expansion of $(1+x)^n$, for $n = 1, 2, 3, \ldots$.

Here is another way to look at the numbers in Pascal's triangle.

Imagine Pascal's triangle as paths going down a hill. At each crossing, there are two choices for how to continue downward.

A red and and a blue path are marked on the diagram below, starting at the top of the triangle, marked **start**, and going downward toward the crossing marked **end**. Can you see that there are 8 other paths from the **start** to the **end**? In fact, the numbers in Pascal's triangle tell you exactly how many paths there are from the top of the triangle to the position of the number.



Thinking about path counting may help you to see why each entry of Pascal's triangle is the sum of the entries diagonally above. Why are there '1' s going down both sides of the triangle? Do the rows read the same way backwards as forwards? Why?

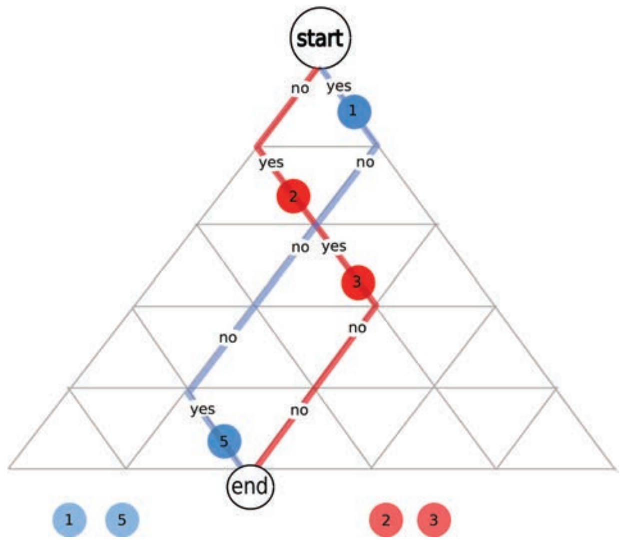
Here is a third way that Pascal's triangle is commonly used.

Consider 5 balls numbered $1, 2, 3, 4, 5$.

How many ways are there of choosing 2 of the 5 balls?

You can think of Pascal's triangle as a decision tree. As you go down the paths, you have two choices at each crossing. Think of each descent from one row to another as a choice made. First you choose whether to take ball 1, next whether to take ball 2, and so on. If choosing the right path means "yes" and choosing the left path means "no", then the red path is the choice of balls 2 and 3 and the blue path is the choice of 1 and 5.

It follows that the number of ways to choose 2 out of 5 balls (or *5 choose 2*) is the same as the number of ways to choose paths going downward from the top of Pascal's triangle to the crossing marked **end**. So 5 choose 2 equals 10.



In general to find n choose k, you go to the nth row down from the start, and move k crossings to the right from the leftmost crossing. Can you list the 6 ways of choosing 2 balls out of 4?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