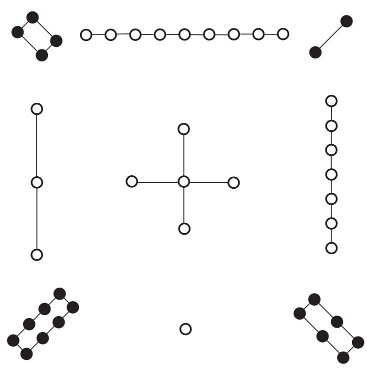
## Chapter 1 RECREATIONAL MATHEMATICS

*Recreational problems have survived, not because they were fostered by the textbook writers, but because of their inherent appeal to our love of mystery.*

**Vera Sanford**

Before taking up the noteworthy mathematical thinkers and their memorable problems, a brief overview of the history of mathematical recreations may benefit the reader. For more historical details see, *e.g.*, the books [6], [118], [133, Vol. 4], [153, Ch. VI], [167, Vol. II]. According to V. Sanford [153, Ch. VI], recreational mathematics comprises two principal divisions: those that depend on object manipulation and those that depend on computation.



| | | |
|---|---|---|
| 4 | 9 | 2 |
| 3 | 5 | 7 |
| 8 | 1 | 6 |

Figure 1.1. The oldest magic square—lo-shu

Perhaps the oldest known example of the first group is the magic square shown in the figure above. Known as *lo-shu* to Chinese mathematicians around 2200 B.C., the magic square was supposedly constructed during the reign of the Emperor Yii (see, *e.g.*, [61, Ch. II], or [167, Vol. I, p. 28]). Chinese myth [27] holds that Emperor Yii saw a tortoise of divine creation swimming in the Yellow River with the *lo-shu*, or magic square figure, adorning its shell. The figure on the left shows the *lo-shu* configuration where the numerals from 1 to 9 are composed of knots in strings with black knots for even and white knots for odd numbers.

The Rhind (or Ahmes) papyrus,[1] dating to around 1650 B.C., suggests that the early Egyptians based their mathematics problems in puzzle form. As these problems had no application to daily life, perhaps their main purpose was to provide intellectual pleasure. One of the earliest instances, named "*As I was going to St. Ives*", has the form of a nursery rhyme (see [153]):

> "Seven houses; in each are 7 cats; each cat kills 7 mice; each mouse would have eaten 7 ears of spelt; each ear of spelt will produce 7 hekat. What is the total of all of them?"[2]

The ancient Greeks also delighted in the creation of problems strictly for amusement. One name familiar to us is that of Archimedes, whose the *cattle problem* appears on pages 41 to 43. It is one of the most famous problems in number theory, whose complete solution was not found until 1965 by a digital computer.

The classical Roman poet Virgil (70 B.C.–19 B.C.) described in the *Aeneid* the legend of the Phoenician princess Dido. After escaping tyranny in her home country, she arrived on the coast of North Africa and asked the local ruler for a small piece of land, only as much land as could be encompassed by a bull's hide. The clever Dido then cut the bull's hide into the thinnest possible strips, enclosed a large tract of land and built the city of Carthage that would become her new home. Today the problem of enclosing the maximum area within a fixed boundary is recognized as a classical *isoperimetric problem*. It is regarded as the first problem in a new mathematical discipline, established 17 centuries later, as calculus of variations. Jacob Steiner's elegant solution of Dido's problem is included in this book.

Another of the problems from antiquity is concerned with a group of men arranged in a circle so that if every $k$th man is removed going around the circle, the remainder shall be certain specified (favorable) individuals. This problem, appearing for the first time in Ambrose of Milan's book *ca.* 370, is known as the Josephus problem, and it found its way not just into later European manuscripts, but also into Arabian and Japanese books. Depending on the time and location where the particular version of the Josephus problem was raised, the survivors and victims were sailors and

[1] Named after Alexander Henry Rhind (1833–1863), a Scottish antiquarian, layer and Egyptologist who acquired the papyrus in 1858 in Luxor (Egypt).

[2] T. Eric Peet's translation of *The Rhind Mathematical Papyrus,* 1923.

smugglers, Christians and Turks, sluggards and scholars, good guys and bad guys, and so on. This puzzle attracted attention of many outstanding scientists, including Euler, Tait, Wilf, Graham, and Knuth.

As Europe emerged from the Dark Ages, interest in the arts and sciences reawakened. In eighth-century England, the mathematician and theologian Alcuin of York wrote a book in which he included a problem that involved a man wishing to ferry a wolf, a goat and a cabbage across a river. The solution shown on pages 240–242 demonstrates how one can solve the problem accurately by using graph theory. River-crossing problems under specific conditions and constraints were very popular in medieval Europe. Alcuin, Tartaglia, Trenchant and Leurechon studied puzzles of this type. A variant involves how three couples should cross the river in a boat that cannot carry more than two people at a time. The problem is complicated by the jealousy of the husbands; each husband is too jealous to leave his wife in the company of either of the other men.

Four centuries later, mathematical puzzles appear in the third section of Leonardo Fibonacci's *Liber Abaci*, published in 1202. This medieval scholar's most famous problem, the *rabbit problem*, gave rise to the unending sequence that bears his name: the Fibonacci sequence, or Fibonacci numbers as they are also known, 1, 1, 2, 3, 5, 8, 13, 21, 34, ... (see pages 12–13).

Yet another medieval mathematician, ibn Khallikan (1211–1282), formulated a brain teaser requiring the calculation of the total number of wheat grains placed on a standard $8 \times 8$ chessboard. The placement of the grains must respect the following distribution: 1 grain is placed on the first square, 2 grains on the second, 4 on the third, 8 on the fourth, and so on, doubling the number for each successive square. The resulting number of grains is $2^{64} - 1$, or 18,446,744,073,709,551,615. Ibn Khallikan presented this problem in the form of the tale of the Indian king Shirham who wanted to reward the Grand Vizier Sissa ben Dahir for having invented chess. Sissa asked for the number of grains on the chessboard if each successive position is the next number in a geometric progression. However, the king could not fulfill Sissa's wish; indeed, the number of grains is so large that it is far greater than the world's annual production of wheat grains. Speaking in broad terms, ibn Khallikan's was one of the earliest chess problems.

Ibn Kallikan's problem of the number of grains is a standard illustration of geometric progressions, copied later by Fibonacci, Pacioli, Clavius and Tartaglia. Arithmetic progressions were also used in these entertaining problems. One of the most challenging problems appeared in Buteo's book *Logistica* (Lyons, 1559, 1560):[3]

[3]The translation from Latin is given in [153], p. 64.

"A mouse is at the top of a poplar tree 60 braccia[4] high, and a cat is on the ground at its foot. The mouse descends 1/2 of a braccia a day and at night it turns back 1/6 of a braccia. The cat climbs one braccia a day and goes back 1/4 of a braccia each night. The tree grows 1/4 of a braccia between the cat and the mouse each day and it shrinks 1/8 of a braccia every night. In how many days will the cat reach the mouse and how much has the tree grown in the meantime, and how far does the cat climb?"

At about the same time Buteo showed enviable knowledge of the general laws of permutations and combinations; moreover, he constructed a combination lock with movable cylinders displayed in Figure 1.2.[5]

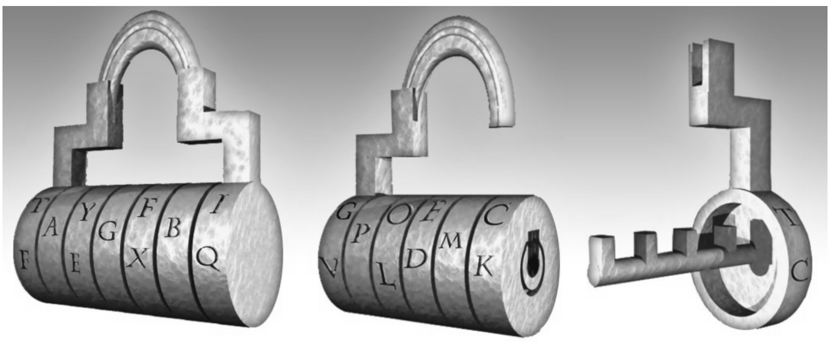
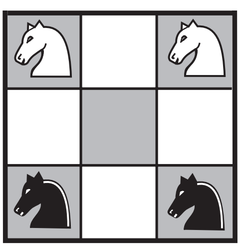


Figure 1.2. Buteo's combination lock (1559)

In 1512 Guarini devised a chessboard problem in which the goal is to effect the exchange of two black and two white knights, with each pair placed at the corners of a $3 \times 3$ chessboard (see figure left), in the minimum number of moves. The solution of this problem by using graph theory is shown on pages 274–276. People's interest in chess problems and the challenge they provide has lasted from the Middle Ages, through the Renaissance and to the present day.



While the Italian mathematicians Niccolo Tartaglia (1500–1557) and Girolamo Cardano (1501–1576) labored jointly to discover the explicit formula for the solution of cubic algebraic equations, they also found time for recreational problems and games in their mathematical endeavors. Tartaglia's *General Trattato* (1556) described several interesting tasks; four of which,

[4] *Braccia* is an old Italian unit of length.

[5] Computer artwork, sketched according to the illustration from Buteo's *Logistica* (Lyons, 1559, 1560).

the weighing problem, the division of 17 horses, the wine and water problem, and the ferryboat problem, are described on pages 20, 24, 25 and 173.

Girolamo Cardano was one the most famous scientists of his time and an inventor in many fields. Can you believe that the joint connecting the gear box to the rear axle of a rear wheel drive car is known to the present day by a version of his name—the cardan shaft? In an earlier book, *De Subtilitate* (1550), Cardano presented a game, often called the *Chinese ring puzzle* (Figure 1.3), that made use of a bar with several rings on it that remains popular even now. The puzzle's solution is closely related to Gray's error-correcting binary codes introduced in the 1930s by the engineer Frank Gray. The Chinese ring puzzle also bears similarities to the *Tower of Hanoi*, invented in 1883 by Edouard Lucas (1842–1891), which is also discussed later in the book.

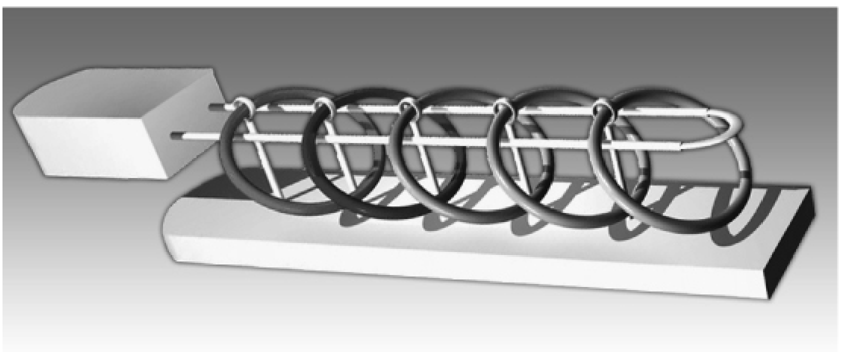


Figure 1.3. Chinese ring puzzle

Many scholars consider *Problèms Plaisans et Délectables,* by Claude Gaspar Bachet (1581–1638), to be the first book on mathematical puzzles and tricks. Most of the famous puzzles and curious problems invented before the seventeenth century may be found in Bachet's delightful book. In addition to Bachet's original "delectable" problems, the book contains puzzles by Alcuin of York, Pacioli, Tartaglia and Cardano, and other puzzles of Asian origin. Bachet's book, first published in 1612 and followed by the second edition published in 1624, probably served as the inspiration for subsequent works devoted to mathematical recreation.

Other important writers on the subject include the Jesuit scholar Jean Leurechon (1591–1670), who published under the name of Hendrik van Etten, and Jacques Ozanam (1640–1717). Etten's work, *Mathematical Recreations, or a Collection of Sundry Excellent Problems Out of Ancient and Modern Philosophers Both Useful and Recreative*, first published in French in 1624 with an English translation appearing in 1633, is a compilation of mathematical problems interspersed with mechanical puzzles and experiments in hydrostatics and optics that most likely borrowed heavily from Bachet's work.

Leonhard Euler (1707–1783), one of the world's greatest mathematicians whose deep and exacting investigations led to the foundation and development of new mathematical disciplines, often studied mathematical puzzles and games. Euler's results from the *seven bridges of Königsberg* problem (pages 230–232) presage the beginnings of graph theory. The *thirty-six officers problem* and orthogonal Latin squares (or Eulerian squares), discussed by Euler and later mathematicians, have led to important work in combinatorics. Euler's conjecture on the construction of mutually orthogonal squares found resolution nearly two hundred years after Euler himself initially posed the problem. These problems, and his examination of the chessboard *knight's re-entrant tour problem*, are described on pages 188 and 258. A knight's re-entrant path consists of moving a knight so that it moves successively to each square once and only once and finishes its tour on the starting square. This famous problem has a long history and dates back to the sixth century in India. P. M. Roget's half-board solution (1840), shown in Figure 1.4, offers a remarkably attractive design.

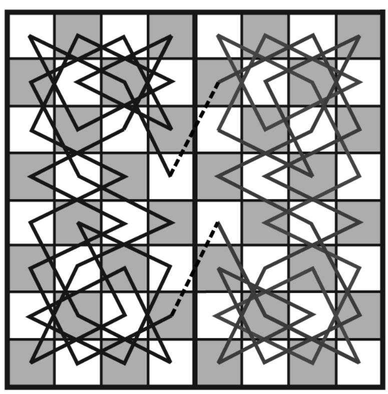


FIGURE 1.4. Knight's re-entrant path—Roget's solution

In 1850 Franz Nauck posed another classic chess problem, the *eight queens problem*, that calls for the number of ways to place eight queens on a chessboard so that no two queens attack each other. Gauss gave a solution of this problem, albeit incomplete in the first attempts. Further details about the *eight queens problem* appear on pages 269–273. In that same year, Thomas P. Kirkman (1806–1895) put forth the *schoolgirls problem* presented on pages

189 to 192. Several outstanding mathematicians, Steiner, Cayley and Sylvester among them, dealt with this combinatorial problem and other related problems. Although some of these problems remain unsolved even now, the subject continues to generate important papers on combinatorial design theory.

In 1857 the eminent Irish mathematician William Hamilton (1788–1856) invented the *icosian game* in which one must locate a path along the edges of a regular dodecahedron that passes through each vertex of the dodecahedron once and only once (see pages 234–237). As in Euler's Königsberg bridges problem, the Hamiltonian game is related to graph theory. In modern terminology, this task requires a Hamiltonian cycle in a certain graph and it is one of the most important open problems not only in graph theory but in the whole mathematics. The Hamiltonian cycle problem is closely connected to the famous *traveling salesman problem* that asks for an optimal route between some places on a map with given distances.

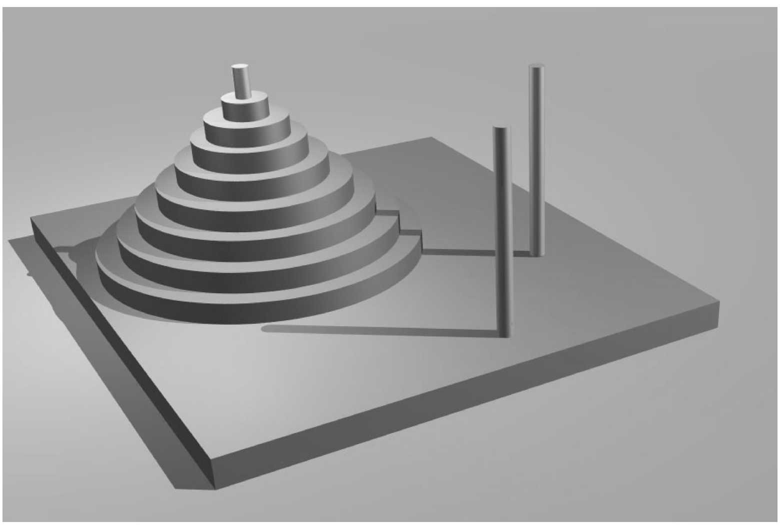
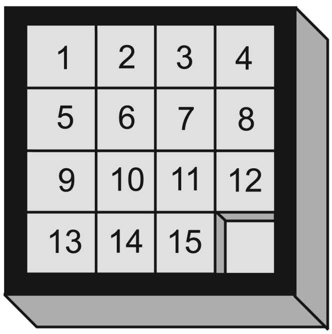


Figure 1.5. The Tower of Hanoi

The French mathematician François Edouard Lucas, best known for his results in number theory, also made notable contributions to recreational mathematics, among them, as already mentioned, the *Tower of Hanoi* (Figure 1.5), which is covered on pages 196–199, and many other amusing puzzles. Lucas' four-volume book *Récréations Mathématiques* (1882–94), together with Rouse Ball's, *Mathematical Recreations and Problems*, published in 1892, have become classic works on recreational mathematics.

No discussion of recreational mathematics would be complete without including Samuel Loyd (1841–1911) and Henry Ernest Dudeney (1857–1931), two of the most renowned creators of mathematical diversions. Loyd and Dudeney launched an impressive number of games and puzzles that remain as popular now as when they first appeared. Loyd's ingenious toy-puzzle the "15 Puzzle" (known also as the "Boss Puzzle", or "Jeu de Taquin") is popular even today. The "15 Puzzle" (figure below) consists of a square divided into 16 small squares and holds 15 square blocks numbered from 1 to 15. The task is to start from a given initial arrangement and set these numbered blocks into the required positions (say, from 1 to 15), using the vacant square for moving blocks. For many years after its appearance in 1878, people all over the world were obsessed by this toy-puzzle. It was played in taverns, factories, in homes, in the streets, in the royal palaces, even in the Reichstag (see page 2430 in [133, Vol. 4].




Martin Gardner (b. 1914 Tulsa, OK), most certainly deserves mention as perhaps the greatest twentieth-century popularizer of mathematics and mathematical recreations. During the twenty-five years in which he wrote his *Mathematical Games* column for the *Scientific American,* he published quantities of amusing problems either posed or solved by notable mathematicians.