Part 4

Problems for Children 5 to 15 Years Old

Problems

I wrote these problems in Paris in the spring of 2004. Some Russian residents of Paris had asked me to help cultivate a culture of thought in their young children. This tradition in Russia far surpasses similar traditions in the West.

I am deeply convinced that this culture is developed best through early and independent reflection on simple, but not easy, questions, such as are given below. (I particularly recommend Problems 1, 3, and 13.)

My long experience has shown that C-level students, lagging in school, can solve these problems better than outstanding students, because the survival in their intellectual "Kamchatka" at the back of the classroom "demanded more abilities than are requisite to govern Empires", as Figaro said of himself in the Beaumarchais play. A-level students, on the other hand, cannot figure out "what to multiply by what" in these problems. I have even noticed that five year olds can solve problems like this better than can school-age children, who have been ruined by coaching, but who, in turn, find them easier than college students who are busy cramming at their universities. (And Nobel prize or Fields Medal winners are the worst at all in solving such problems.)

1. Masha was seven kopecks short of the price of an alphabet book, and Misha was one kopeck short. They combined their money to buy one book to share, but even then they did not have enough. How much did the book cost?

2. A bottle with a cork costs \$1.10, while the bottle alone costs 10 cents more than the cork. How much does the cork cost?

3. A brick weighs one pound plus half a brick. How many pounds does the brick weigh?

4. A spoonful of wine from a barrel of wine is put into a glass of tea (which is not full). After that, an equal spoonful of the (non-homogeneous) mixture from the glass is put back into the barrel. Now there is a certain volume of "foreign" liquid in each vessel (wine in the glass and tea in the barrel). Is the volume of foreign liquid greater in the glass or in the barrel?

5. Two elderly women left at dawn, one traveling from A to B and the other from B to A. They were heading towards one another (along the same road). They met at noon, but did not stop, and each of them kept walking at the same speed as before. The first woman arrived at B at 4 PM, and the second arrived at A at 9 PM. At what time was dawn on that day?

6. The hypotenuse of a right-angled triangle (on an American standardized test) is 10 inches, and the altitude dropped to it is 6 inches. Find the area of the triangle.

American high school students had been successfully solving this problem for over a decade. But then some Russian students arrived from Moscow, and none of them was able to solve it as their American peers had (by giving 30 square inches as the answer). Why not?

7. Victor has 2 more sisters than he has brothers. How many more daughters than sons do Victor's parents have?

8. There is a round lake in South America. Every year, on June 1, a Victoria Regia flower appears at its center. (Its stem rises from the bottom, and its petals lie on the water like those of a water lily). Every day the area of the flower doubles, and on July 1, it finally covers the entire lake, drops its petals, and its seeds sink to the bottom. On what date is the area of the flower half that of the lake?

9. A peasant must take a wolf, a goat and a cabbage across a river in his boat. However the boat is so small that he is able to take only one of the three on board with him. How can he transport all three across the river? (The wolf cannot be left alone with the goat, and the goat cannot be left alone with the cabbage.)

10. During the daytime a snail climbs 3cm up a post. During the night it falls asleep and slips down 2cm. The post is 10m high, and a delicious sweet is waiting for the snail on its top. In how many days will the snail get the sweet?

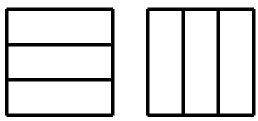
11. A hunter walked from his tent 10 km. south, then turned east, walked straight eastward 10 more km, shot a bear, turned north and after another 10 km found himself by his tent. What color was the bear and where did all this happen?

12. High tide occurred today at 12 noon. What time will it occur (at the same place) tomorrow?

13. Two volumes of Pushkin, the first and the second, are side-by-side on a bookshelf. The pages of each volume are 2cm thick, and the front and back covers are each 2mm thick. A bookworm has gnawed through (perpendicular to the pages) from the first page of volume 1 to the last page of volume 2. How long is the bookworm's track? [This topological problem with an incredible answer–4 mm–is totally impossible for academicians, but some preschoolers handle it with ease.]

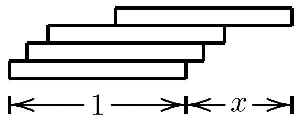
14. Viewed from above and from the front, a certain object (a polyhedron) gives the shapes shown. Draw its shape as viewed from the side. (Hidden edges of the polyhedron are to be shown as dotted lines.)

15. How many ways are there to break the number 64 up into the sum of ten natural numbers, none of which is greater than 12? Sums which differ only in the order of the addends are not counted as different.

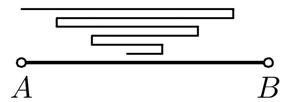


Top view Front view

To Problem 14



To Problem 16



To Problem 17

16. We have a number of identical bars (say, dominoes). We want to stack them so that the highest hangs out over the lowest by a length equal to x bar-lengths. What is the largest possible value of x?

17. The distance between towns A and B is 40km. Two cyclists leave from A and B simultaneously traveling towards one another, one at a speed of 10km/h and the other at a speed of 15km/h. A fly leaves A together with the first cyclist, and flies towards the second at a speed of 100km/h. The fly reaches the second cyclist, touches his forehead, then flies back to the first, touches his forehead, returns to the second, and so on until the cyclists collide with their foreheads and squash the fly. How many kilometers has the fly flown altogether?

18. Vanya solved a problem about two pre-school age children. He had to find their ages (which are integers), given the product of their ages.

Vanya said that this problem could not be solved. The teacher praised him for a correct answer, but added to the problem the condition that the name of the older child was Petya. Then Vanya could solve the problem right away. Now you solve it.

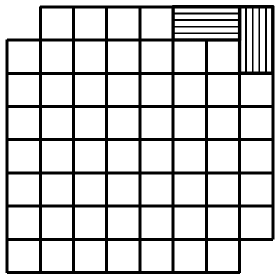
19. Is the number 140359156002848 divisible by 4206377084?

20. One domino covers two squares of a chessboard. Cover all the squares except for its two opposite corners (on the same diagonal) with 31 dominoes. (A chessboard consists of $8 \times 8 = 64$ squares.)

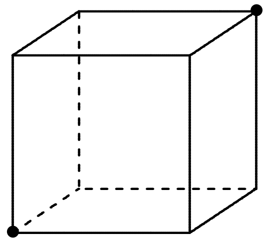
21. A caterpillar wants to slither from the front left corner of the floor of a cubical room to the opposite corner (the right rear corner of the ceiling). Find the shortest route for such a journey along the walls of the room.

22. You have two vessels of volumes 5 liters and 3 liters. Measure out one liter, leaving the liquid in one of the vessels.

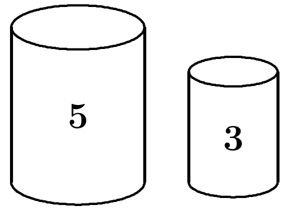
23. There are five heads and fourteen legs in a family. How many people and how many dogs are in the family?



To Problem 20



To Problem 21



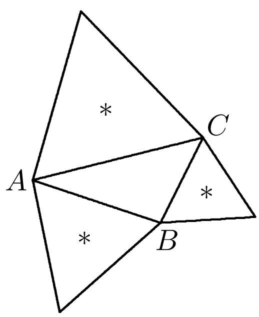
To Problem 22

24. Equilateral triangles are constructed externally on sides AB, BC, and CA of a triangle ABC. Prove that their centers (marked by asterisks on the diagram) form an equilateral triangle.

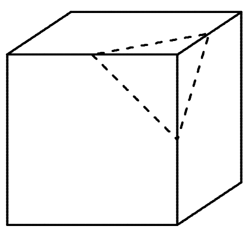
25. What polygons may be obtained as sections of a cube cut off by a plane? Can we get a pentagon? A heptagon? A regular hexagon?

26. Draw a straight line through the center of a cube so that the sum of the squares of the distances to it from the eight vertices of the cube is (a) maximal, (b) minimal (as compared with other such lines).

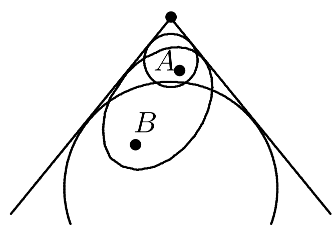
27. A right circular cone is cut by a plane along a closed curve. Two spheres inscribed in the cone are tangent to the plane, one at point A and the other at point B. Find a point C on the cross-section such that the sum of the distances $CA + CB$ is (a) maximal, (b) minimal.



To Problem 24



To Problem 25



To Problem 27

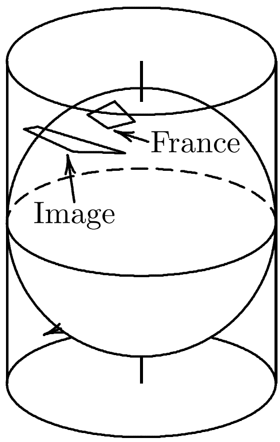
28. The Earth's surface is projected onto a cylinder formed by the lines tangent to the meridians at the points where they intersect the equator. The projection is made along rays parallel to the plane of the equator and passing through the axis of the earth that connects its north and south poles. Will the area of the projection of France be greater or less than the area of France itself?

29. Prove that the remainder upon division of the number 2^{p-1} by an odd prime p is 1 (for example: $2^2 = 3a + 1$, $2^4 = 5b + 1$, $2^6 = 7c + 1$, $2^{10} - 1 = 1023 = 11 \cdot 93$).

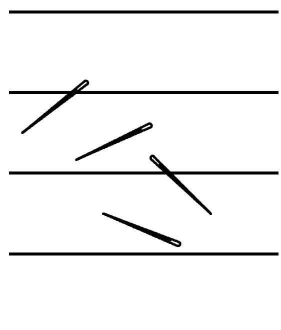
30. A needle 10 cm. long is thrown randomly onto ruled paper. The distance between neighboring lines on the paper is also 10 cm. This is repeated N (say, a million) times. How many times (approximately, up to a few per cent error) will the needle fall so that it intersects a line on the paper?

One can perform this experiment with $N = 100$ instead of a million throws. (I did this when I was 10 years old.)

[The answer to this problem is surprising: $\frac{2}{\pi}N$. Moreover, even for a curved needle of length $a \cdot 10$cm. the number of intersections observed over N throws will be approximately $\frac{2a}{\pi}N$. The number $\pi \approx \frac{355}{113} \approx \frac{22}{7}$.]



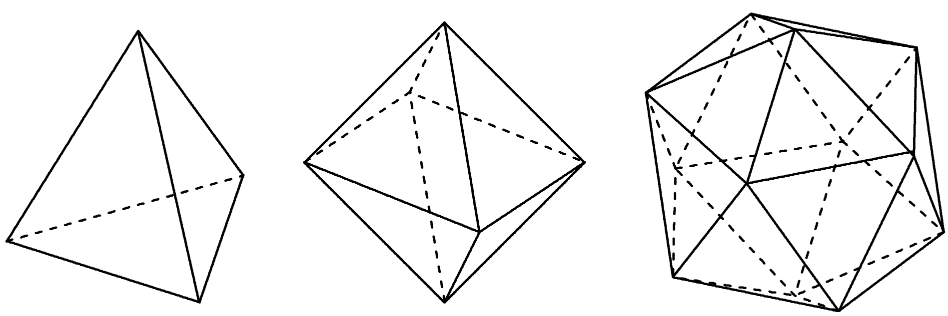
To Problem 28



To Problem 30

31. Some polyhedra have only triangular faces. Some examples are the Platonic solids: the (regular) tetrahedron (4 faces), the octahedron (8 faces), and the icosahedron (20 faces). The faces of the icosahedron are all identical, it has 12 vertices, and it has 30 edges.

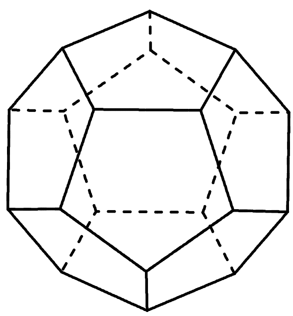
Is it true that for any such solid (a bounded convex polyhedron with triangular faces) the number of faces is equal to twice the number of vertices minus four?



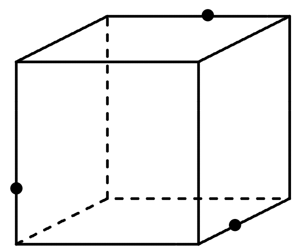
tetrahedron (tetra = 4) | octahedron (octa = 8) | icosahedron (icosa = 20)

32. There is one more Platonic solid (there are 5 of them altogether): a dodecahedron. It is a convex polyhedron with twelve (regular) pentagonal faces, twenty vertices and thirty edges (its vertices are the centers of the faces of an icosahedron).

Inscribe five cubes in a dodecahedron, whose vertices are also vertices of the dodecahedron, and whose edges are diagonals of faces of the dodecahedron. (A cube has 12 edges, one for each face of the dodecahedron). [This construction was invented by Kepler to describe his model of the planets.]



To Problem 32



To Problem 33^{bis}

33. Two regular tetrahedra can be inscribed in a cube, so that their vertices are also vertices of the cube, and their edges are diagonals of the cube's faces. Describe the intersection of these tetrahedra.

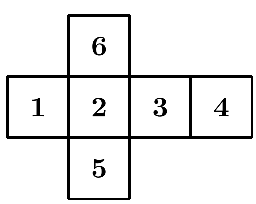
What fraction of the cube's volume is the volume of this intersection?

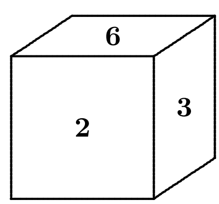
33^{bis}. Construct the section cut of a cube cut off by the plane passing through three given points on its edges. [Draw the polygon along which the plane intersects the faces of the cube.]

34. How many symmetries does a tetrahedron have? A cube? An octahedron? An icosahedron? A dodecahedron? A symmetry of a figure is a transformation of this figure preserving lengths.

How many of these symmetries are rotations, and how many are reflections in planes (in each of the five cases listed)?

35. How many ways are there to paint the six faces of similar cubes with six colors (1,. . . ,6) [one color per face] so that no two of the colored cubes obtained are the same (that is, no two can be transformed into each other by a rotation)?

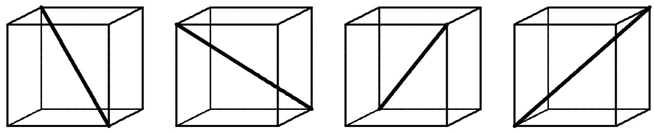




36. How many different ways are there to permute n objects?

For $n = 3$ there are six ways: $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(3, 2, 1)$. What if the number of objects is $n = 4$? $n = 5$? $n = 6$? $n = 10$?

37. A cube has 4 major diagonals (that connect its opposite vertices). How many different permutations of these four objects are obtained by rotations of a cube?



38. The sum of the cubes of several integers is subtracted from the cube of the sum of these numbers. Is this difference always divisible by 3?

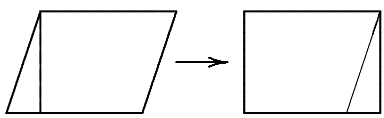
39. Answer the same question for the fifth powers and divisibility by 5, and for the seventh powers and divisibility by 7.

40. Calculate the sum

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{99 \cdot 100}$$

(with an error of not more than 1% of the correct answer).

41. If two polygons have equal areas, then they can be cut into a finite number of polygonal parts which may then be rearranged to obtain both the first and second polygons. Prove this. [For spatial solids this is not the case: the cube and the tetrahedron of equal volumes cannot be cut this way!]



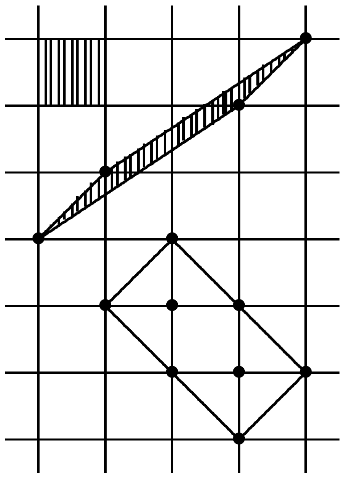
42. Four lattice points on a piece of graph paper are the vertices of a parallelogram. It turns out that there are no other lattice points either on the sides of the parallelogram or inside it. Prove that the area of such a parallelogram is equal to that of one of the squares of the graph paper.

43. Suppose, in Problem 42, there turn out to be a lattice points inside the parallelogram, and b lattice points on its sides. Find its area.

44. Is the statement analogous to the result of problem 43 true for parallelepipeds in 3-space?

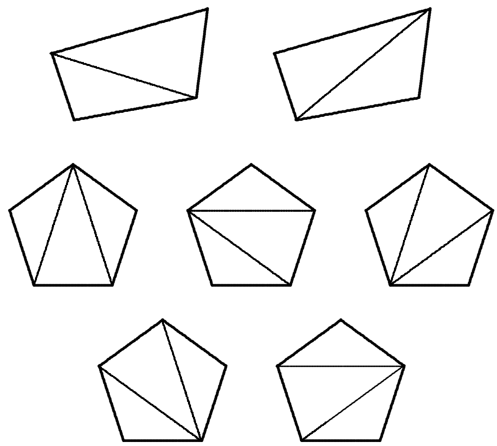
45. The Fibonacci ("rabbit") numbers are the sequence $(a_1 = 1)$, $1, 2, 3, 5, 8, 13, 21, 34, \ldots$, for which $a_{n+2} = a_{n+1} + a_n$ for any $n = 1, 2, \ldots$. Find the greatest common divisor of the numbers a_{100} and a_{99}.

46. Find the number of ways to cut a convex n-gon into triangles by cutting along non-intersecting diagonals. (These are the *Catalan* numbers, $c(n)$). For example, $c(4) = 2$, $c(5) = 5$, $c(6) = 14$. How can one find $c(10)$?



$a = 2, b = 2$

To Problems 42, 43



To Problem 46

47. There are n teams participating in a tournament. After each game, the losing team is knocked out of the tournament, and after $n-1$ games the team left is the winner of the tournament.

A schedule for the tournament may be written symbolically as (for example) $((a,(b,c)),d)$. This notation means that there are four teams participating. First b plays c, then the winner plays a, then the winner of this second game plays d.

How many possible schedules are there if there are 10 teams in the tournament?

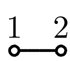
For 2 teams, we have only (a,b), and there is only one schedule.

For 3 teams, the only possible schedules are $((a,b),c)$, or $((a,c),b)$, or $((b,c),a)$, and are 3 possible schedules.

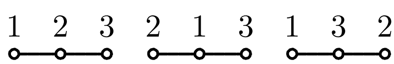
For 4 teams we have 15 possible schedules:

$$(((a,b),c),d);\quad (((a,c),b),\ d);\quad (((a,d),b),c);\quad (((b,c),a),d);$$
$$(((b,d),a),c);\quad (((c,d),a),b);\quad (((a,b),d),c);\quad (((a,c),d),b);$$
$$(((a,d),c),b);\quad (((b,c),d),a);\quad (((b,d),c),a);\quad (((c,d),b),a);$$
$$((a,b),(c,d));\quad ((a,c),\ (b,d));\quad ((a,d),(b,c)).$$

48. We connect n points $1, 2, \ldots, n$ with $n-1$ segments to form a tree. How many different trees can we get? (Even the case $n = 5$ is interesting!)

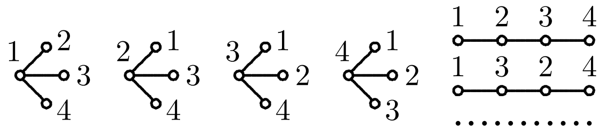
$n = 2$: 

the number $= 1$;

$n = 3$: 

the number $= 3$;

49. A permutation $(x_1, x_2, \ldots, x_n)$ of the numbers $\{1, 2, \ldots, n\}$ is called a *snake* (of length n) if $x_1 < x_2 > x_3 < x_4 > \cdots$.

$n = 4$:  the number $= 16$.

Examples. $n = 2$, only $1 < 2$, the number $= 1$;

$n = 3$, $\left.\begin{array}{l} 1 < 3 > 2 \\ 2 < 3 > 1 \end{array}\right\}$, the number $= 2$;

$n = 4$, $\left.\begin{array}{l} 1 < 3 > 2 < 4 \\ 1 < 4 > 2 < 3 \\ 2 < 3 > 1 < 4 \\ 2 < 4 > 1 < 3 \\ 3 < 4 > 1 < 2 \end{array}\right\}$, the number $= 5$;

Find the number of snakes of length 10.

50. Let s_n denote the number of snakes of length n, so that

$$s_1 = 1,\ s_2 = 1,\ s_3 = 2,\ s_4 = 5,\ s_5 = 16,\ s_6 = 61.$$

Prove that the Taylor series for the tangent function is

$$\tan x = 1\frac{x^1}{1!} + 2\frac{x^3}{3!} + 16\frac{x^5}{5!} + \cdots = \sum_{k=1}^{\infty} s_{2k-1}\frac{x^{2k-1}}{(2k-1)!}.$$

51. Find the sum of the series

$$1 + 1\frac{x^2}{2!} + 5\frac{x^4}{4!} + 61\frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} s_{2k}\frac{x^{2k}}{(2k)!}.$$

52. For $s > 1$, prove the identity

$$\prod_{p=2}^{\infty} \frac{1}{1 - \dfrac{1}{p^s}} = \sum_{n+1}^{\infty} \frac{1}{n^s}.$$

(The product is over all prime numbers p, and the summation over all natural numbers n.)

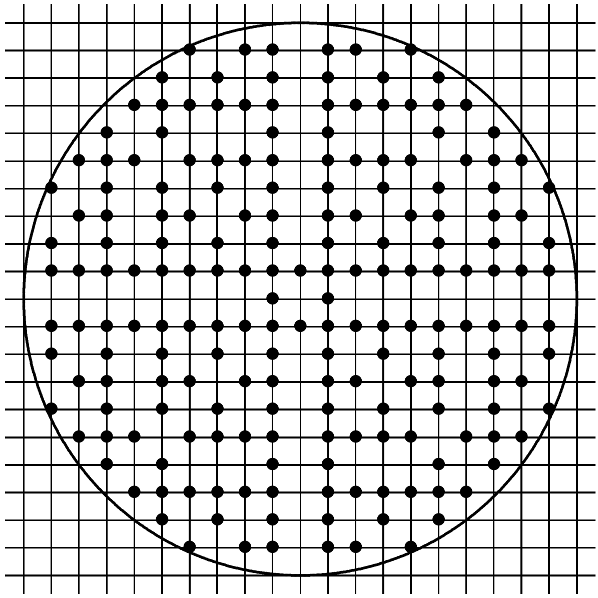
53. Find the sum of the series

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$\left(\text{Prove that it is equal to } \dfrac{\pi^2}{6}, \text{ or approximately } \dfrac{3}{2}\right)$.

54. Find the probability that the fraction $\dfrac{p}{q}$ is in lowest terms.

This probability is defined as follows: in the disk $p^2 + q^2 \leq R^2$, we count the number $N(R)$ of points with integer coordinates p and q not having a common divisor greater than 1. Then we take the limit of the ratio $N(R)/M(R)$, where $M(R)$ is the total number of integer points in the disk ($M \sim \pi R^2$).



$N(10) = 192$
$M(10) = 316$
$N/M = 192/316$
≈ 0.6076

55. The sequence of Fibonacci numbers was defined in problem 45. Find the limit of the ratio a_{n+1}/a_n as n approaches infinity:

$$\frac{a_{n+1}}{a_n} = 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \dots .$$

Answer: "The golden ratio", $\dfrac{\sqrt{5}+1}{2} \approx 1.618$.

This is the ratio of the sides of a postcard which stays similar to itself if we snip off a square whose side is the smaller side of the postcard.

How is the golden ratio related to a regular pentagon and a five-pointed star?

56. Calculate the value of the infinite continued fraction

$$1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}},$$

where $a_{2k} = 1$, $a_{2k+1} = 2$.

That is, find the limit as n approaches infinity of

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

57. Find the polynomials $y = \cos 3(\arccos x)$, $y = \cos 4(\arccos x)$, $y = \cos n(\arccos x)$, where $|x| \leq 1$.

58. Calculate the sum of the k^{th} powers of the n complex n^{th} roots of unity.

59. On the $(x, y)-$plane, draw the curves defined parametrically:

$$\{x = \cos 2t,\ y = \sin 3t\},\ \ \{x = t^3 - 3t,\ y = t^4 - 2t^2\}.$$

60. Calculate (with an error of not more than 10% of the answer)

$$\int_0^{2\pi} \sin^{100} x\ dx.$$

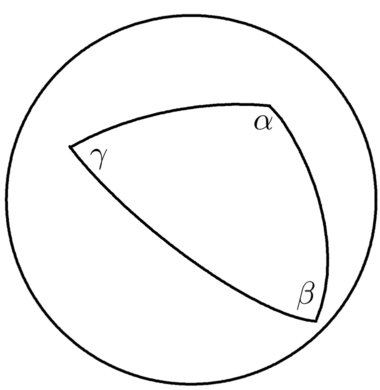
61. Calculate (with an error of not more than 10% of the answer)

$$\int_1^{10} x^x\ dx.$$

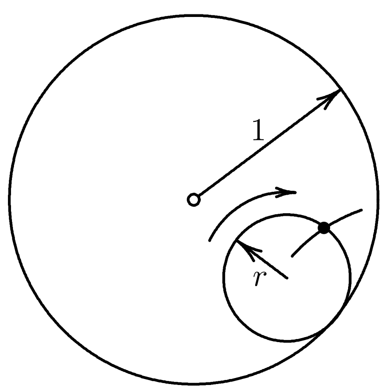
62. Find the area of a spherical triangle with angles (α, β, γ) on a sphere of radius 1. (The sides of such a triangle are great circles; that is, cross-sections of the sphere formed by planes passing through its center).

Answer: $S = \alpha + \beta + \gamma - \pi$. (For example, for a triangle with three right angles, $S = \pi/2$, that is, one-eighth of the total area of the sphere).

63. A circle of radius r rolls (without slipping) inside a circle of radius 1. Draw the whole trajectory of a point on the rolling circle (this trajectory is called a hypocycloid) for $r = 1/3$, $r = 1/4$ for $r = 1/n$, for $r = p/q$, and for $r = 1/2$.



To Problem 62



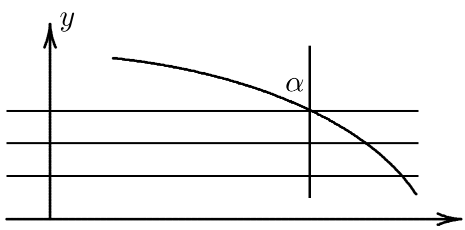
To Problem 63

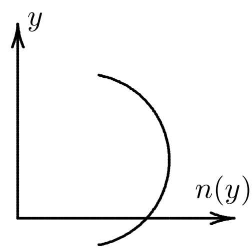
64. In a class of n students, estimate the probability that two students have the same birthday. Is this a high probability? Or a low one?

Answer: (Very) high if the number of the pupils is (well) above some number n_0, (very) low if it is (well) below n_0, and what this n_0 actually is (when the probability $p \approx 1/2$) is what the problem is asking.

65. Snell's law states that the angle α made by a ray of light with the normal to layers of a stratified medium satisfies the equation $n(y)\sin\alpha =$ const, where $n(y)$ is the index of refraction of the layer at height y. (The quantity n is inversely proportional to the speed of light in the medium if we take its speed in a vacuum to be 1. In water $n = 4/3$).

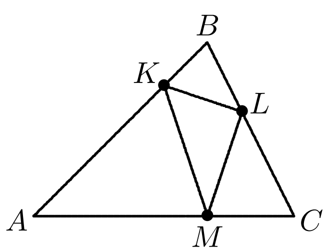
Draw the rays forming the light's trajectories in the medium "air above a desert", where the index $n(y)$ has a maximum at a certain height. (See the diagram on the right.)





(A solution to this problem explains the phenomenon of mirages to those who understand how trajectories of rays emanating from objects are related to their images).

66. In an acute angled triangle ABC inscribe a triangle KLM of minimal perimeter (with its vertex K on AB, L on BC, M on CA).



Hint: The answer for non-acute angled triangles is not nearly as beautiful as the answer for acute angled triangles.

67. Calculate the average value of the function $1/r$ (where $r^2 = x^2 + y^2 + z^2$ is the distance to the origin from the point with coordinates (x, y, z)) on the sphere of radius R centred at the point (X, Y, Z).

Hint: The problem is related to Newton's law of gravitation and Coulomb's law in electricity. In the two-dimensional version of the problem, the given function should be replaced by $\ln r$, and the sphere by a circle.

68. The fact that $2^{10} = 1024 \approx 10^3$ implies that $\log_{10} 2 \approx 0.3$. Estimate by how much they differ, and calculate $\log_{10} 2$ to three decimal places.

69. Find $\log_{10} 4$, $\log_{10} 8$, $\log_{10} 5$, $\log_{10} 50$, $\log_{10} 32$, $\log_{10} 128$, $\log_{10} 125$, and $\log_{10} 64$ with the same precision.

70. Using the fact that $7^2 \approx 50$, find an approximate value for $\log_{10} 7$.

71. Knowing the values of $\log_{10} 64$ and $\log_{10} 7$, find $\log_{10} 9, \log_{10} 3$, $\log_{10} 6, \log_{10} 27$, and $\log_{10} 12$.

72. Using the fact that $\ln(1+x) \approx x$ (where ln means $\log_e$), find $\log_{10} e$ and $\ln 10$ from the relation[16]

$$\log_{10} a = \frac{\ln a}{\ln 10}$$

and from the values of $\log_{10} a$ computed earlier (for example, for $a = 128/125$, $a = 1024/1000$ and so on).

Solutions to Problems 67–71 will give us, after a half hour of computation, a table of four-digit logarithms of any numbers using products of numbers whose logarithms have been already found as points of support and the formula

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

for corrections. (This is how Newton compiled a table of 40-digit logarithms!).

73. Consider the sequence of powers of two: 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, ... Among the first twelve numbers, four have decimal numerals starting with 1, and none have decimal numerals starting with 7.

Prove that in the limit as $n \to \infty$ each digit will be met with as the first digit of the numbers 2^m, $0 \le m \le n$, with a certain average frequency: $p_1 \approx 30\%, p_2 \approx 18\%, \dots, p_9 \approx 4\%$.

74. Verify the behavior of the first digits of powers of three: 1, 3, 9, 2, 8, 2, 7, ... Prove that, in the limit, here we also get certain frequencies and that the frequencies are same as for the powers of two. Find an exact formula for $p_1, \dots,\ p_9$.

Hint: The first digit of a number x is determined by the fractional part of the number $\log_{10} x$. Therefore one has to consider the sequence of fractional parts of the numbers ma, where $a = \log_{10} 2$.

Prove that these fractional parts are uniformly distributed over the interval from 0 to 1: of the n fractional parts of the numbers ma, $0 \le m < n$,

[16]Euler's constant $e = 2.71828\cdots$ is defined as the limit of the sequence $\left(1 + \frac{1}{n}\right)^n$ as $n \to \infty$. It is equal to the sum of the series $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$. It can also be defined by the given formula for $\ln(1+x)$: $\lim\limits_{x\to 0} \frac{\ln(1+x)}{x} = 1$.

a subinterval A will contain the quantity $k_n(A)$ such that as $n \to \infty$, $\lim(k_n(A)/n) =$ the length of the subinterval A.

75. Let $g : M \to M$ be a smooth map of a bounded domain M onto itself which is one-to-one and preserves areas (volumes in the multi-dimensional case) of domains.

Prove that in any neighborhood U of any point of M and for any N there exists a point x such that $g^T x$ is also in U for a certain integer $T > N$ (the "Recurrence Theorem").

76. Let M be the surface of a torus (with coordinates $\alpha \pmod{2\pi}, \beta \pmod{2\pi}$), and let $g(\alpha,\ \beta) = (\alpha + 1, \beta + \sqrt{2})$. Prove that for every point x of M the sequence of points $\{g^T(x)\}, T = 1, 2, \dots$ is everywhere dense on the torus.

77. In the notation of problem 76, let

$$g(\alpha,\ \beta) = (2\alpha + \beta, \alpha + \beta) \pmod{2\pi}.$$

Prove that there is an everywhere dense subset of the torus consisting of periodic points x (that is, such that $g^{T(x)} = x$ for some integer $T(x) > 0$).

78. In the notation of Problem 77 prove that, for almost all points x of the torus, the sequence of points $\{g^T(x)\},\ T = 1, 2, \dots$ is everywhere dense on the torus (that is, the points x without this property form a set of measure zero).

79. In Problems 76 and 78, prove that the sequence $\{g^T(x)\},\ T = 1, 2, \dots$ is distributed over the torus uniformly: if a domain A contains $k_n(A)$ points out of the n points with $T = 1, 2, \dots, n$, then

$$\lim_{n\to\infty} \frac{k_n(A)}{n} = \frac{\operatorname{mes} A}{\operatorname{mes} M}$$

(for example, for a Jordan measurable domain A of measure mes A).

Note to Problem 13. In posing this problem, I have tried to illustrate the difference in approaches to research by mathematicians and physicists in my invited paper in the journal "Advances in Physical Sciences" for the 2000 Centennial issue. My success far surpassed the goal I had in mind: the editors, unlike the preschool students on the experience with whom I based my plans, could not solve the problem. So they changed it to fit my answer of 4mm. in the following way: instead of "from the first page of the first volume to the last page of the second", they wrote "from the *last* page of the first volume to the *first* page of the second".

This true story is so implausible that I am including it here: the proof is the editors' version published by the journal.

Solutions to Selected Problems*

6. Such a triangle cannot exist: if the hypotenuse is 10 inches long, then the triangle can be inscribed into a half-disc of diameter 10 inches, and its altitude cannot exceed 5 inches.

8. On June 30.

10. Answer: after 998 days. Indeed, at the end of the first day the snail will be 3 centimeters high, at the end of the second day it will be 4 centimeters high, and so on. At the end of 998-th day the snail will be 10 meters high.

13. The answer 4 millimeters may seem unexpected, but look at Figure 41.

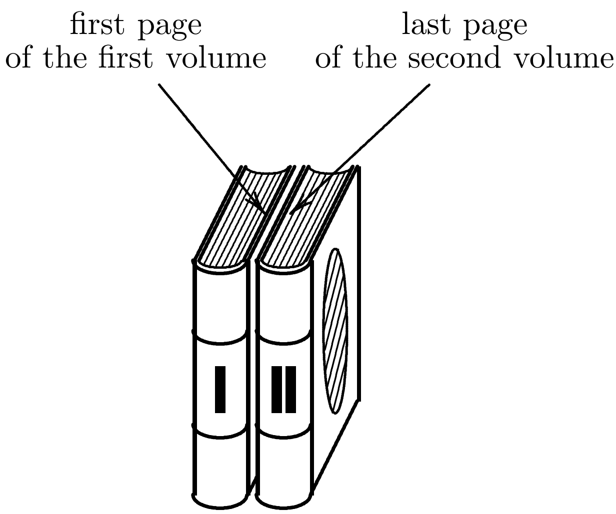


Figure 41. To Solution 13.

14. There are many such bodies. See an illustration (Figure 42) of one obtained from a cube by removing two triangular prisms. A side view is also shown.

15. Answer: 4,447. There are many different ways to count this number without listing all the partitions (although a computer program can do this in a fraction of a second). For example, one can use the following trick. Let

*Composed by Dmitry Fuchs

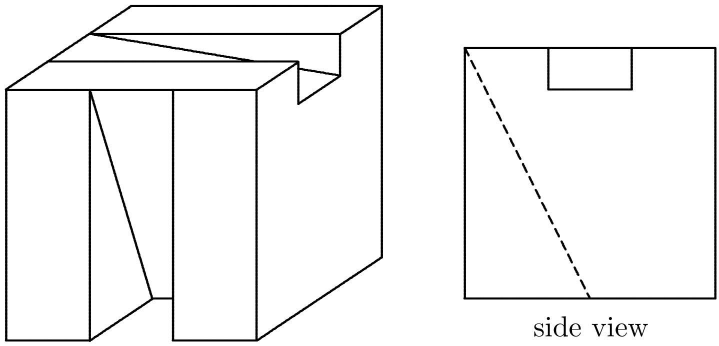


FIGURE 42. To Solution 14.

$P(n; m, k)$ be the number of partitions $n = a_1 + \cdots + a_k$ where the integers a_i satisfy the inequalities $m \geq a_1 \geq \cdots \geq a_k \geq 0$. What we need to find is $P(43; 11, 9)$. There is an obvious equality

$$P(n; m, k) = P(n; m, k-1) + P(n-k; m-1, k)$$

(which is obtained when we count separately partitions $m = a_1 + \cdots + a_k$ with $a_k = 0$ and $a_k \geq 1$). We apply it sufficiently many times:

$$\begin{aligned} P(43; 11, 9) &= P(43; 11, 8) + P(34; 10, 9) \\ &= P(43; 11, 7) + P(35; 10, 8) + P(34; 10, 8) + P(25; 9, 9) = \cdots \end{aligned}$$

(using, when necessary, the fact that $P(n; m, k) = 0$, if $n > mk$), and eventually we obtain

$$\cdots = 4447\, P(0; 0, 1) = 4447.$$

16. The length is unlimited. Indeed, consider a tower of $n+1$ identical plates of length 1, and introduce notations $x_1, \ldots, x_n$ as shown in Figure 43.

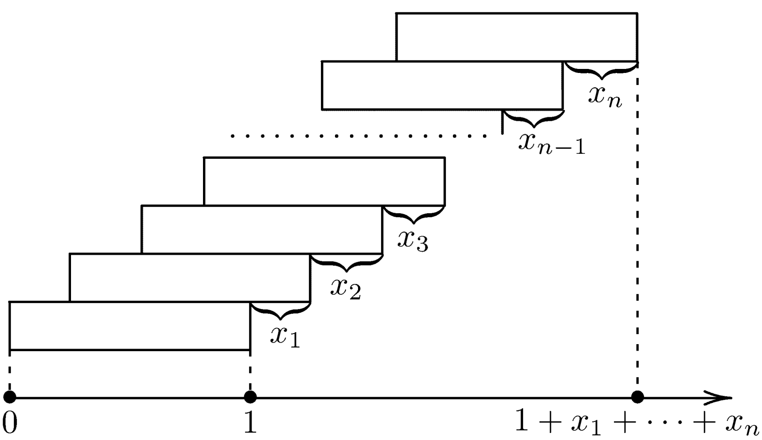


FIGURE 43. To Solution 16.

The length of the canopy is $x_1 + \cdots + x_n$.

For this tower being stable, it is needed that for $k = 1, \dots, n$ the mass center of the union of plates ## $k+1, \dots, n+1$ be located over some point of the plate #k. The horizontal coordinate of the mass center of the plate #k is $\frac{1}{2} + x_1 + \cdots + x_{k-1}$. Since all the plates are identical, the horizontal coordinate of the mass center of the union described is the mean of the horizontal coordinates of mass centers of the plates in this union, that is,

$$\frac{\left(\frac{1}{2} + x_1 + \cdots + x_k\right) + \cdots + \left(\frac{1}{2} + x_1 + \cdots + x_n\right)}{n-k+1}$$
$$= \frac{1}{2} + x_1 + \cdots + x_k + \frac{n-k}{n-k+1}\, x_{k+1} + \cdots + \frac{1}{n-k+1}\, x_n.$$

This last sum should be less than the horizontal coordinate of the right end of the plate #k, that is, less than $1 + x_1 + \cdots + x_{k-1}$. We arrive at the inequality

$$x_k + \frac{n-k}{n-k+1}\, x_{k+1} + \cdots + \frac{1}{n-k+1}\, x_n < \frac{1}{2},$$

that is,

$$(n-k+1)x_k + (n-k)x_{k+1} + \cdots + x_n < \frac{n-k+1}{2}. \tag{$*$}$$

Let us assume that for all j, $x_j < \dfrac{1}{2(n-j+1)}$. Then each of the $n-k+1$ summands in the left hand side of the inequality $(*)$ is less that one half, and the inequality holds.

We see that the length of the canopy can be any number less than

$$\frac{1}{2n} + \frac{1}{2(n-1)} + \cdots + \frac{1}{4} + \frac{1}{2} = \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).$$

It is well known that the sum in parentheses is unbounded, when n grows. Thus, the canopy may be arbitrarily long.

17. The cyclists met $\dfrac{40}{10+15} = 1.6$ hours after they started. Hence, the fly traveled $1.6 \cdot 100 = 160$ kilometers.

18. For Vanya, it was important to know not the name of the older boy, but rather the fact that the ages of the two boys were different.

19. The smaller number is divisible by 7, since 42, 63, 77, and 84 are divisible by 7. The bigger number is not divisible by 7, since 14, 35, 91, 56, and 28 are divisible by 7, but 48 is not divisible by 7. Therefore, the smaller number does not divide the bigger number.

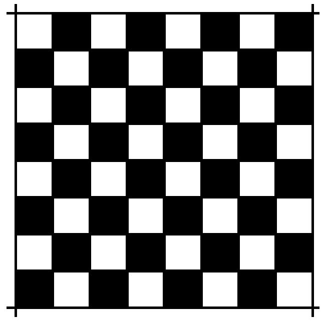


FIGURE 44. To Solution 20.

20. Consider the standard coloring of the chess board (see Figure 44). It is obvious that a domino covers one white square and one black square. But the domain we want to cover contains unequal numbers of white and black squares (30 and 32). Therefore, it cannot be covered with non-overlapping dominoes.

21. The shortest path has length $a\sqrt{5}$ where a is the length of the edge of the cube. Moreover, there are 6 different paths of this length (see the left side of Figure 45).

First, any path can be shortened if it contains a connected non-straight part within any face of the cube: just replace this part by a straight interval with the same ends. This means that a shortest path must consist of straight intervals within the faces with both ends on edges. The starting point belongs to three faces, and the movement has to begin by a straight interval within one of them. This interval ends at one of the two edges opposite to the starting point, and this endpoint belongs to a face containing the endpoint. For shortness sake, we must go from this point straight to the endpoint. Thus, our shortest path belongs to the union of two adjacent faces. The right side of Figure 45 shows that, to be the shortest, the path must pass through the midpoint of the connecting edge.

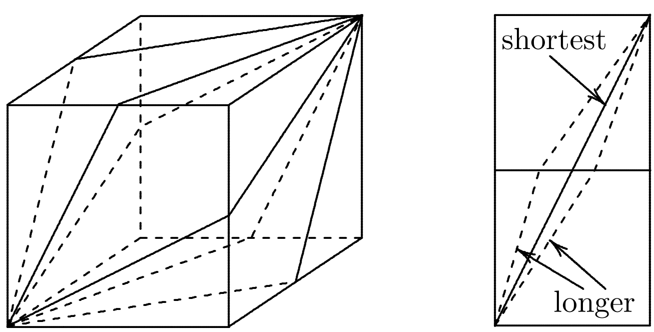


FIGURE 45. To Solution 21.

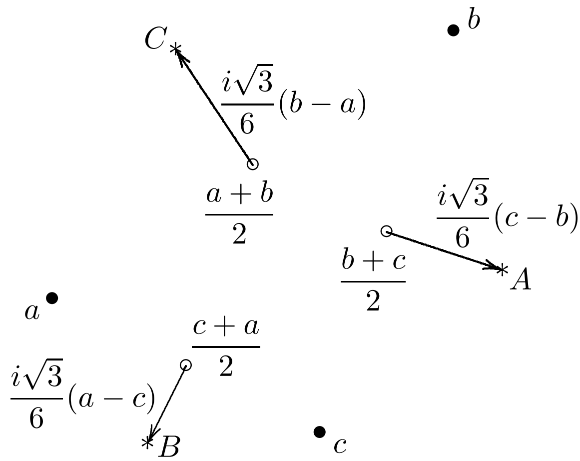


FIGURE 46. To Solution 24.

24. Let us think of the given triangle as drawn on the complex plane. Let a, b, c be complex numbers corresponding to its vertices. Then the midpoints of the sides will be $\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2}$, and the vectors from these midpoints to the vertices of the new triangle, which we denote by A, B, C, will be obtained from the sides $b-a, c-b, a-c$ by counterclockwise rotation by $90°$ (that is, multiplication by i) and multiplication by $\frac{\sqrt{3}}{6}$ (which is one third of the ratio of lengths of an altitude and a side in an equilateral triangle). All this is shown in Figure 46 above.

Thus,

$$A = \frac{b+c}{2} + \frac{i\sqrt{3}}{6}(c-b) = \frac{3-i\sqrt{3}}{6}b + \frac{3+i\sqrt{3}}{6}c,$$
$$B = \frac{c+a}{2} + \frac{i\sqrt{3}}{6}(a-c) = \frac{3-i\sqrt{3}}{6}c + \frac{3+i\sqrt{3}}{6}a,$$
$$C = \frac{a+b}{2} + \frac{i\sqrt{3}}{6}(b-a) = \frac{3-i\sqrt{3}}{6}a + \frac{3+i\sqrt{3}}{6}b,$$

and

$$A - B = -\frac{3+i\sqrt{3}}{6}a + \frac{3-i\sqrt{3}}{6}b + \frac{i\sqrt{3}}{3}c,$$
$$B - C = -\frac{3+i\sqrt{3}}{6}b + \frac{3-i\sqrt{3}}{6}c + \frac{i\sqrt{3}}{3}a,$$
$$C - A = -\frac{3+i\sqrt{3}}{6}c + \frac{3-i\sqrt{3}}{6}a + \frac{i\sqrt{3}}{3}b.$$

To prove that ABC is an equilateral triangle, we need to check that $B - C$ is obtained from $A - B$ by a counterclockwise rotation by $120°$, that is, by multiplication by $\cos 120° + i \sin 120° = \dfrac{-1 + i\sqrt{3}}{2}$; this is checked by an immediate calculation. (In the same way, we can check that $C - A = \dfrac{-1 + i\sqrt{3}}{2}(B-C)$ and $A-B = \dfrac{-1 + i\sqrt{3}}{2}(C-A)$, but we do not need this.)

25. The section is a polygon whose sides are intersections of the plane with faces. Since the cube has 6 faces, the number of sides of the polygon cannot exceed 6. For any $n \leq 6$ there is an n-gonal section (see Figure 47 below).

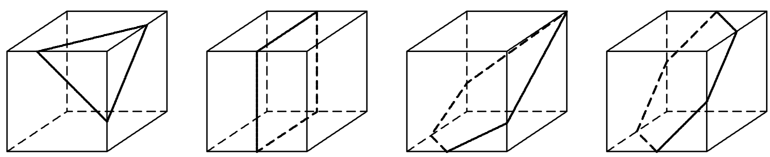


FIGURE 47. To Solution 25.

Hexagonal, quadrilateral, and triangular sections may be regular (shown in the picture); in particular, a regular hexagonal section can be obtained if (and only if) the plane passes through the center of the cube and is orthogonal to a big diagonal. A pentagonal section may appear when the plane passes through one of the vertices; it is never regular. With a certain abuse of language, we can say that 2-gonal section (an interval), 1-gonal section (a point), and 0-gonal section (the empty set) are also possible.

26. This sum does not depend on the line: it is always equal to $4\ell^2$ where ℓ is the length of the edge of the cube.

To prove this, let us first find the square of the distance from an arbitrary point (a, b, c) in space to an arbitrary line passing through the origin. The parametric equations of this line have the form $x = \alpha t, y = \beta t, z = \gamma t$ where we can assume that $\alpha^2 + \beta^2 + \gamma^2 = 1$.

Let $(\alpha t, \beta t, \gamma t)$ be the base of the perpendicular dropped from the point (a, b, c) to our line. Figure 48 shows that $t = \|(a, b, c)\| \cos \varphi$; thus, t is the dot product of the vectors (a, b, c) and (α, β, γ), that is, $t = a\alpha + b\beta + c\gamma$. Hence, the square of the distance from our point to our line is

$$\begin{aligned}&(a - \alpha t)^2 + (b - \beta t)^2 + (c - \gamma t)^2\\ &\qquad = a^2 + b^2 + c^2 - 2(a\alpha + b\beta + c\gamma)t + (\alpha^2 + \beta^2 + \gamma^2)t^2\\ &\qquad = a^2 + b^2 + c^2 - (a\alpha + b\beta + c\gamma)^2.\end{aligned}$$

Assume now that the eight vertices of the cube have coordinates $(\pm 1, \pm 1, \pm 1)$. Then the center of the cube is the origin, and we can apply the previous formula. Our sum of the squares is

$$\sum\nolimits_{\pm}(3 - (\pm\alpha \pm \beta \pm \gamma)^2) = 24 - \sum\nolimits_{\pm}(\pm\alpha \pm \beta \pm \gamma)^2.$$

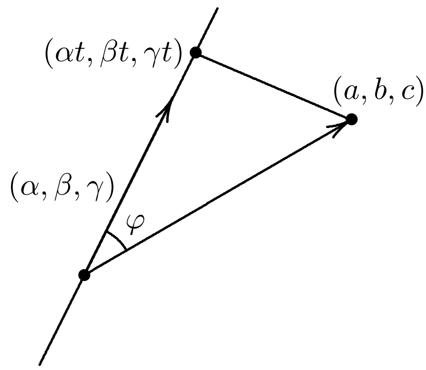


FIGURE 48. To Solution 26.

The last sum contains 8 times $\alpha^2 + \beta^2 + \gamma^2 = 1$, and each of the double product $2\alpha\beta, 2\alpha\gamma, 2\beta\gamma$ appears 8 times, 4 times with the sign $+$ and four times with the sign $-$. Thus, the final result is

$$24 - 8 = 16 = 4\ell^2 \text{ (since } \ell = 2).$$

Remark. Similar statements hold for all regular polyhedra.

27. The sum $CA + CB$ does not depend on the choice of the point C. The following proof was, probably, known to Appolonius (third century BC).

In addition to the drawing accompanying the statement of this problem, let us show the circles of tangency of the spheres to the cone. Choose an arbitrary point C on the section, and draw a line through this point and the vertex of the cone. Let A', B' be the intersection (tangency) points of

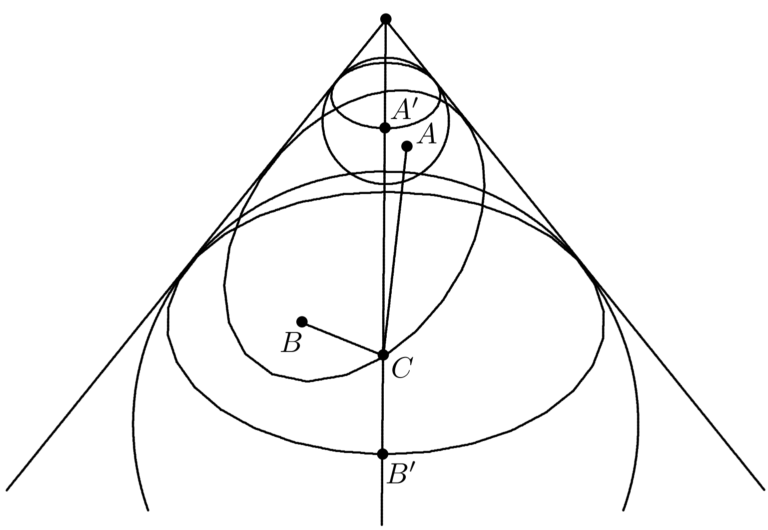


FIGURE 49. To Solution 27.

this line with the spheres. Then $CA = CA'$ (these are two tangents to a sphere from the same point), and similarly $CB = CB'$. Hence $CA + CB = CA' + CB' = A'B'$, and the latter, obviously, does not depend on the choice of C.

28. The area will be precisely equal to the area of France, and a similar thing holds for any area on the sphere. The following proof belongs to Cavalieri (1598–1647).

Take a point of the sphere, and let φ be the latitude of this point measured in radians $\left(\text{so } -\dfrac{\pi}{2} < \varphi < \dfrac{\pi}{2}\right)$. Then our projection onto the cylinder $\dfrac{1}{\cos\varphi}$ times stretches the parallel through this point and, in a proximity of this point, compresses the meridian (approximately) $\dfrac{1}{\cos\varphi}$ times (a picture below explains this).

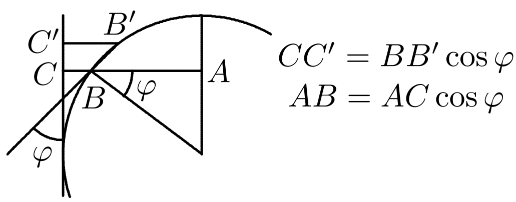


FIGURE 50. To Solution 28.

29. Use the binomial formula:

$$2^p = (1+1)^p = 1 + p + \frac{p(p-1)}{2} + \cdots + \frac{p(p-1)}{2} + p + 1.$$

The part of the last sum between the two 1's is divisible by p and is even (since 2^p is even). So, $2^p - 2 = Np$ where N is even. Divide the last equality by 2: $2^{p-1} - 1 = \dfrac{N}{2}p$. We see that $2^{p-1} - 1$ is divisible by p.

Remark. A more general result states that if p is prime and q is not divisible by p, then q^{p-1} has a remainder 1 upon division by p. This is called the *Little Fermat Theorem*.

30. It is not important that the length of the needle and the distance between the line are both 10 *cm*; all we need to know is that they are equal to each other. To make further formulas simpler, we assume that this length and this distance are both 2. Moving the needle horizontally does not change whether it intersects the line; so, we can assume that the midpoint of the needle is on a chosen vertical line. Similarly, moving the needle vertically by the distance of an even integer does not change whether it intersects the line; so, we can assume that the distance on the midpoint of the needle is between -1 and 1. With these assumptions, the position of the needle is described by two numbers: h, $-1 \le h \le 1$, which is the vertical coordinate

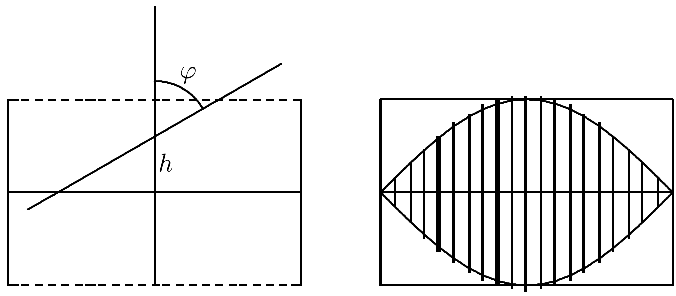


FIGURE 51. To Solution 30.

of the midpoint of the needle, and φ, $-\dfrac{\pi}{2} \le \varphi \le \dfrac{\pi}{2}$, the angle formed by the needle with the vertical direction (see the left side of Figure 51). We may assume that these two numbers are randomly chosen. It is clear that the needle may intersect only the line on this diagram (if the needle is vertical and $h = \pm 1$, then the needle also hits the other line, but this does not affect the probability). It is clear also that the needle intersects the line if and only if $\cos\varphi > |h|$.

In the plane (φ, h), the domain of all possible positions of the needle in the rectangle $-\dfrac{\pi}{2} \le \varphi \le \dfrac{\pi}{2}, -1 \le h \le 1$ of the area 2π and the domain of positions with an intersection with a line (shadowed in the right image of Figure 51) is bounded by the graphs $h = \pm\cos\varphi$ of the area 4 (it is proved by elementary calculus: $2\int_{-\pi/2}^{\pi/2} \cos\varphi \, d\varphi = 2\sin\varphi|_{-\pi/2}^{\pi/2} = 2(1-(-1)) = 4$). Thus, our probability is $\dfrac{4}{2\pi} = \dfrac{2}{\pi}$.

31. This follows from the famous Euler Theorem (proved, in fact, by Descartes 100 years before Euler) which states that if V, E, and F are numbers of vertices, edges, and faces of a convex polyhedron, then $V - E + F = 2$. If all the faces of the polyhedron are triangles, then $2E = 3F$. Indeed, let P be the number of all pairs (a face, an edge of this face); then $P = 3F$ and $P = 2E$. Thus, the Euler Theorem implies $V - \dfrac{3}{2}F + F = 2$ which becomes, after the multiplication by 2, $2V - 3F + 2F = 4$, that is, $2V = F + 4$.

32. In each of the twelve faces of the dodecahedron, we choose one the (five) diagonals. The first one we choose in an arbitrary way (in an arbitrarily chosen face), and then choose the rest of them using the following rule: if two faces are adjacent to each other, then the chosen diagonals either are both parallel to the common edge, or make different angles with the common face. See Figure 52.

The diagonals chosen form a cube, since any rotation of the dodecahedron which takes one of the chosen diagonals into another one takes the whole family of the chosen diagonals into itself.

There are five such cubes, because every face has five diagonals.

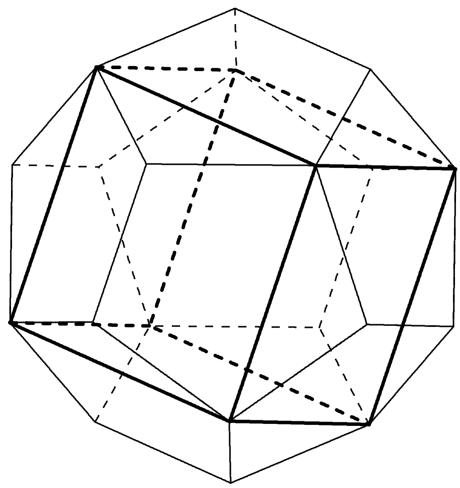


FIGURE 52. To Solution 32.

33. The intersection of the two tetrahedra is an octahedron whose six vertices are the centers of the faces of the cube (see Figure 53). Indeed, the two diagonals of the every face of the cube are edges of two different tetrahedra. So, their crossing point, which is the center of the cube, belongs to the intersection of the two tetrahedra. Consequently, the convex hull of the six-point set of the centers of the faces of the cube, which is the octahedron described, is contained in the intersection of the tetrahedra. To prove that this intersection contains nothing else, we notice that the eight triangular faces of the octahedron are contained in the eight triangular faces of the tetrahedra as triangles formed by the midpoints of the edges of the faces of the tetrahedra.

To evaluate the volume of the octahedron, we notice that the octahedron is the union of two quadrilateral pyramids, with the (square) base whose area is one half of the area of the face of the cube, and whose altitude is one half of the edge of the cube. Thus, the volume of each pyramid is $\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}$ of the volume of the cube, and the volume of the whole octahedron is $\frac{1}{6}$ of the volume of the cube.

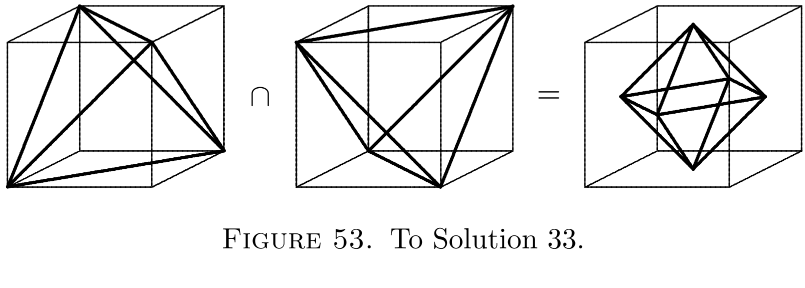


FIGURE 53. To Solution 33.

34. Let our regular polyhedron have a vertices, and let the number of edges converging at each vertex be b. Then the total number of symmetries is $2ab$, and ab of them are rotations. Indeed, fix a vertex A of our polyhedron. Then A can be taken by a rotational symmetry of the polyhedron into any other vertex B, and if B is specified, then all the symmetries which take A into B can be obtained from one of them by b rotations and b reflections. Thus:

– for a tetrahedron, there are $2 \cdot 4 \cdot 3 = 24$ symmetries, 12 of which are rotations;

– for a cube, there are $2 \cdot 8 \cdot 3 = 48$ symmetries, 24 of which are rotations;

– for an octahedron, there are $2 \cdot 6 \cdot 4 = 48$ symmetries, 24 of which are rotations;

– for an icosahedron, there are $2 \cdot 12 \cdot 5 = 120$ symmetries, 60 of which are rotations;

– for a dodecahedron, there are $2 \cdot 20 \cdot 3 = 120$ symmetries, 60 of which are rotations.

The number of reflections in planes is equal to the number of planes of symmetry. It is 6 for a tetrahedron, 9 for a cube and an octahedron, and 15 for an icosahedron and a dodecahedron.

35. There are 30 ways. Indeed, if we do not allow rotations, then there are $6 \cdot 5 \cdot \ldots \cdot 1 = 720$ colorings (we enumerate the faces of the cube by numbers $1, 2, \ldots, 6$ in an arbitrary way, then choose one of 6 colors for the face #1, one of 5 remaining colors for the face #2, and so on). No rotation (which is not the identity) takes any coloring into itself. There are 24 rotations of the cube (see Problem 34). So, the whole set of 720 colorings falls into the union of sets of rotationally equivalent colorings, each contains 24 colorings. Thus, up to a rotation, there are $720/24 = 30$ different colorings.

36. There are $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$ ways. In particular, for $n = 4, 5, 6, 10$, there are $24, 120, 720, 3628800$ ways.

37. Every symmetry of the cube yields a permutation of the four diagonals, and every permutation of the diagonals corresponds to two different symmetries. [Indeed, there are two symmetries, which take every diagonal into itself: the identity and the antipodal map (the reflection in the center). Hence, every permutation of the diagonals corresponds to two symmetries, which are obtained from each other by a composition with the antipodal map.] Precisely one of these two symmetries is a rotation. Thus, there is a one-to-one correspondence between rotations of the cube and permutation of the diagonals.

38. It is true, and it can be proven by induction with respect to the number n of integers. If $n = 1$, then the difference is 0, it is divisible by 3. Assume that for the sum of $n - 1$ integers the statement is true. Let $a_1, \ldots, a_n$ be the given integers, and let $b = a_1 + \cdots + a_{n-1}$. Then

$$(a_1 + \cdots + a_n)^3 = (b + a_n)^3 = b^3 + 3b^2 a_n + 3ba_n^2 + a_n^3,$$

and we see that

$$(a_1 + \cdots + a_n)^3 - b^3 - a_n^3$$

is divisible by 3. Hence

$$(a_1+\cdots+a_n)^3-a_1^3-\cdots-a_n^3 = [(a_1+\cdots+a_n)^3-b^3-a_n^3]+[b^3-a_1^3-\cdots-a_{n-1}^3],$$

and of two summands in square brackets, the first is divisible by 3 by the statement above, and the second is divisible by 3 by the induction hypothesis.

39. It is true, and the proof is almost the same as in the previous solution. The only difference is that we need to replace the formula for $(b + a_n)^3$ by one of the formulas

$$(b + a_n)^5 = b^5 + 5b^4 a_n + 10b^3 a_n^2 + 10b^2 a_n^3 + 5ba_n^4 + a_n^5,$$
$$(b + a_n)^7 = b^7 + 7b^6 a_n + 21b^5 a_n^2 + 35b^4 a_n^3 + 35b^3 a_n^4 + 21b^2 a_n^5 + 7ba_n^6 + a_n^7.$$

Similar arguments show that we can replace 3 (or 5, or 7) by any prime number, but for a composite exponent this may be not true: $(1+1)^4 - 1^4 - 1^4 = 14$ is not divisible by 4, $(1 + 1)^6 - 1^6 - 1^6 = 62$ is not divisible by 6.

40. It is

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{99} - \frac{1}{100}\right) = 1 - \frac{1}{100} = \frac{99}{100}$$

precisely (no error).

41. It is sufficient to prove that every polygon P can be cut by straight lines into pieces, of which one can assemble a rectangle of size $1 \times \text{area}(P)$. This can be done in five steps. (See Figure 54.)

Step One. Any polygon can be cut into several triangles. This allows us to assume that the polygon P itself is a triangle.

Step Two. We cut the triangle into two pieces by a line through midpoints of two sides, and from these pieces we assemble a parallelogram.

Step Three. We cut a triangle off the parallelogram by a line AB where A is a vertex, B is a point on a side opposite to A such that the length r of AB is rational. Then we attach this triangle to the other side of the parallelogram. As a result, we obtain a different parallelogram with one of the sides being of a rational length.

Step Four. Cut the parallelogram by parallel lines perpendicular to the edge AB of rational length r at the distance r from each other; two of these lines pass through A and B. Then rearrange the pieces to assemble a rectangle with one of the edges being AB.

Step Five. Let $r = \dfrac{p}{q}$. Divide the side AB into q equal pieces and divide the perpendicular side into p equal pieces. Then divide the rectangle into pq equal small rectangles by lines parallel to the sides. These pq small rectangles can be rearranged into a new rectangle with one side of length 1.

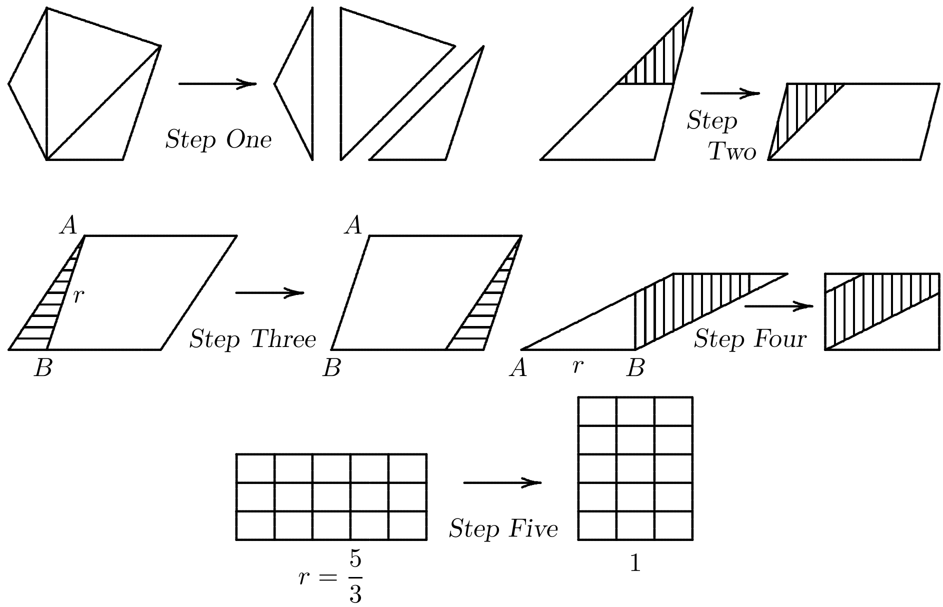


FIGURE 54. To Solution 41.

42. See Solution to Problem 43.

43. The area is $a + \dfrac{b}{2} + 1$. In the situation of Problem 42, $a = b = 0$, so the area equals 1, as stated.

Let s be the area of our parallelogram and $c = a + \dfrac{b}{2} + 1$; we want to prove that $s = c$.

Consider the tiling of the plane by parallelograms obtained from our parallelogram by shifting by the vectors $m_1 s_1 + m_2 s_2$ where s_1 and s_2 are the sides (considered as vectors) and m_1, m_2 are integers (see Figure 55).

Choose one of the four vertices of our parallelogram and call it the *prime vertex*. Then every tile of our tiling has a prime vertex. These prime vertices form a sublattice of the standard lattice (formed by the vertices of the graph paper). Fix some point in the plane and denote by $d(R)$ the (closed) disc centered at this point. Let $N(R)$ be the number of points of the standard lattice contained in $d(R)$, and let $M(R)$ be the number of points of our sublattice contained in $d(R)$.

Let us denote by $S(R)$ the union of tiles whose prime vertices lie in $d(R)$, and denote by ℓ the length of the longer diagonal of our parallelogram (we assume that the length of the side of the cell of the graph paper is 1). Then area, $S(R) = sM(R)$. Obviously, $d(R - \ell) \subset S(R) \subset d(R + \ell)$, so

$$\pi(R - \ell)^2 \leq sM(R) \leq \pi(R + \ell)^2. \tag{1}$$

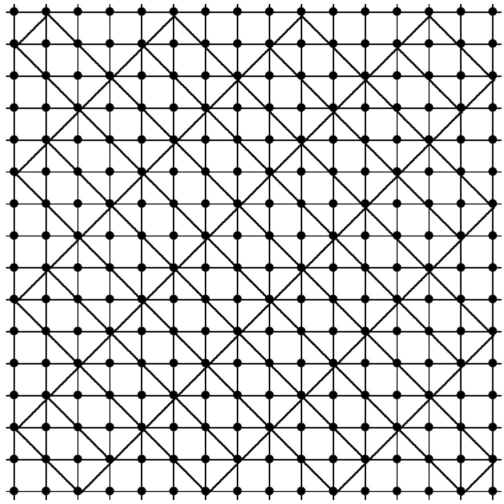


Figure 55. To Solution 43.

The same arguments applied to the tiling by the cells of the graph paper yield the inequalities

$$\pi(R-\sqrt{2})^2 \le N(R) \le \pi(R+\sqrt{2})^2 \tag{2}$$

(since the area of a cell is 1 and the length of the diagonal of a cell is $\sqrt{2}$).

Next, let us consider the product $cM(R)$. This can be regarded as the sum over all the tiles T in $S(R)$ where the summand corresponding to T is, in turn, the sum over all the vertices of the graph paper within T of summands equal to 1 for points inside the T, to $\frac{1}{2}$ for points inside the edges of T, and to $\frac{1}{4}$ for the vertices of T. This shows that the total contribution of a vertex of the graph paper to $cM(R)$ never exceeds 1, it is 1 for the vertices in $d(R-2\ell)$, and it can be positive only for vertices in $d(R+\ell)$. Thus

$$N(R-2\ell) \le cM(R) \le N(R+\ell), \tag{3}$$

which gives, in combination with (2),

$$\pi(R-2\ell-\sqrt{2})^2 \le cM(R) \le \pi(R+\ell+\sqrt{2})^2. \tag{4}$$

From (1) and (4), we can deduce for $\dfrac{c}{s} = \dfrac{cM(R)}{sM(R)}$

$$\frac{(R-2\ell-\sqrt{2})^2}{(R+\ell)^2} \le \frac{c}{s} \le \frac{(R+\ell+\sqrt{2})^2}{(R-\ell)^2}. \tag{5}$$

Since both the first and the third fractions in (5) become arbitrarily close to 1 when R grows, (5) shows that $\dfrac{c}{s}$ must be 1.

44. Yes, and the statement of Problem 43 as well. The latter means that if P is a parallelepiped in space whose vertices all have integer coordinates, and if a, b, and c are the number of points with integer coordinates, respectively, inside P, inside the faces of P, and inside the edges of P, then

$$\text{volume}\,(P) = a + \frac{b}{2} + \frac{c}{4} + 1.$$

If there are no points with integer coordinates in P (including the boundary) besides the vertices, that is, if $a = b = c = 0$, then the volume of P is 1. The proof is a replica of the proof in Solution to Problem 43. Similar facts hold in any dimension.

45. For any positive integers a, b, it is true that $(a+b, b) = (a, b)$. Using this, we have:

$$\begin{aligned}(a_{100}, a_{99}) \quad &= (a_{98} + a_{99}, a_{99}) = (a_{98}, a_{99}) = (a_{99}, a_{98})\\ &= (a_{97} + a_{98}, a_{98}) = (a_{97}, a_{98}) = (a_{98}, a_{97})\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &= (a_1 + a_2, a_2) = (a_1, a_2) = (1, 1) = 1.\end{aligned}$$

46. There is a recursion formula for $c(n)$ (in this formula, by definition, $c(2) = 1$; we can try to justify it by saying that a two-gon is divided by diagonals into the union of 0 triangles in one way). The formula:

$$c(n) = c(2)c(n-1) + c(3)c(n-2) + c(4)c(n-3) + \cdots + c(n-1)c(2).$$

To prove that, we first choose one of the sides of the n-gon. Then, for every partition of the n-gon into $n-2$ triangles there is a triangle containing the chosen side; to specify it, we need to choose one of the $n-2$ vertices not belonging to the chosen side. Figure 56 shows how it looks for a hexagon (the chosen side is the bottom side). If we remove the chosen side, then our n-gon falls into the union of an m-gon and an $(n+1-m)$-gon (with $m = 2, 3, \ldots, n-1$). To complete the partition of the n-gon into triangles by diagonals, we need to do this for both the m-gon and the $(n+1-m)$-gon, which can be done in $c(m)c(n+1-m)$ ways (for a fixed m). Whence our formula.

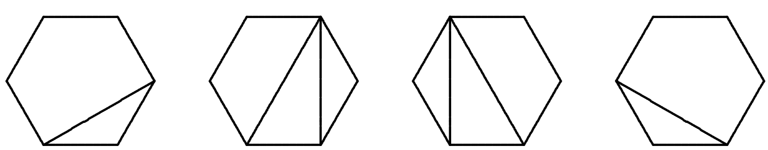


FIGURE 56. To Solution 46.

Now we use our formula for computations:

$$\begin{aligned}
&c(2) = 1,\\
&c(3) = 1 \cdot 1 = 1,\\
&c(4) = 1 \cdot 1 + 1 \cdot 1 = 2,\\
&c(5) = 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 = 5,\\
&c(6) = 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 1 + 5 \cdot 1 = 14,\\
&c(7) = 1 \cdot 14 + 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 + 14 \cdot 1 = 42,\\
&c(8) = 1 \cdot 42 + 1 \cdot 14 + 2 \cdot 5 + 5 \cdot 2 + 14 \cdot 1 + 42 \cdot 1 = 132,\\
&c(9) = 1 \cdot 132 + 1 \cdot 42 + 2 \cdot 14 + 5 \cdot 5 + 14 \cdot 2 + 42 \cdot 1 + 132 \cdot 1 = 429,\\
&c(10) = 1 \cdot 429 + 1 \cdot 132 + 2 \cdot 42 + 5 \cdot 14 + 14 \cdot 5\\
&\qquad\qquad\qquad\qquad +42 \cdot 2 + 132 \cdot 1 + 429 \cdot 1 = 1430.
\end{aligned}$$

Remark. The Catalan numbers have the property to appear in many very different combinatorial contexts. An interested reader can get familiar with some of them by Wikipedia. This source contains also an explicit (not recursion) formula for these numbers:

$$c(n) = \frac{(2n-4)!}{(n-1)!(n-2)!}.$$

47. The number of different schedules for a tournament of $n \geq 2$ teams is

$$1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)$$

(the product of all odd numbers from 1 to $2n-3$). Let us prove this by induction. If $n = 2$, then there is only one schedule, which agrees with our formula. Assume that the result holds for $n-1$ teams, that is, there are $1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-5)$ schedules for $n-1$ teams. Imagine that the n-th team entered the tournament after the schedule for the $n-1$ teams had been already established. To include the new team, we need to do one of two things: either choose one of the $n-2$ games of the existing schedule and have the new team to play with one of the participants of the chosen game and then have the winner to play with the other participant (for this, we have $2(n-2)$ options); or have the new team play with the winner of the last game (one option for this). Thus a schedule for $n-1$ teams may be turned into a schedule for n teams in $2(n-2)+1 = 2n-3$ ways, and the total number of schedules for n teams is

$$[1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-5)] \cdot (2n-3),$$

which completes our induction.

48. The answer is n^{n-2}. This fact is called the *Cayley formula* and has several known proofs, neither of which is short and elementary. Below, we restrict ourselves to proving a recursion formula for the number of trees.

Let T_n be the number of trees with vertices $1, 2, \ldots, n$. Consider the set of such trees *with one edge marked.* The number of elements of such set is, obviously, $(n-1)T_n$. Let us count the number of elements of this set in a different way. First, we choose a marked edge; for this, there are

$\binom{n}{2} = \dfrac{n(n-1)}{2}$ options. If we remove, from a tree, the marked edge (but not its endpoints!), then our tree falls into a disjoint union of two trees, with the numbers of vertices m and $n-m$ (where $m = 0, 1, 2, \ldots, n$). For the trees, there are T_m and T_{n-m} options, but also we need to specify the vertices of, say, the first tree, for which there are $\binom{n-2}{m-1}$ options. Thus, the whole amount of options is $\sum_{m=0}^{n} \binom{n-2}{m-1} T_m T_{n-m}$, and we obtain the formula

$$(n-1)T_n = \frac{n(n-1)}{2} \sum_{m=0}^{n} \binom{n-2}{m-1} T_m T_{n-m},$$

or

$$2T_n = n \sum_{m=0}^{n} \binom{n-2}{m-1} T_m T_{n-m}. \tag{1}$$

In particular,

$$\begin{aligned}
T_1 &= 1,\\
2T_2 &= 2(1\cdot 1) = 2,\ T_2 = 1,\\
2T_3 &= 3(1\cdot 1 + 1\cdot 1) = 6,\ T_3 = 3,\\
2T_4 &= 4(1\cdot 3 + 2\cdot 1\cdot 1 + 3\cdot 1) = 32,\ T_4 = 16,\\
2T_5 &= 5(1\cdot 16 + 3\cdot 1\cdot 3 + 3\cdot 3\cdot 1 + 1\cdot 16\cdot 1) = 250,\ T_5 = 125,\\
2T_6 &= 6(1\cdot 125 + 4\cdot 1\cdot 16 + 6\cdot 3\cdot 3 + 4\cdot 16\cdot 1 + 125\cdot 1) = 2592,\ T_6 = R1296.
\end{aligned}$$

We leave to the reader to check that the numbers $T_n = n^{n-2}$ satisfy the relation (1).

49. For a snake of length n, mark the term x_k equal to n (see Figure 57).

Obviously, k must be even; indeed, if k is odd, then x_k must be less than at least one of x_{k-1} and x_{k+1} (whichever exists), which is impossible, if $x_k = n$. To the left and to the right of x_k, there are snakes of lengths $k-1$ and $n-k$, but (1) the left one should be read from the right to the left and (2) these snakes are permutations not of the set $\{1, 2, \ldots, n\}$, but of two complementary subsets of the set $\{1, 2, \ldots, n-1\}$ consisting, respectively, of $k-1$ and $n-k$ elements. Thus, for a snake of the length n, we need to specify an even number k, $1 \le k \le n$, and for this k, a set of $k-1$ numbers between 1 and $n-1$ (there are $\binom{n-1}{k-1}$ options for that), and two snakes of lengths $k-1$ and $n-k$ ($s(k-1)$ and $s(n-k)$ options, respectively). We arrive at the recursion formula

$$s(n) = \sum_{\substack{1 \le k \le n \\ k \text{ is even}}} \binom{n-1}{k-1} s(k-1) s(n-k) \tag{1}$$

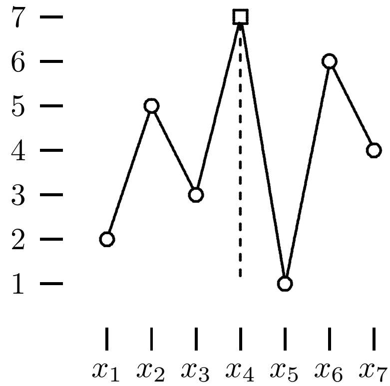


FIGURE 57. To Solution 49.

(where we assume $s(0) = s(1) = 1$). From this:

$$\begin{aligned}
s(2) &= 1,\\
s(3) &= 2s(1)s(1) = 2,\\
s(4) &= 3s(1)s(2) + s(3)s(0) = 5,\\
s(5) &= 4s(1)s(3) + 4s(3)s(1) = 16,\\
s(6) &= 5s(1)s(4) + 10s(3)s(2) + s(5)s(0) = 61,\\
s(7) &= 6s(1)s(5) + 20s(3)s(3) + 6s(5)s(1) = 232,\\
s(8) &= 7s(1)s(6) + 35s(3)s(4) + 21s(5)s(2) + s(7)s(0) = 1345,\\
s(9) &= 8s(1)s(7) + 56s(3)s(5) + 56s(5)s(3) + 8s(7)s(1) = 7296,\\
s(10) &= 9s(1)s(8) + 72s(3)s(6) + 126s(5)s(4) + 36s(7)s(2) + s(9)s(0) = 46617.
\end{aligned}$$

50. The function $y = \tan x$ satisfies the differential equation

$$y' = y^2 + 1,$$

and this, together with the condition $y(0) = 0$, uniquely determines the function $y = \tan x$. Since $y = \tan x$ is an odd function, its Taylor series involves only odd powers of x. Let

$$\tan x = \sum_{k=1}^{\infty} \frac{a_k}{(2k-1)!} x^{2k-1};$$

we want to prove that $a_k = s(2k-1)$. The derivation and squaring the power series above shows that

$$(\tan x)' = \sum_{k=1}^{\infty} \frac{a_k}{(2k-2)!} x^{2k-2},$$

$$(\tan x)^2 = \sum_{k=2}^{\infty} \left[\sum_{\substack{p+q=k \\ p\geq 1, q\geq 1}} \frac{a_p a_q}{(2p-1)!(2q-1)!} \right] x^{2k-2}.$$

From the differential equation, $a_1 = 1$ and, for $k \geq 2$,

$$\frac{a_k}{(2k-2)!} = \sum_{\substack{p+q=k \\ p\geq 1, q\geq 1}} \frac{a_p a_q}{(2p-1)!(2q-1)!}$$

or

$$a_k = \sum_{\substack{p+q=k \\ p\geq 1, q\geq 1}} \binom{2k-2}{2p-1} a_p a_q.$$

If we plug in this formula $a_k = s(2k-1)$, we will obtain precisely the recursion formula (1) from Solution to Problem 49 for $n = 2k-1$. Thus the numbers a_k satisfy the recursion formula for $s(2k-1)$, so $a_k = s(2k-1)$.

51. Let $f(x)$ the sum of the given series. Then

$$f'(x) = \sum_{k=1}^{\infty} \frac{s(2k)}{(2k-1)!} x^{2k-1},$$

and by formula (1) from Solution to Problem 49 this sum is equal to

$$\sum_{k=1} \sum_{p+q=k} \binom{2k-1}{2p-1} \frac{1}{(2k-1)!} s(2p-1) s(2q) x^{2k-1}$$

$$= \sum_{p=1}^{\infty} \sum_{q=0}^{\infty} \frac{s(2p-1)}{(2p-1)!} x^{2p-1} \cdot \frac{s(2q)}{(2q)!} x^{2q}.$$

Taking into account the result of Problem 50, we arrive at the differential equation

$$f'(x) = f(x) \cdot \tan x,$$

which, together with the condition $f(0) = 1$, uniquely determines $f(x)$. It is easy to check that the function $f(x) = \dfrac{1}{\cos x}$ satisfies the equation and the condition, so this is the sum of the series.

52. Let $p_1 < p_2 < p_3 < \cdots$ be the sequence of all primes. We have:

$$\begin{aligned}\prod_{k=1}^{\infty} \frac{1}{1-\dfrac{1}{p_k^s}} &= \prod_{k=1}^{\infty} \sum_{m_k=0}^{\infty} \frac{1}{p_k^{sm_k}} = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} \cdots \frac{1}{p_1^{sm_1} p_2^{sm_2} p_3^{sm_3} \ldots} \\ &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} \cdots \frac{1}{(p_1^{m_1} p_2^{m_2} p_3^{m_3} \cdots)^s} = \sum_{n=1}^{\infty} \frac{1}{n^s}.\end{aligned}$$

53. There are many different proofs of this fact, which was first observed (but not proved rigorously) by Euler. The reader can find them on Wikipedia. Euler's heuristic arguments were as follows. The function $f(x) = \dfrac{\sin x}{x}$ (equal, by definition, to 1 at 0) has zeroes at all points $x = n\pi, n \neq 0$ and has no other zeroes (even in the complex domain). We can expect that

$$\begin{aligned}\frac{\sin x}{x} &= \cdots \left(1+\frac{x}{2\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\cdots \\ &= \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots\end{aligned}$$

(since the right hand side has the same zeroes as $\dfrac{\sin x}{x}$ and equals 1 at 0; actually, the equality can be proved by instruments of complex analysis). The last product, turned into series, has the form

$$1 - \left(\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}\right) x^2 \pm \cdots;$$

compare this with the Taylor expansion

$$\frac{\sin x}{x} = 1 - \frac{1}{6}x^2 \pm \cdots$$

to get the equality

$$\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} = \frac{1}{6} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Remark. We can use the formulas obtained above to compute $\displaystyle\sum_{n-1}^{\infty} \frac{1}{n^s}$ for all even s. For example, let us do it for $s = 4$. Comparing the coefficients at x^4 in the above infinite product formula for $\dfrac{\sin x}{x}$ and using the Taylor expansion $\dfrac{\sin x}{x} = 1 - \dfrac{1}{6}x^2 + \dfrac{1}{120}x^4 \pm \cdots$, we get

$$\sum_{1 \le p < q < \infty} \frac{1}{p^2q^2\pi^4} = \frac{1}{120} \quad \text{or} \quad \sum_{1 \le p < q < \infty} \frac{1}{p^2q^2} = \frac{\pi^4}{120}.$$

Furthermore, squaring the formula for $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, we get

$$\left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^2 = \sum_{n=1}^{\infty} \frac{1}{n^4} + 2 \sum_{1 \le p < q < \infty} \frac{1}{p^2 q^2} = \frac{\pi^4}{36},$$

from which

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{36} - 2\frac{\pi^4}{120} = \frac{\pi^4}{90}.$$

Similar arguments yield formulas $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$, $\sum_{n=1}^{\infty} \frac{1}{n^8} = \frac{\pi^8}{9450}$, and so on. Actually, $\sum_{n=1}^{\infty} \frac{1}{n^{2s}} = \rho_s \pi^{2s}$, where ρ_s is a rational number which can be found explicitly in terms of the so called *Bernoulli numbers*. However, no formula exists for $\sum_{n=1}^{\infty} \frac{1}{n^t}$ with t odd.

54. To do this, we will need the results of the two previous problems. The probability of the fact that an integer n is not divisible by some p is $1 - \frac{1}{p}$; the probability of the fact that two integers, m and n, do not share a divisor p, is $1 - \frac{1}{p^2}$. A fraction $\frac{m}{n}$ is not cancellable if the integers m and n do not share any prime divisors. Since these events for different primes are clearly independent, the probability of the fact that the fraction $\frac{m}{n}$ is not cancellable is $\prod_p \left(1 - \frac{1}{p^2}\right)$, where the product is taken over all primes. By the statement of Problem 52, the inverse to this product is $\sum_{n=1}^{\infty} \frac{1}{n^2}$, and by the statement of Problem 53, the last (infinite) sum equals $\frac{\pi^2}{6}$. Thus, our probability is $\frac{6}{\pi^2} \approx 0.608$.

55. The problem becomes easy if one assumes that the limit

$$r = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}$$

exists. With this assumption, we take the equality

$$\frac{a_{n-1}}{a_n} + 1 = \frac{a_{n+1}}{a_n}$$

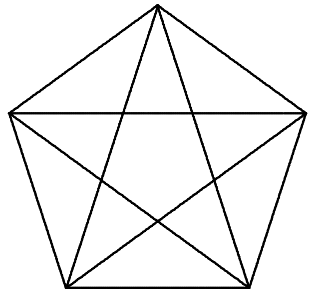


FIGURE 58. To Solution 55.

and apply to both sides $\lim\limits_{n\to\infty}$. We get: $\dfrac{1}{r}+1=r$, that is, $r^2-r-1=0$, and this quadratic equation has only one positive root: the golden ratio $\tau=\dfrac{1+\sqrt{5}}{2}$.

We leave the proof of the existence of the limit to the reader. One of the ways: first prove (by induction) that $a_{n+1}a_{n-1}-a_n^2=(-1)^n$, and then notice that the sequence

$$d_n=\frac{a_{n+1}}{a_n}-\frac{a_n}{a_{n-1}}=\frac{a_{n+1}a_{n-1}-a_n^2}{a_na_{n-1}}=\frac{(-1)^n}{a_na_{n-1}}$$

has alternating signs and $\lim|d_n|=0$; this implies the existence of our limit.

As to the geometric question, the ratio of the lengths of the side and the diagonal of a regular pentagon (which is, simultaneously, the edge of the five-point star inscribed into the pentagon; see Figure 58) is equal to the golden ratio. To prove this, we need to know that $\cos 36^\circ=\dfrac{\tau}{2}$. This follows, in turn, from the equality $\cos(3\cdot 36^\circ)=-\cos(2\cdot 36^\circ)$. This can be regarded as a cubic equation for $\cos 36^\circ$, whose roots are $-1, \dfrac{1\pm\sqrt{5}}{4}$.

56. Again, we leave it to the reader to prove the existence of the limit. If it exists and is equal to r, then

$$r=1+\cfrac{1}{2+\cfrac{1}{r}}\Rightarrow r=\frac{3r+1}{2r+1}\Rightarrow 2r^2-2r-1=0.$$

The positive solution of this quadratic equation is $\dfrac{1+\sqrt{3}}{2}$ which is the answer to our problem.

Remark. The reader who finds the two last problems interesting may want to read Part 1, "Continued Fractions", of this volume, especially the section about the Lagrange Theorem.

57. To do this, we need to know the formulas for sines and cosines of multiple angles (which, certainly, are important and useful by themselves).

Namely, there are polynomials $P_n(u)$ and $Q_n(u)$ such that

$$\sin nx = \sin(x) \cdot Q_n(\cos x) \text{ and } \cos nx = P_n(\cos x).$$

For example, $P_1(u) = u, Q_1(u) = 1, P_2(u) = 2u^2 - 1, Q_2(u) = 2u$. If we know these polynomials, our problem is solved immediately: $\cos(n \arccos x) = P_n(x)$.

Using the relations

$$\begin{aligned} \sin(n+1)x &= \sin nx \cos x + \sin x \cos nx, \\ \cos(n+1)x &= \cos nx \cos x - \sin nx \sin x, \end{aligned}$$

we obtain recursion formulas for the polynomials P_n and Q_n,

$$\begin{aligned} P_{n+1}(u) &= uP_n(u) + (u^2-1)Q_n(u), \\ Q_{n+1}(u) &= uQ_n(u) + P_n(u), \end{aligned}$$

from which we can get the following table of polynomials P_n and Q_n.

n	$P_n(u)$	$Q_n(u)$
1	u	1
2	$2u^2-1$	$2u$
3	$4u^3-3u$	$4u^2-1$
4	$8u^4-8u^2+1$	$8u^3-4u$
5	$16u^5-20u^2+5u$	$16u^4-12u^2+1$
6	$32u^6-48u^4+18u^2-1$	$32u^5-32u^3+6u$
7	$64u^7-112u^5+56u^3-7u$	$64u^6-80u^4+24u^2-1$
...	...	...

This table suggests general formulas for $P_n(u)$ and $Q_n(u)$:

$$P_n(u) = \frac{1}{2} \sum_{j \le n/2} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (2u)^{n-2j},$$

$$Q_n(u) = \sum_{j<(n-1)/2} (-1)^j \binom{n-1-j}{j-1} (2u)^{n-1-2j}.$$

It is very easy to check these formulas using the recursion formulas given above.

58. The n-th roots of 1 are $1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{n-1}$ where

$$\varepsilon = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}.$$

The sum of k-th powers of these roots is $1 + \cdots + 1 = n$, if k is divisible by n, and it is 0 otherwise. The latter follows from the formula for the sum of a geometric sequence:

$$1+\varepsilon^k+\varepsilon^{2k}+\cdots+\varepsilon^{(n-1)k}=\frac{\varepsilon^{kn}-1}{\varepsilon^k-1}=\frac{1-1}{\varepsilon^k-1}=0.$$

59. The curve $x=\cos 2t$, $y=\sin 3t$ is periodic with period 2π; so we can expect that this curve be closed. However, the graph (see Figure 59) does not look closed. The curve starts at the point $(1,0)$ $(t=0)$, then it goes up, to the point $\left(\frac{1}{2},1\right)$ $\left(t=\frac{\pi}{6}\right)$, and then, through the point $(0,0)$ $\left(t=\frac{\pi}{3}\right)$ to the point $(-1,-1)$ $\left(t=\frac{\pi}{2}\right)$. After that, the curve goes back along itself, because

$$\cos 2\left(\frac{\pi}{2}+\alpha\right)=\cos 2\left(\frac{\pi}{2}-\alpha\right)\ \text{ and }\ \sin 3\left(\frac{\pi}{2}+\alpha\right)=\sin 3\left(\frac{\pi}{2}-\alpha\right)$$

reaches $(0,0)$ at $t=\pi$, and then repeats the same path reflected in the x axis from $t=\pi$ to $t=2\pi$.

The curve $x=t^3-3t, y=t^4-2t^2$ has two singularities (cusps) at $t=\pm 1$, since $x'(\pm 1)=y'(\pm 1)=0$. On the graph, these cusps are located at the points $(\pm 2,-1)$ (see Figure 59).

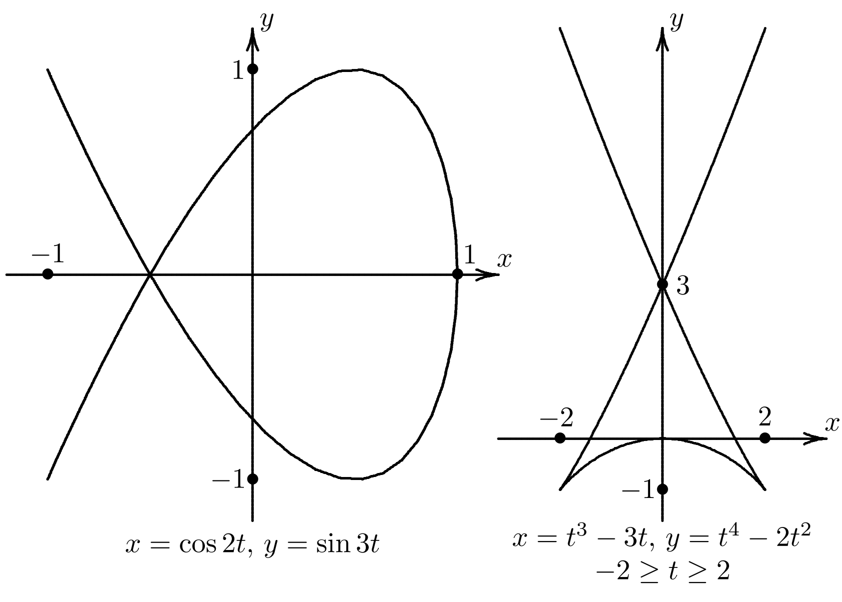


FIGURE 59. To Solution 59.

60. It is not hard to compute this integral precisely. For any $n \geq 2$, integration by parts gives

$$\begin{aligned}\int_0^{2\pi} \sin^n x\,dx &= -\sin^{n-1} x \cos x\Big|_0^{2\pi} - \int_0^{2\pi} (n-1)\sin^{n-2} x \cos x(-\cos x)\,dx \\ &= (n-1)\int_0^{2\pi} \sin^{n-2} x(1-\sin^2 x)\,dx \\ &= (n-1)\int_0^{2\pi} \sin^{n-2} x\,dx - (n-1)\int_0^{\pi} \sin^n x\,dx,\end{aligned}$$

so that

$$n\int_0^{2\pi} \sin^n x\,dx = (n-1)\int_0^{2\pi} \sin^{n-2} x\,dx$$

or

$$\int_0^{2\pi} \sin^n x\,dx = \frac{n-1}{n}\int_0^{2\pi} \sin^{n-2} x\,dx$$

(actually, if n is odd, then this equality becomes $0 = 0$, but this is not important to us now). Since $\int_0^{2\pi} \sin^0 x\,dx = 2\pi$, we have

$$\int_0^{2\pi} \sin^{100} x\,dx = \frac{99}{100}\cdot\frac{97}{98}\cdot\frac{95}{96}\cdot\ldots\cdot\frac{1}{2}\cdot 2\pi = \frac{100!}{2^{100}(50!)^2}\cdot 2\pi.$$

It is not hard to compute this number explicitly using a pocket calculator. The result is

$$0.079589\cdot 2\pi \approx 0.500072.$$

Or one can use Stirling's approximation of factorials, $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$; then our expression is approximated by

$$\frac{\sqrt{2\pi 100}}{(\sqrt{2\pi 50})^2}\frac{1}{2^{100}}\frac{100^{100}}{e^{100}}\frac{e^{100}}{50^{100}}\cdot 2\pi = \frac{\sqrt{2\pi}}{5} \approx 0.501236.$$

Both computations show that the approximation

$$\int_0^{2\pi} \sin^{100} x\,dx \approx 0.5$$

gives an error much less than the problem requested.

61. The logarithmic differentiation gives

$$(x^x)' = x^x(1+\ln x).$$

Hence, by the Fundamental Theorem of Calculus,

$$\int_1^{10} x^x(1+\ln x)\,dx = x^x\big]_1^{10} = 10^{10} - 1^1.$$

It is clear, however, that the function $y = x^x$, $1 \geq x \geq 10$, is concentrated in a proximity of $x = 10$ (see the graph in Figure 60).

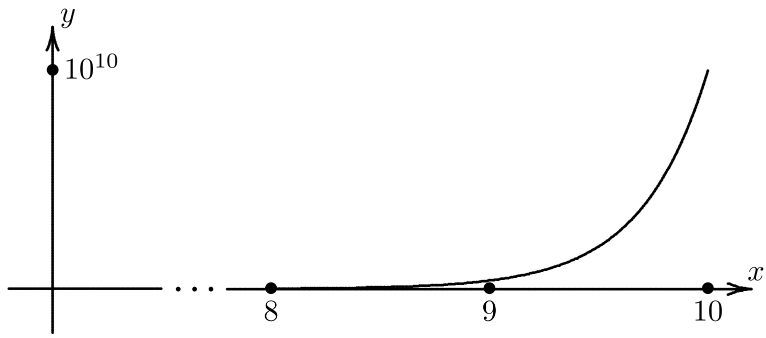


FIGURE 60. To Solution 61.

Hence, if we assume $\int_1^{10} x^x(1+\ln x)\,dx \approx \int_1^{10} x^x(1+\ln 10)\,dx$, then the error will be relatively small (we omit a precise estimate, but an easy computation shows that the relative error will be less than 1%, not 10%). If we assume this, we will get

$$\int_1^{10} x^x\,dx \approx \frac{10^{10}}{1+\ln 10} \approx 3,027,931,065.6.$$

Remark. A computer computation of the given integral gives

$$\int_1^{10} x^x\,dx \approx 3,007,764,122.4.$$

Thus, the approximation $\int_1^{10} x^x\,dx \approx 3\cdot 10^9$ (three billion) is much better, than the problem requested.

62. Draw big circles which contain the sides of our triangle. Let T be our triangle. We denote by S_α the spherical sector bounded by two great semicircles obtained by continuation of the sides of the triangle forming the angle α (so $T \subset S_\alpha$), and define in a similar way spherical sectors S_β and S_γ. See Figure 61. We will use the same notations $T, S_\alpha, S_\beta, S_\gamma$ for the areas of these four domains. Obviously, S_α is $\dfrac{\alpha}{2\pi}$ of the whole sphere, so $S\alpha = 2\alpha$; similarly, $S_\beta = 2\beta$ and $S_\gamma = 2\gamma$.

Consider the three hemispheres bounded by the three great circles which contain T. Their common area is $3\cdot 2\pi = 6\pi$, and they cover the whole sphere minus the triangle T' antipodal to T; thus, they cover the area $4\pi - T$. However, they overlap: each of the differences $S_\alpha - T, S_\beta - T$, and $S_\gamma - T$ is covered twice, and T is covered thrice. Hence, to obtain the area covered by the three hemispheres, we need to subtract from 6π each of $S_\alpha - T$, $S_\beta - T$, $S_\gamma - T$, and also $2T$. This yields the relation

$$4\pi - T = 6\pi - (2\alpha - T) - (2\beta - T) - (2\gamma - T) - 2T,$$

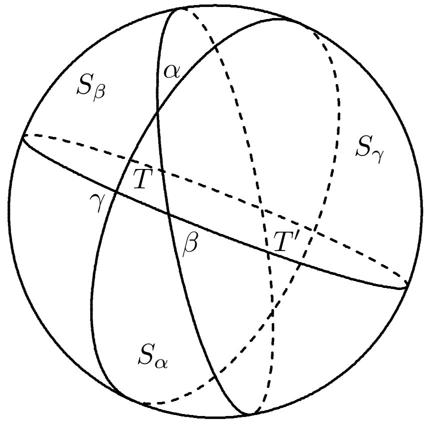


FIGURE 61. To Solution 62.

which gives, after cancellations, the equality

$$T = \alpha + \beta + \gamma - \pi.$$

63. Below are drawings of hypocycloids for four values of r.

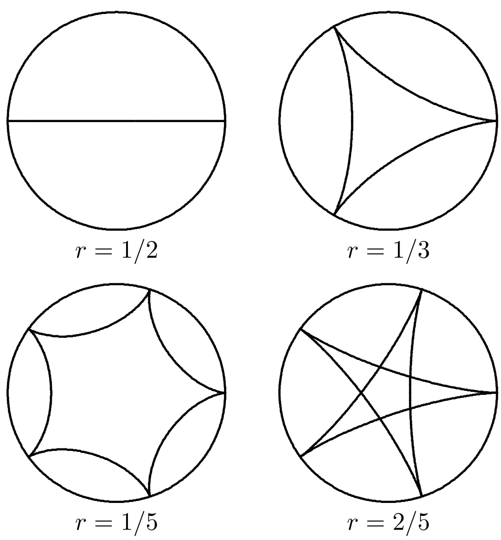


FIGURE 62. To Solution 63.

64. Let N be the number of days in a year. For different values of the number n of students in class, let us find the probability $p(n)$ of *no coincidences* in birthdays. If $n = 1$, then the probability is 1. If we add one student, then "no coincidences" means that the birthday of the newcomer is not the same as that of the first one; thus $p(2) = \dfrac{N-1}{N}$. For the third student, we get a new condition, independent of the previous one: the birthday of the new one should not fall on the birthdays of the previous two. Thus, $p(3) = \frac{N-1}{N}\frac{N-2}{N}$. And so on:

$$p(n) = \frac{N-1}{N}\frac{N-2}{N}\cdots\frac{N-(n-1)}{N} = \left(1-\frac{1}{N}\right)\cdots\left(1-\frac{n-1}{N}\right).$$

From this:

$$\ln p(n) = \ln\left(1-\frac{1}{N}\right) + \ln\left(1-\frac{2}{N}\right) + \cdots + \ln\left(1-\frac{n-1}{N}\right).$$

Using the approximation $\ln(1-t) = -t$ (for t small enough), we find that

$$\ln p(n) \approx -\frac{1+2+\cdots+(n-1)}{N} = -\frac{n(n-1)}{2N}.$$

We want to find the value n_0 of n for which the last expression is close to $\ln\frac{1}{2}$, that is, $n_0(n_0-1) \approx 2N\ln 2$. When $N = 365$, the right hand side of the last formula is $\approx 506 = 23 \cdot 22$, so we can take $n_0 = 23$. To make this approximate calculation more convincing, let us observe some calculator values of the probability $1 - p(n)$ of existing of a pair of students with the same birthday:

$n = 10$	11.7%;	$n = 30$	70.6%;
$n = 15$	25.3%;	$n = 40$	89.1%;
$n = 23$	50.7%;	$n = 50$	97.4%.

66. The vertices of the triangle KLM of the minimal perimeter are the bases of the altitudes AL, BM, and CK. To prove this, consider an arbitrary triangle KLM inscribed into the triangle ABC and then reflect the triangle ABC first in the side BC, and in the image BA' of the side BA. These reflections map the side LK (of the inscribed triangle) onto LK' and the side KM onto $K'M'$. (See the left side of Figure 63.)

The perimeter of the triangle KLM is equal to the length of the polygonal line $MLK'M'$, and it is clear that this perimeter is not minimal, if this polygonal line is not straight. If it is straight, then $\angle CLM = \angle BLK' = \angle BLK$. Similarly, we must have $\angle LK'B = \angle M'K'A'$ which is the same as $\angle LKB = \angle MKA$. Also, the equality $\angle AMK = \angle CML$ must be true, because the polygonal line $LK'M'L'$ (obtained by one more reflection of the triangle) must be also straight. All these equalities of angles hold if K, L, and M are bases of altitudes, as shown in the drawing on the right.

If the triangle is obtuse, then this construction does not go through, since two of the three altitudes lie outside of the triangle. In this case the inscribed triangle of the minimal perimeter is the degenerate triangle AKA where A is the vertex of the obtuse angle and AK is the altitude from this vertex.

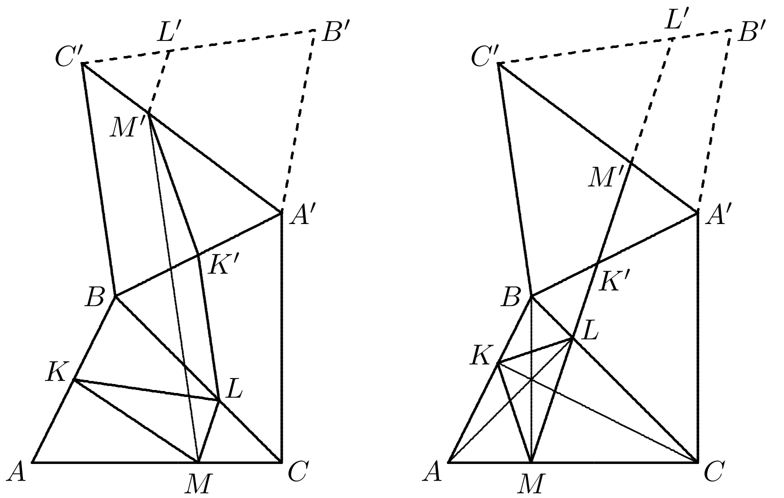


FIGURE 63. To Solution 66.

67. Let $\rho = \sqrt{X^2 + Y^2 + Z^2}$ (that is, the value of our function at the point (X, Y, Z)). Then the average of the function $\frac{1}{r}$ with respect to the sphere of radius R centered at (X, Y, Z) is equal to $\frac{1}{\rho}$ if $\rho \geq R$ and is equal to $\frac{1}{R}$ if $\rho \leq R$. Let us prove this.

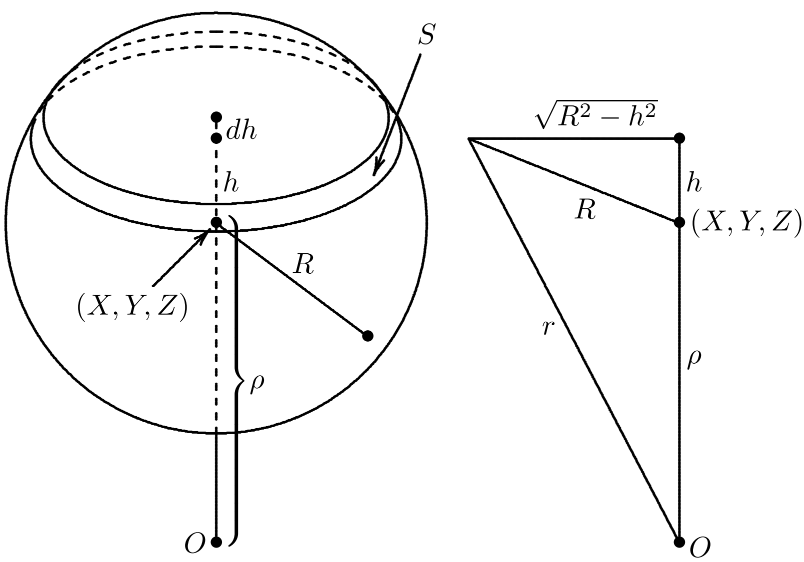


FIGURE 64. To Solution 67.

The geometric data of the problem is shown on the left side of Figure 64 (in this picture, $\rho > R$). So, h varies in the interval $-R \leq h \geq R$. Let

S be the spherical belt between planes perpendicular to the ray from O to (X, Y, Z) at the levels h and $h + dh$. It is known (see Solution to Problem 28) that the area of S is the same as the area of its projection onto the cylinder circumscribed about the sphere, that is, it is equal to $2\pi R \cdot dh$. Our function is nearly constant within S (we assume dh very small); the right side of Figure 64 shows that

$$r^2 = (\rho + h)^2 + (R^2 - h^2) = \rho^2 + 2\rho h + R^2.$$

To find the average value of the function with respect to the sphere, we need to find the integral of this function over the sphere and divide it over the area $4\pi R^2$ of the sphere. Thus, our average is

$$\frac{1}{4\pi R^2} \int_{-R}^{R} \frac{2\pi R\, dh}{\sqrt{\rho^2 + 2\rho h + R^2}} = \frac{1}{2R} \int_{-R}^{R} \frac{dh}{\sqrt{\rho^2 + 2\rho h + R^2}}.$$

Make a substitution $\rho^2 + 2\rho h + R^2 = u$. Then $du = 2\rho dh$, and if $h = \pm R$, then $u = \rho^2 \pm 2\rho R + R^2 = (\rho \pm R)^2$. We continue computing the average:

$$= \frac{1}{2R} \int_{(\rho-R)^2}^{(\rho+R)^2} \frac{du}{2\rho\sqrt{u}} = \frac{1}{4R\rho} \cdot 2\sqrt{u}\,\Big]_{(\rho-R)^2}^{(\rho+R)^2}$$

$$= \frac{1}{2R\rho}((\rho + R) - |\rho - R|) = \begin{cases} \dfrac{1}{\rho}, & \text{if } \rho \geq R, \\ \dfrac{1}{R}, & \text{if } \rho \leq R, \end{cases}$$

as was stated.

Remark. If $\rho = R$, the integral becomes improper (the integrand is ∞ for $h = -R$), but this does not affect its value $\dfrac{1}{\rho} = \dfrac{1}{R}$.

68. (Here and below, we use the notation log for $\log_{10}$.) $10 \cdot \log 2 = \log 2^{10} = \log 1024 = \log 1000 + \log 1.024 = 3 + \log 1.024$. It is not hard to check that $\log 1.024$ is very close to 0.01 ($\sqrt[3]{10} \approx 2.1544$ and, taking 5 times the square root of this number, we see that $\sqrt[96]{10} \approx 1.0243$). Hence, $10 \log 2 \approx 3.01$ and $\log 2 \approx 0.301$.

Actually, the difference between $\log 1.024$ and 0.01 is much less than 0.001, so the approximation of $\log 2$ by 0.301 is better than just three decimal places. The calculator value of $\log 2$ is 0.30103.

69. (It is reasonable to have in mind the last remark in the previous solution.)

$\log 4 = 2\log 2 \approx 0.602$;
$\log 8 = 3\log 2 \approx 0.903$;
$\log 5 = 1 - \log 2 \approx 0.699$;
$\log 50 = 1 + \log 5 \approx 1.699$;
$\log 32 = 5\log 2 \approx 1.505$;
$\log 128 = 7\log 2 \approx 2.107$;
$\log 125 = 3\log 5 \approx 2.097$;
$\log 64 = 6\log 2 \approx 1.808$.

70. A rough estimate would be $\log 7 = \dfrac{1}{2}\log 50 \approx \dfrac{1}{2}\cdot 1.699$. We could obtain a better approximation, if we notice that $\dfrac{50}{49} \approx 1.02$:

$$\log 50 \approx \log 49 + \log 1.02 \approx 2\log 7 + 0.01,$$

so

$$\log 7 \approx \frac{1}{2}(1.699 - 0.01) \approx 0.845.$$

71. Roughly, $6\log 2 = \log 64 \approx \log 63 = \log 7 + \log 9$, from which $\log 9 \approx 6\log 2 - \log 7 \approx 6\cdot 0.301 - 0.845 = 0.961$. To get a better approximation, we notice that the function $\log(1+x)$ is 0 at 0 and, as any smooth function, is almost linear for small values of x. Since we know that $\log 1.024 \approx 0.01$ and $\dfrac{64}{63} \approx 1.016$, we can conclude that $\log 6463 \approx \dfrac{0.016}{0.024}\log 1.024 \approx 0.007$. Hence,

$$6\log 2 = \log 63 + \log\frac{64}{63} \approx \log 7 + \log 9 + 0.007,$$
$$\log 9 \approx 6\cdot 0.301 - 0.845 - 0.007 = 0.954.$$

Furthermore,

$$\log 3 = \log 9/2 \approx 0.477; \qquad \log 6 = \log 2 + \log 3 \approx 0.778;$$
$$\log 27 = 3\log 3 \approx 1.431; \qquad \log 12 = 2\log 2 + \log 3 \approx 1.079.$$

72. $\log 1.024 \approx 0.01$, while $\ln 1.024 \approx 0.024$. Therefore,

$$\ln 10 \approx \frac{0.024}{0.01} = 2.4, \text{ and } \log e = \frac{1}{\ln}10 \approx \frac{1}{2.4} \approx 0.42.$$

Remark. As usual, multiple approximations lead to significant errors. The calculator values of $\ln 10$ and $\log e$ are slightly different: $\ln 10 \approx 2.3026$ and $\log e \approx 0.4343$.

73, 74. We begin with a computer result. For a digit $n = 1, 2, \ldots, 9$, we denote by $d_2(n)$ the number of powers of 2, among $2^1, 2^2, 2^3, \ldots, 2^{10000}$, with the first digit n, by $d_3(n)$ the similar number for powers of 3, and put $L(n) = [10000\cdot(\log(n+1) - \log(n))]$. Here are the values of $d_2(n), d_3(n)$ and $L(n)$:

n	$d_2(n)$	$d_3(n)$	$L(n)$
1	3011	3007	3010
2	1760	1764	1761
3	1250	1247	1249
4	969	968	969
5	791	792	792
6	670	669	669
7	580	582	580
8	511	513	512
9	458	458	458

This table speaks for itself. It strongly suggests that if $d_2(n, N)$ is the number of powers of 2, among $2^1, 2^2, \ldots, 2^N$ with the first digit n, then

$$\lim_{N\to\infty} \frac{d_2(n, N)}{N} = \log(n+1) - \log n,$$

and the same is true for powers of 3. Actually, as we explain below, this is true for powers of any positive real number a such that $\log a$ is irrational.

Lemma. *Let α be a positive irrational number, and let γ_m be the fractional part of the number $m\alpha$ (that is, $\gamma_m = m\alpha - [m\alpha]$). Choose any interval $[c, d] \subset [0, 1]$, and denote by $F(n)$ the number of elements of the set $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ which are contained in $[c, d]$. Then*

$$\lim_{n\to\infty} \frac{F(n)}{n} = d - c.$$

Proof. Let $[c', d'] \subset [0, 1]$ be the interval obtained from $[c, d]$ by shifting right by $k\alpha$ (within the whole number line) and then shifting left by some integer (in other words, $c + k\alpha - c'$ and $d + k\alpha - d'$ are equal integers; in particular, $d' - c' = d - c$). Let γ_i' and $F'(n)$ be defined for $[c', d']$ in the same way as γ_i, and let $F(n)$ be defined for $[c, d]$. Then $\gamma_m' = \gamma_{m+k}$, which shows that, for any n, $|F'(n) - F(n)| \le k$. Thus, for n large, the ratios $\dfrac{F(n)}{n}$ and $\dfrac{F'(n)}{n}$ are very close to each other.

Next we prove that for n large the ratio $\dfrac{F(n)}{n}$ does not exceed twice the length of the interval $[c, d]$. Suppose that $\dfrac{1}{r} \le d - c < \dfrac{1}{r-1}$ where r is an integer and $r \ge 2$. In the case $r = 2$ we have nothing to prove (since twice the length is at least 1); so we may assume that $r \ge 3$. We can find mutually disjoint intervals, $[c_1, d_1], \ldots, [c_{r-1}, d_{r-1}]$, each of the form $[c', d']$ for some k. For an n really big, the ratios $\dfrac{F_i(n)}{n}$ (calculated for intervals $[c_i, d_i]$) are almost the same, and since their sum does not exceed 1 (because $F_1(n) + \cdots + F_{r-1}(n) \le n$), each of them, for big n, will be less than any number greater $\dfrac{1}{r-1}$, in particular, some number less than $\dfrac{2}{r} \le 2(d - c)$; this is what we wanted to prove.

Therefore, if we slightly change c and d, then for n large the ratios $\dfrac{F(n)}{n}$ will stay almost unchanged. In particular, if two intervals have the same length, then these ratios for them are almost the same for n large.

From this we easily deduce our statement. For the intervals $\left[0, \dfrac{1}{r}\right]$, $\left[\dfrac{1}{r}, \dfrac{2}{r}\right], \ldots, \left[\dfrac{r-1}{r}, 1\right]$ the ratios $\dfrac{F(n)}{n}$ are almost the same for n large, and their sum is 1. Thus, $\displaystyle\lim_{n\to\infty} \frac{F(n)}{n} = \frac{1}{r} =$ length, and this is true for any interval of the length $\frac{1}{r}$. Consequently, it is true for any interval of rational length, and hence, by the remark above, for any interval at all. $\square$

The lemma implies our statement. Let γ_n be the fractional part of $n \log a$. The first digit of a^n is 1, if γ_n lies in the interval $[0, \log 2]$; it is 2, if γ_n lies in $[\log 2, \log 3]$; and so on. Our limit relation follows.

Remark. This distribution of the first digits can be observed not only for the sequence of powers of a fixed real number, but, in some sense, for any naturally defined sequence of numbers. (The reader who likes experimenting, can take, for example, the list of all cities in California. The populations will have first digits distributed in the same way.) This phenomenon is called "Benford's Law", after F. Benford who described it in 1938. (F. Benford was, in turn, inspired by observations made by S. Newcomb in 1881.) A rigorous mathematical explanation of Benford's Law is still missing. A good reference for this is the article "Benford's Law strikes back: no simple explanation in sight for mathematical gem" by A. Berge and T. P. Hill [**18***].

75. The domains $U, g(U), g^2(U), \dots$ cannot be all disjoint, since the area of M is finite, and the area of U is positive. Hence, some intersection $g^m(U) \cap g^n(U)$ with $m > n$ is non-empty. So, there exist $x, y \in U$ such that $g^m(x) = g^n(y)$. Applying g backward n times, we get $g^{m-n}(x) = y \in U$, and we can take $T = m - n$.

76. First, notice that this density property does not depend on the choice of $x \in M$. Indeed, if $x = (\xi, \eta) \bmod 2\pi$ and $x' = (\xi', \eta') \bmod 2\pi$, then the set $\{x', gx', g^2x', \dots\}$ can be obtained from $\{x, gx, g^2x, \dots\}$ by the transformation $(\alpha, \beta) \mapsto (\alpha - \xi + \xi', \beta - \eta + \eta') \bmod 2\pi$ which, obviously, would not affect the density. Second, we remark that to prove our statement we need to know that $1, \sqrt{2}$, and π are not comeasurable, that is, for no non-zero triple of integers p, q, r the number $p + q\sqrt{2} + r\pi$ is zero. The last fact follows from the statements that $\sqrt{2}$ is irrational and π is transcendental; these statements are broadly known, but the proof of the second one is not elementary, and we will not give it here.

Fix a small $\varepsilon > 0$, and let $x = (\pi, \pi)$. Cover the square $\{0 \le x \le 2\pi, 0 \le y \le 2\pi\} \subset \mathbb{R}^2$ by a finite family of disks $d_1, \dots, d_N$ of diameters $< \varepsilon$. For every n, the point g^n belongs to some d_r modulo 2π. Since the number of disks is finite, it is true that for some $k, \ell > k$, the points $g^k x, g^\ell x$ belong (modulo 2π) to the same d_r. Hence, $g^n x$, where $n = \ell - k$, lies in the disk of radius $\varepsilon/2$ centered at x. We will prove the following: the set $\{x, g^n x, g^{2n} x, g^{3n} x, \dots\}$ is ε-dense in the torus, that is, for every point of the torus, there exists a point of our set at the distance $< \varepsilon$ from this point. Since ε is arbitrary, this shows that the set $S = \{x, gx, g^2x, \dots\}$ is dense in the torus.

Modulo 2π, $g^n x$ is $y = (\pi + n + 2M\pi, \pi + n\sqrt{2} + 2N\pi)$ where M, N are integers and the distance δ from y to x is less than ε. Let L be the line in the plane passing through the points x and y. If we take on L the points at the distance $0, \delta, 2\delta, \ldots$ from x, then modulo 2π this will be our set S. Thus, S may be regarded as an $\varepsilon/2$-dense subset of the line L.

The line L intersects the horizontal line $y = 3\pi$ at some point $(\pi+\lambda, 3\pi)$ where $\lambda = 2\pi\dfrac{n+2M\pi}{n\sqrt{2}+2N\pi}$. It is important for us that $\dfrac{\lambda}{\pi}$ is not rational. But indeed, if $\dfrac{\lambda}{\pi} = \dfrac{p}{q}$, then

$$\frac{p}{q} = \frac{2n+4M\pi}{n\sqrt{2}+2N\pi},$$

so

$$p(n\sqrt{2}+2N\pi) = q(2n+4M\pi),$$

which obviously contradicts to the above-mentioned fact that 1, $\sqrt{2}$, and π are not comeasurable.

Notice now in Figure 65 that the horizontal shift of the line L by λ units produces the same line L' as the vertical shift of L by -2π units:

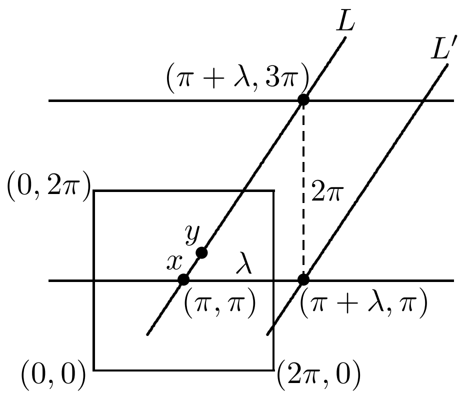


FIGURE 65. To Solution 76.

Obviously, from the point of view of the torus, L and L' are the same line (because L' is obtained from L by a vertical shift by -2π). Consequently, for arbitrary integers p, q, a horizontal shift of the line L by $p\lambda + 2q\pi$ does not change it as a subset of the torus. In particular, the set S can be regarded as an $\varepsilon/2$-dense subset of any such line. (Our proof will show, actually, that this set is dense, but we will not need that.)

Next, let us prove that the set of numbers of the form $p\lambda + 2q\pi$ is $\varepsilon/2$-dense in the real line. It is important that we already know that all these points are pairwise different (since λ/π is irrational). The proof is similar to first step of the current proof. Obviously, there are infinitely many points of our set in the interval $[0, 2\pi]$ (for every p there exists a q such that $p\lambda+2q\pi \in [0, 2\pi]$). Cover the interval $[0, \pi]$ by intervals $i_1, \ldots, i_N$ of lengths $< \varepsilon$. Since the number of intervals is finite, it is possible to find two different points of our set, $p_1\lambda + 2q_1\pi$ and $p_2\lambda + 2q_2\pi$ which belong to the same interval; we may assume that the second of these numbers is greater, than the first one. Then the number $p\lambda + 2q\pi$ where $p = p_2 - p_1$ and $q = q_2 - q_1$ lies in the interval $[0, \varepsilon]$, and the points $mp\lambda + 2mq\pi$ with $0 < m < 2\pi/\varepsilon$ form an $\varepsilon/2$-dense subset of $[0, 2\pi]$.

Now we can finish our proof. For every point (α, β) of the torus, there is a line (in the plane) at the distance less that $\varepsilon/2$ from (α, β) which contains S as an $\varepsilon/2$-dense subset. The point of this subset closest to (α, β) is at the distance $< \varepsilon$ from (α, β). This is all we need.

77. Every point of the form $(r\pi, s\pi)$ with rational r and s is periodic. Indeed, let q be a common denominator of r and s, so our point (α, β) has the form $\left(\dfrac{a}{q}\pi, \dfrac{b}{q}\pi\right)$ with non-negative integer $a < 2q$ and $b < 2q$. Then, all points $g(\alpha, \beta), g^2(\alpha, \beta), g^3(\alpha, \beta), \ldots$ have the same form (maybe, with different pairs a, b). But there are finitely many (at most $4q^2$) such points. Hence there must be an equality $g^m(\alpha, \beta) = g^n(\alpha, \beta)$ for some $m > n$. Our transformation g is invertible $g^{-1}(\alpha, \beta) = (\alpha - \beta, -\alpha + 2\beta)$. Apply n times g^{-1} to the equality $g^m(\alpha, \beta) = g^n(\alpha, \beta)$, and we will get $g^{m-n}(\alpha, \beta) = (\alpha, \beta)$. It is obvious that the set of points of this form is dense in the torus.

79. The solution is similar to our solution of Problem 74.