1

Surprising right triangles

The formula giving the sides of all right triangles with sides of integer lengths is well known. However, that formula would appear to be quite irrelevant to both problems of this section.

Problem 1. Find all right triangles with sides of integer lengths for which the hypotenuse is one unit longer than one of the legs.

Denote the lengths of the legs by a and b; the length of the hypotenuse is then $b+1$, say. Pythagoras' theorem gives $a^2+b^2=(b+1)^2$, whence $a^2=2b+1$, which implies that a must be odd. Writing $a=2k+1$, we obtain $4k^2+4k+1=2b+1$, whence $b=2k(k+1)$. Hence there are infinitely many such triangles, and the triples of the lengths of their sides are given by $(a,b,c)=\big(2k+1, 2k(k+1), 2k^2+2k+1\big)$, where k ranges over the natural numbers. Here are the first few such triples.

a	3	5	7	9	11	13	15
b	4	12	24	40	60	84	112
c	5	13	25	41	61	85	113

Thus this problem turned out to be quite easy. However, in mathematics it often happens that making what seems like a small change in the formulation of the problem results in a considerable increase in difficulty.

Problem 2. Find all right triangles with sides of integer lengths and with one leg one unit shorter than the other.

One such triangle quickly comes to mind, namely the "Egyptian" one with sides of lengths 3, 4 and 5. But can you find even one more such triangle? Denoting the length of the shorter leg by a and the length of the hypotenuse by c, we obtain the equation

$$a^2 + (a+1)^2 = c^2, \text{ or } 2a^2 + 2a + 1 = c^2.$$

From this we see that c must be odd, so we write $c = 2k + 1$, and obtain the equation $a(a+1) = 2k(k+1)$. But what do we do now? We might have recourse to a computer in order to find a few more solutions, such as: $(20, 21, 29)$ and $(119, 120, 169)$, but this sheds no light on the general problem.

Let's rewrite our equation as follows:

$$4a^2 + 4a + 2 = 2c^2, \quad \text{or} \quad (2a+1)^2 + 1 = 2c^2.$$

Setting $2a + 1 = x$ and $c = y$, we find we have arrived at *Pell's equation*, so-called:

$$x^2 - 2y^2 = -1. \tag{1}$$

One obvious solution of this equation is the pair $(x, y) = (1, 1)$.

With each pair (x, y) of integers solving equation (1), we associate the number $x + y\sqrt{2}$. Suppose now that (x_1, y_1) and (x_2, y_2) are two solutions of that equation. Consider the number

$$\begin{aligned} x + y\sqrt{2} &= (x_1 + y_1\sqrt{2})(x_2 + y_2\sqrt{2}) \\ &= x_1x_2 + 2y_1y_2 + (x_1y_2 + x_2y_1)\sqrt{2}. \end{aligned}$$

We then have

$$\begin{aligned} x^2 - 2y^2 &= (x_1x_2 + 2y_1y_2)^2 - 2(x_1y_2 + x_2y_1)^2 \\ &= x_1^2x_2^2 + 4x_1x_2y_1y_2 + 4y_1^2y_2^2 - 2x_1^2y_2^2 - 4x_1y_2x_2y_1 - 2x_2^2y_1^2 \\ &= x_1^2\left(x_2^2 - 2y_2^2\right) - 2y_1^2\left(x_2^2 - 2y_2^2\right) = \left(x_1^2 - 2y_1^2\right)\left(x_2^2 - 2y_2^2\right). \end{aligned}$$

Since by assumption $x_1^2 - 2y_1^2 = x_2^2 - 2y_2^2 = -1$, it follows that the pair (x, y) satisfies the equation

$$x^2 - 2y^2 = 1 \tag{2}$$

On the other hand, if (x_1, y_1) is a solution of equation (1), but (x_2, y_2) is a solution of equation (2), then the pair (x, y) will be a solution of equation (1). Since we know one such solution, namely $(1, 1)$, it follows that if we define the natural numbers x_n and y_n via $x_n + y_n\sqrt{2} = (1 + \sqrt{2}\,)^n$, then for odd n the pair (x_n, y_n) will be a solution of equation (1). Computing (easiest done on a computer), we obtain the following table:

n	1	3	5	7	9	11	13
x_n	1	7	41	239	1393	8119	47321
y_n	1	5	29	169	985	5741	33461

from which we quickly obtain the following table of lengths of sides of right triangles satisfying the condition of Problem 2:

a	3	20	119	696	4059	23660
b	4	21	120	697	4060	23661
c	5	29	169	985	5741	33461

As is clear from the numbers appearing in this tale, the increase in difficulty over Problem 1 is striking.

The answer to the obvious question as to whether the above procedure yields *all* solutions, is given by the following result.

Theorem 1.1. *The formula $x + y\sqrt{2} = \pm(1+\sqrt{2})^k$, where $k \in \mathbb{Z}$ and x and y are integers, yields all pairs (x, y) of integers satisfying equations* (1) *and* (2). *More precisely, if the integer k is even, then the pair (x, y) is a solution of equation* (2), *while if k is odd (x, y) is a solution of equation* (1) (*and these account for all solutions of those equations*).

For the proof we shall need two lemmas.

Lemma 1.2. *Let a and b be integers such that the number $a + b\sqrt{2}$ lies in the interval $(1, 1+\sqrt{2})$. Then the pair (a, b) cannot be a solution of either of the equations Pell* (1) *or* (2).

Proof. We argue "by contradiction". Thus we assume that $1 < a + b\sqrt{2} < 1+\sqrt{2}$ and $a^2 - 2b^2 = \pm 1$. Then since $a - b\sqrt{2} = \frac{\pm 1}{a+b\sqrt{2}}$ and $a + b\sqrt{2} > 1$ by assumption, we infer that $-1 < a - b\sqrt{2} < 1$. Adding the two pairs of inequalities then yields $0 < 2a < 2 + 2\sqrt{2}$. Since a is an odd integer, it follows that $a = 1$. Hence $1 < 1 + b\sqrt{2} < 1 + \sqrt{2}$, which is impossible since b is an integer. ■

The second lemma is immediate from the algebraic manipulations preceding the statement of Theorem 1.1.

Lemma 1.3. *If the pairs (x_1, y_1) and (x_2, y_2) are solutions of equation* (1) *or equation* (2) *(but not necessarily of the same one of these two equations)* *and* $a + b\sqrt{2} = (x_1 + y_1\sqrt{2})(x_2 + y_2\sqrt{2})$, *then the pair* (a, b) *is likewise a solution of one of the equations* (1) *or* (2). ■

Proof of Theorem 1.1. Write $\omega = 1 + \sqrt{2}$ and suppose (x, y) is a solution of either equation (1) or (2). We wish to show that $x + y\sqrt{2} = \omega^n$ for some integer n. The proof breaks up into cases.

Case 1. Assume first that $x + y\sqrt{2} > 1$ and $x + y\sqrt{2} \neq \omega^n$ for any integer n. In this case there must exist a natural number k such that $\omega^k < x + y\sqrt{2} < \omega^{k+1}$. Writing $a + b\sqrt{2} = (x + y\sqrt{2})\omega^{-k}$, we then have $1 < a + b\sqrt{2} < \omega$, so by Lemma 1.2 the pair (a, b) is not a solution of either of the equations (1) or (2). However, since $\omega^{-1} = -1 + \sqrt{2}$ and the pair $(-1, 1)$ is a solution of equation (1), it follows from Lemma 1.3 that the pair (a, b) *is* a solution of one of the equations (1) or (2). This contradiction completes the proof.

Case 2. Now suppose $0 < x + y\sqrt{2} < 1$ and assume for instance that (x, y) is a solution of equation (1). Since $x^2 - 2y^2 = (x - y\sqrt{2})(x + y\sqrt{2}) = -1$ we must have $-x + y\sqrt{2} > 1$. Note that $(-x, y)$ is also a solution of equation (1). Hence by the previous case, we have $-x + y\sqrt{2} = \omega^n$ for some natural number n, whence $x + y\sqrt{2} = (-x + y\sqrt{2})^{-1} = \omega^{-n}$. The case that (x, y) is a solution of equation (2) is similar.

Case 3. If $x + y\sqrt{2} < 0$ then $-x - y\sqrt{2} > 0$ and one or other of the preceding cases applies. ■

Corollary 1.1. *The formula* $x + y\sqrt{2} = (3 + 2\sqrt{2})^n = (1 + \sqrt{2})^{2n}$, *where* n *is a non-negative integer, gives all solutions of equation* (2) *in natural numbers.*

To see this, it is enough to observe that if n is a negative integer and $x + y\sqrt{2} = (3 + 2\sqrt{2})^n$, then $x + y\sqrt{2} = (3 - 2\sqrt{2})^{-n}$, so $y < 0$. ■

The following statement, providing the solution of Problem 2, is proved similarly.

Corollary 1.2. *The formula* $x + y\sqrt{2} = (1 + \sqrt{2})^{2n-1}$, $n \in \mathbb{N}$, *furnishes all natural solutions of equation* (1). ■

2
Surprisingly short solutions of geometric problems

Presenting lines, circles and other plane curves in terms of equations provides us with an opportunity for *calculating*. And moreover sometimes, as you shall shortly see, the translation "from geometry to algebra" yields shorter *proofs*; the solution of Problem 2 of the present section affords an illustration of this. To facilitate understanding of that solution, we first analyze the standard solution of a different problem.

Problem 1. Consider the circles with centers at the points $O_1(-1, 1)$ and $O_2(3, 2)$ and respective radii $r_1 = 3$ and $r_2 = 2$. Find the equation of the straight line through the points of intersection of these circles.

The circles in question have equations $(x + 1)^2 + (y - 1)^2 = 9$ and $(x - 3)^2 + (y - 2)^2 = 4$ (see Figure 1). Hence their points (x, y) of

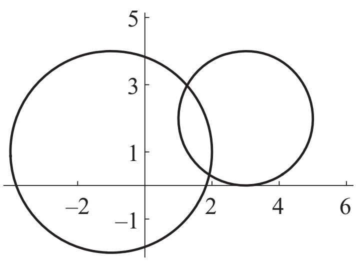

Figure 1

intersection are the solutions of the system

$$\begin{cases} x^2 + y^2 + 2x - 2y - 7 = 0, \\ x^2 + y^2 - 6x - 4y + 9 = 0. \end{cases}$$

Of course, this system can be solved. Subtracting the second equation from the first yields the equivalent system

$$\begin{cases} 4x + y = 8, \\ x^2 + y^2 - 6x - 4y + 9 = 0. \end{cases}$$

Substituting $y = 8 - 4x$ in the second equation then gives the equation

$$17x^2 - 54x + 41 = 0,$$

yielding $x = \frac{27 \pm 4\sqrt{2}}{17}$. Hence $y = \frac{28 \mp 16\sqrt{2}}{17}$. Thus the points of intersection of the given circles have coordinates $(x, y) = (\frac{27 \pm 4\sqrt{2}}{17}, \frac{28 \mp 16\sqrt{2}}{17})$.

We now find the equation of the straight line through these two points. We have

$$\frac{x - \frac{27+4\sqrt{2}}{17}}{\frac{8\sqrt{2}}{17}} = \frac{y - \frac{28-16\sqrt{2}}{17}}{-\frac{32\sqrt{2}}{17}} \quad \text{or} \quad 4\left(x - \frac{27+4\sqrt{2}}{17}\right) = -y + \frac{28-16\sqrt{2}}{17},$$

$$\text{or } 4x - \frac{108+16\sqrt{2}}{17} = -y + \frac{28-16\sqrt{2}}{17},$$

$$\text{or } 4x + y - \frac{136}{17} = 0, \ \text{ or } 4x + y - 8 = 0.$$

And what did we obtain? The same equation as appeared immediately at the beginning of the calculation! So perhaps rather than just *calculating* we should have done some *thinking*?!

Set $f_1(x, y) = x^2 + y^2 + 2x - 2y - 7$ and $f_2(x, y) = x^2 + y^2 - 6x - 4y + 9$. Since it follows from $f_1(x, y) = 0$ and $f_2(x, y) = 0$ that $f_1(x, y) = f_2(x, y)$, we infer that the points of intersection of the circles must satisfy the equation $f_1(x, y) = f_2(x, y)$, which is an equation of degree one, and therefore an equation of a straight line. Hence this equation must be a equation of the straight line through the points of intersection of the given circles.

Figure 2 shows three pairwise intersecting circles and the three straight lines through the points of intersection of pairs of the circles. This figure was drawn by means of a computer, which calculated the points of intersection of the circles and drew lines through the appropriate pairs of points. We see that in the diagram these three straight lines all pass through a single point, that is, are concurrent. Although it is certainly true that this general fact can

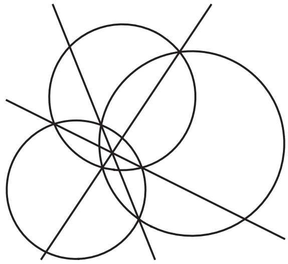

Figure 2

be established by purely geometric means, you will now see how very short (and natural) its algebraic proof is.

Problem 2. Suppose we are given three pairwise intersecting circles in the plane. For each pair of circles, consider the line through the two points of intersection of those circles. Prove that if no two of these three lines are parallel, then they are concurrent.

Each of the given circles has an equation of the form $f_i(x, y) = 0$, $i = 1, 2, 3$, where

$$f_i(x, y) = (x - x_i)^2 + (y - y_i)^2 - r_i^2, \quad i = 1, 2, 3.$$

As was shown in the solution to Problem 1 above, the lines ℓ_{12}, ℓ_{13} and ℓ_{23}, through the points of intersection of pairs of circles, have equations $f_1(x, y) = f_2(x, y)$, $f_1(x, y) = f_3(x, y)$ and $f_3(x, y) = f_2(x, y)$.

Let $M(x_0, y_0)$ denote the point of intersection of the lines ℓ_{12} and ℓ_{13}. Since the equations $f_1(x_0, y_0) = f_2(x_0, y_0)$ and $f_1(x_0, y_0) = f_3(x_0, y_0)$ together imply $f_2(x_0, y_0) = f_3(x_0, y_0)$, it follows that the point M also lies on the line ℓ_{23}, so that the three lines do indeed all pass through a single point.

An elegant argument, is it not? Later on, in Theme 8, we shall use modifications and generalizations of it. We conclude the section with another problem.

Problem 3. Prove that the four points of intersection of the two parabolas $y = 2x^2 + 2x - 3$ and $x = 3 - 2y - y^2$ lie on a circle.

Rewrite the equations of the given parabolas in the form

$$2x^2 + 2x - 3 - y = 0 \quad \text{and} \quad y^2 + x + 2y - 3 = 0.$$

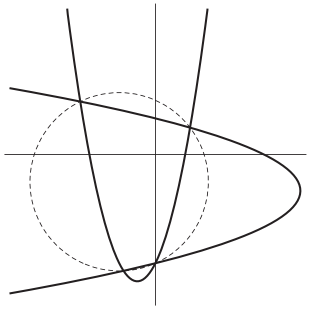

Figure 3

Adding twice the second of these equations to the first yields the equation $2x^2 + 2y^2 + 4x + 3y - 9 = 0$, which is the equation of a circle and is satisfied by all points of intersection of the given two curves (Figure 3).

3

A natural assertion with a surprising proof

It is well-known that for any triangle, the sum of the lengths of any two sides is greater than the length of the remaining side. Moreover, for any three positive numbers with the property that the sum of any two is greater than the third, there is a triangle with sides of lengths equal to the given numbers. It's strange that not even the *three*-dimensional analogues of these statements are to be found in the relevant mathematical literature.

We begin with the solution of the following problem.

Problem 1. Prove that the sum of the areas of any three faces of a tetrahedron is greater than the area of the fourth.

Consider any tetrahedron $ABCD$ and denote by P the (orthogonal) projection of the vertex D onto the plane of the face ABC. The triangles ABP, BCP and ACP are then the projections of the faces ABD, BCD and ACD respectively. Denote by θ the angle between the base plane and the side face ABD. Since by definition this angle is the angle between two half-lines, it must lie in the interval $(0, \pi)$. We first prove that $S_{ABP} = S_{ABD} \cdot |\cos\theta|$ (where S denotes area). Since $HP = HD \cdot |\cos\theta|$ (see Figure 4),

$$S_{ABP} = \frac{1}{2} AB \cdot HP = \frac{1}{2} AB \cdot HD \cdot |\cos\theta| = S_{ABD} \cdot |\cos\theta|.$$

Thus $S_{ABP} < S_{ABD}$. We conclude that the area of the projection of any of the three side faces is less that the actual area of that face.

Now suppose first that the point P is inside (or on the boundary of) the triangle ABC (as in Figure 5a). In this case we have

$$S_{ABC} = S_{ABP} + S_{BCP} + S_{ACP} < S_{ABD} + S_{BCD} + S_{ACD},$$

as we wished to prove. We now examine the other possibilities.

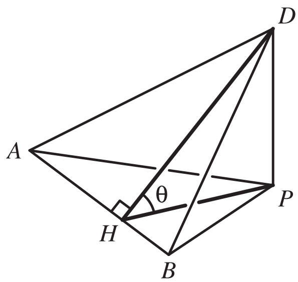

Figure 4

The projection of the tetrahedron onto the plane of its face ABC may be a triangle or a quadrilateral. If it is a triangle there are two possibilities: either P, the projection of the vertex D, is inside the triangle ABC, which is the case we have already dealt with, or one of the vertices A, B, or C lies inside the triangle formed by P and the other two of A, B, C. We may assume the situation is as in Figure 5b, in which case we have $S_{ABC} \leq S_{ABP} < S_{ABD} < S_{ABD} + S_{BCD} + S_{ACD}$. The final case is that where the points A, B, C and P are the vertices of a convex quadrilateral, with the vertex diagonally opposite P being A, say, as in Figure 5c. In this case we have $S_{ABC} \leq S_{ABP} + S_{ACP} < S_{ABD} + S_{ACD} < S_{ABD} + S_{BCD} + S_{ACD}$.

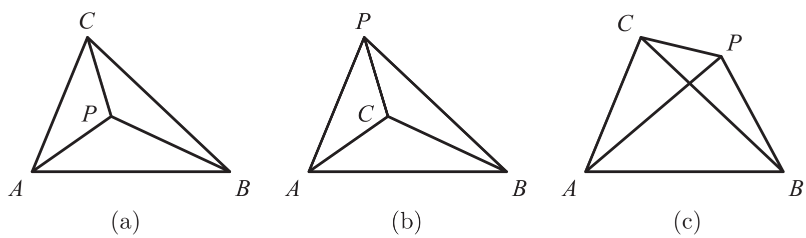

Figure 5

It is natural to ask if the converse statement is true. This Theme is devoted to the proof that this is indeed the case.

Theorem 3.1. *Given any four positive numbers with the property that the sum of any three is greater than the fourth, there is a tetrahedron with faces of areas equal to the given numbers.*

The proof is based on the following result.

Problem 2. For a given tetrahedron, let $\boldsymbol{n}_i$, $i = 1, 2, 3, 4$, denote the vector perpendicular to the ith face, of length equal to the area of that face and directed outwards from the tetrahedron. Then $\boldsymbol{n}_1 + \boldsymbol{n}_2 + \boldsymbol{n}_3 + \boldsymbol{n}_4 = \mathbf{0}$.

A solution of this problem "in one line" will be given later on in the exposition of Theme 14. The argument we give here is significantly longer, but "more elementary".

We shall show that the projection of the sum $\boldsymbol{n}_1 + \boldsymbol{n}_2 + \boldsymbol{n}_3 + \boldsymbol{n}_4$ on any line perpendicular to the plane of a face is zero, whence the desired conclusion. To this end, we introduce a coordinate system with the property that the face ABC lies in the Oxy-plane, with the z-axis oriented so that the vector $\boldsymbol{n}_4$ perpendicular to that face points in the negative z-direction. Consider the vector $\boldsymbol{n}_1$ perpendicular to the face ABD. The angle θ_{AB} between it and the Oz-axis is equal to the angle between the faces ABC and ABD meeting in the edge AB of the tetrahedron. Hence the projection of that vector on the Oz-axis (that is, the z-component of that vector) is equal to $|\boldsymbol{n}_1| \cos\theta_{AB} = S_{ABD} \cos\theta_{AB} = \pm S_{ABP}$ (in the notation of Problem 1), the sign depending on whether θ_{AB} is acute or obtuse.

Assume first that the angle between every two faces is acute. In this case the projection of each vertex on the plane of the opposite face lies inside that face (as in Figure 6). Since the angles at the edges AB, BC, and AC are acute in this case, it follows that the projection of the sum $\boldsymbol{n}_1 + \boldsymbol{n}_2 + \boldsymbol{n}_3$ on the Oz-axis is equal to $S_{ABP} + S_{BCP} + S_{ACP} = S_{ABC}$. Then since the z-coordinate of the vector $\boldsymbol{n}_4$ is $-S_{ABC}$, we conclude that the projection of the sum of all four vectors is zero.

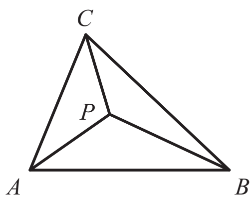

Figure 6

Next suppose that the angle at the edge BC of the tetrahedron is obtuse and those at the edges AB and AC are acute. In this case the points A, B, P and C are the vertices of a quadrilateral (as in Figure 7). In view of the

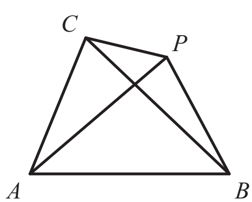

Figure 7

obtuseness of the angle at BC, the z-coordinate of the vector $\boldsymbol{n}_i$ perpendicular to the face BCD is $-S_{BCP}$, so the z-coordinate of the sum of the four vectors is

$$S_{ABP} - S_{BCP} + S_{ACP} - S_{ABC}.$$

Since in this situation we have $S_{ABP} + S_{ACP} = S_{BCP} + S_{ABC}$, we once again have that the projection of the sum of the four vectors on a line perpendicular to the face ABC is zero.

By now it will, one hopes, have become clear how to complete the argument. Suppose the vertex C lies in the triangle ABP (as in Figure 8), which will occur if the angles at the edges AC and BC are both obtuse. In this case we shall have that the z-coordinate of the sum of the vectors $\boldsymbol{n}_i$ is equal to $S_{ABP} - S_{ACP} - S_{BCP} - S_{ABC} = 0$.

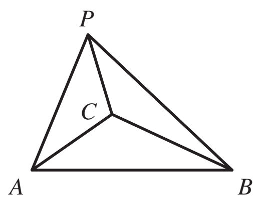

Figure 8

And now we are ready to prove the main result.

Proof of Theorem 3.1. Let a_1, a_2, a_3 and a_4 be positive numbers such that the sum of any three is greater than the fourth. This condition suffices for the existence of a convex quadrilateral with sides of lengths a_1, a_2, a_3 and a_4. By "bending" this quadrilateral along a diagonal, we obtain a non-planar closed curve made up of four straight segments of the given lengths. Imagine "arrows" drawn along these edges in order; the sum of the four vectors in 3-space thus defined will then be zero (see Figure 9). Lay out from the origin of coordinate 3-space rays parallel to these four vectors and for each

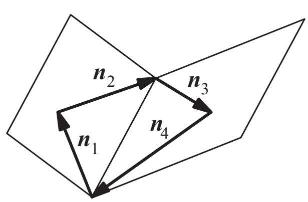

Figure 9

such ray choose a plane intersecting it in a point away from the origin and perpendicular to it. In this way we obtain a tetrahedron with faces on the chosen planes.

Denote by $\boldsymbol{m}_1$, $\boldsymbol{m}_2$, $\boldsymbol{m}_3$ and $\boldsymbol{m}_4$ outwardly directed vectors perpendicular to the faces of this tetrahedron and of lengths equal to the respective areas of the faces. By construction, we have $\boldsymbol{n}_i \parallel \boldsymbol{m}_i$, $i = 1, 2, 3, 4$, and by Problem 2 we also have $\boldsymbol{m}_1 + \boldsymbol{m}_2 + \boldsymbol{m}_3 + \boldsymbol{m}_4 = \mathbf{0}$.

We shall need the following auxiliary result.

Lemma 3.2. *Let $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$ and $\boldsymbol{n}_4$ be non-coplanar vectors satisfying $\boldsymbol{n}_1 + \boldsymbol{n}_2 + \boldsymbol{n}_3 + \boldsymbol{n}_4 = \mathbf{0}$, and let $\boldsymbol{m}_1$, $\boldsymbol{m}_2$, $\boldsymbol{m}_3$ and $\boldsymbol{m}_4$ be such that $\boldsymbol{m}_1 + \boldsymbol{m}_2 + \boldsymbol{m}_3 + \boldsymbol{m}_4 = \mathbf{0}$ and $\boldsymbol{m}_i \parallel \boldsymbol{n}_i$, $i = 1, 2, 3, 4$. Then there exists a number α such that $\boldsymbol{m}_i = \alpha \boldsymbol{n}_i$, $i = 1, 2, 3, 4$.*

Proof. Observe first that it follows from the assumptions of the lemma that no three of the vectors $\boldsymbol{n}_i$, $i = 1, 2, 3, 4$ are coplanar. For, if for instance $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ were all parallel to some plane, then the condition that the sum of all four $\boldsymbol{n}_i$ is zero would imply that the vector $\boldsymbol{n}_4$ was also parallel to that plane. Since $\boldsymbol{n}_i \parallel \boldsymbol{m}_i$, we have $\boldsymbol{m}_i = \alpha_i \boldsymbol{n}_i$. Then

$$\begin{aligned}\alpha_4(\boldsymbol{n}_1 + \boldsymbol{n}_2 + \boldsymbol{n}_3) = -\alpha_4 \boldsymbol{n}_4 - \boldsymbol{m}_4 &= \boldsymbol{m}_1 + \boldsymbol{m}_2 + \boldsymbol{m}_3 \\ &= \alpha_1 \boldsymbol{n}_1 + \alpha_2 \boldsymbol{n}_2 + \alpha_3 \boldsymbol{n}_3,\end{aligned}$$

whence $(\alpha_1 - \alpha_4)\boldsymbol{n}_1 + (\alpha_2 - \alpha_4)\boldsymbol{n}_2 + (\alpha_3 - \alpha_4)\boldsymbol{n}_3 = \mathbf{0}$. However, since the vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ are not coplanar, we must then have $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$, and the lemma is proved. ■

To complete the proof of the Theorem 3.1 it now suffices to observe that the areas S_i of the faces of our tetrahedron satisfy $S_i = |\boldsymbol{m}_i| = \alpha |\boldsymbol{n}_i| = \alpha a_i$. Hence there is a tetrahedron similar to this one with the given numbers a_1, a_2, a_3 and a_4 as the areas of its faces. ■

We shall continue the discussion of such tetrahedra subsequently in the context of Theme 15. To conclude the present section, we note that, from a methodological point of view, the lemma used in the proof can be reformulated in purely geometrical terminology and in this form suggested to the students as a problem for them to solve independently.

Problem 3. Suppose that the edges and diagonal issuing from some vertex of a parallelepiped are pairwise parallel to the edges and diagonal issuing from a vertex of another parallelepiped. Prove that then the lengths of the edges of these parallelepipeds are proportional.

4

Surprising answers

There are certain problems whose solutions are surprising because they contradict our intuition. A typical example of this is afforded by our first problem, which, although very simple, yields an answer so strange that one feels compelled to go back and check the calculation.

Problem 1. The ends of a rope of length 20.1 feet are attached to hooks a distance 20 feet apart fixed at the same height in a wall. What distance will the midpoint of the the rope be displaced vertically by a weight hung at that midpoint?

Figure 10

The answer is, of course, the length of the shorter leg of a right triangle with hypotenuse 10.05 ft and the other leg 10 ft (Figure 10). Hence

$$h = \sqrt{10.05^2 - 10^2} = \sqrt{0.05 \cdot 20.05} = \sqrt{1.0025} \approx 1.00125.$$

Thus the weighted rope will hang down by more than a foot!

The next problem is no more difficult, and has a no less surprising answer.

Problem 2. A rope is tied around the Earth's equator and then lengthened by 6 feet. How high can the rope be raised off the equator to the same height all the way round? In particular, could a mouse creep under it?

Let R be the radius of the Earth and h the height of the rope above the surface. On the one hand, the length of the rope is $2\pi(R+h)$, and on the other, $2\pi R + 6$. Hence $2\pi(R+h) = 2\pi R + 6$, whence $h = \frac{3}{\pi} \approx 0.96$ ft. Thus not only could a mouse creep under the rope, but also a cat, and even some breeds of dog.

It is surprising also that the answer is independent of the radius. Thus if, for example, a rope were stretched around Jupiter's equator, lengthened by 6 feet, and then raised uniformly above the surface, the gap between the rope and Jupiter's equator would be the same as for the Earth.

We now turn to the main problem of this section.

Problem 3. Now suppose that the rope of length 6 feet greater than the distance around the Earth's equator is pulled away from the Earth's surface at just one point. How high above the surface can that point of the rope be pulled?

Figure 11 shows the shape of the rope. It is in contact with the surface of the Earth nearly everywhere, and those two portions of it where it isn't, form straight-line segments tangential to the surface. We write x for the size of

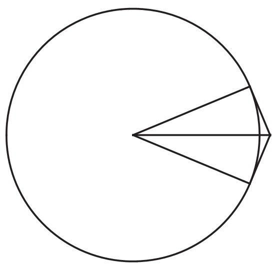

Figure 11

the angle formed by the radius from the Earth's center to one of the points of tangency and the line segment joining the Earth's center to the point where the rope has an "angle". The length of the rope is made up of the length of that part in contact with the Earth's surface and that of the two straight-line segments, that is,

$$R(2\pi - 2x) + 2R\tan x.$$

Since by assumption the length of the rope is $2\pi R + 6$, we obtain the equation

$$R(2\pi - 2x) + 2R\tan x = 2\pi R + 6, \text{ or } \tan x - x = \frac{3}{R},$$

where $R \approx 2.1 \cdot 10^7$ feet. Thus $\frac{1}{R}$ is very small.

Clearly the equation $\tan x - x = a$ cannot be solved exactly, so we shall need to make an approximation. But then, since here we have $a \approx 1.42857 \cdot 10^{-7}$ "very small indeed", to what degree of accuracy do we need to make our approximate calculation?

As it turns out we don't actually need to do any such calculation; it's all much simpler. The desired height h can be found from the equation

$$h + R = \frac{R}{\cos x}, \text{ whence } h = R\,\frac{1-\cos x}{\cos x}.$$

We use the following approximations, good for small values of x:

$$\tan x \approx x + \frac{x^3}{3} \quad \text{and} \quad \cos x \approx 1 - \frac{x^2}{2}\,. \tag{1}$$

The approximations (1) follow from *Taylor's theorem*, which we shall introduce in Theme 24. In this section we shall prove them by more *ad hoc* means.

Thus, assuming these approximations, we may replace the equation $\tan x - x = \frac{3}{R}$ by the equation $\frac{x^3}{3} = \frac{3}{R}$, which has solution $x = \sqrt[3]{\frac{9}{R}}$. The formula $h = R\,\frac{1-\cos x}{\cos x}$ may likewise be replaced by $h = R\,\frac{x^2}{2}$, yielding, with the approximate value of x just obtained,

$$h = \frac{R}{2}\sqrt[3]{\frac{81}{R^2}} = \frac{3}{2}\sqrt[3]{3R} \approx 596.859.$$

Thus we conclude that at its point of greatest height above the Earth's surface the rope is almost 600 feet high!

Of course, this argument is not rigorous. After all, given a statement that one expression is approximately equal to another, it is essential to know *how* accurate the approximation is. One might say that our argument was carried out "on a physical level of rigor". However, as it turns out the use of more precise methods of approximation, our rough approximation did in fact yield a satisfactory accurate answer.

In order to establish the approximations (1), we first remind the reader of what one might call "the first nontrivial limit":

$$\lim_{x\to 0}\frac{\sin x}{x} = 1. \tag{2}$$

The limit expression (2) means that for small x one has the approximate equality $\sin x \approx x$. Then since $1 - \cos x = 2\sin^2\frac{x}{2}$, we infer that

$$1 - \cos x \approx 2\cdot\frac{x^2}{4} = \frac{x^2}{2},$$

so that $\cos x \approx 1 - \frac{x^2}{2}$, the second of the approximations in (1).

The first approximation in (1) has a more roundabout proof. For this we need "Cauchy's mean-value theorem", a basic result of the differential calculus.

Theorem 4.1 (Cauchy)**.** *Let $f(x)$ and $g(x)$ be defined and continuous on the interval $[a, b]$ and differentiable on the interval (a, b). Suppose also that $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists a number $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. Consider the auxiliary function

$$h(x) = (f(x) - f(a))(g(b) - g(a)) - (g(x) - g(a))(f(b) - f(a)).$$

It is easy to see that $h(a) = h(b) = 0$. Hence by Rolle's theorem there exists a number $c \in (a, b)$ for which $h'(c) = 0$. Then since

$$h'(x) = f'(x)(g(b) - g(a)) - g'(x)(f(b) - f(a)),$$

it follows that

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)),$$

whence the desired conclusion. ■

We now prove a series of lemmas leading to the desired approximation.

Lemma 4.2. *If $\frac{f(x)}{g(x)} \to 1$ as $x \to 0$, then for small x the approximate equality $\int_0^x f(t)\,dt \approx \int_0^x g(t)\,dt$ holds.*

Write $F(x) = \int_0^x f(t)\,dt$ and $G(x) = \int_0^x g(t)\,dt$. The lemma asserts that

$$\lim_{x\to 0} \frac{F(x)}{G(x)} = 1.$$

Since $F'(x) = f(x)$ and $G'(x) = g(x)$, it follows from Cauchy's mean value theorem that

$$\lim_{x\to 0} \frac{F(x)}{G(x)} = \lim_{c\to 0} \frac{f(c)}{g(c)} = 1. \qquad ■$$

Lemma 4.3. *The approximation $\sin x \approx x - \frac{x^3}{6}$ is valid for small x.*

Since

$$x - \sin x = \int_0^x (1 - \cos t)\,dt$$

and, as has already been proved,

$$\frac{1 - \cos x}{x^2} \to \frac{1}{2} \text{ as } x \to 0,$$

it follows from Lemma 4.2 that

$$x - \sin x \approx \int_0^x \frac{t^2}{2}\, dt = \frac{x^3}{6}. \qquad \blacksquare$$

Lemma 4.4. *The approximation* $\tan x \approx x + \frac{x^3}{3}$ *is valid for small* x.

In fact, since

$$\tan x - \sin x = \sin x \left(\frac{1}{\cos x} - 1 \right) = \frac{\sin x \cdot 2 \sin^2 \frac{x}{2}}{\cos x} \approx \frac{x^3}{2},$$

we have that

$$\tan x \approx \sin x + \frac{x^3}{2} \approx x - \frac{x^3}{6} + \frac{x^3}{2} = x + \frac{x^3}{3}. \qquad \blacksquare$$

5

A surprising connection between three sequences

In Theme 1 we introduced pairs (a_n, b_n) of natural numbers defined by the equation $a_n + b_n\sqrt{2} = (1+\sqrt{2})^n$. It is actually easier to calculate these numbers using the recurrence relation they satisfy.

Lemma 5.1. *The sequences (a_n) and (b_n) satisfy the the recurrence relations $a_n = a_{n-1} + 2b_{n-1}$, $b_n = a_{n-1} + b_{n-1}$, $a_1 = b_1 = 1$.*

Proof. Here one merely observes that

$$\begin{aligned} a_n + b_n\sqrt{2} &= (1+\sqrt{2})^n = (1+\sqrt{2})(1+\sqrt{2})^{n-1} \\ &= (1+\sqrt{2})(a_{n-1} + b_{n-1}\sqrt{2}) \\ &= a_{n-1} + 2b_{n-1} + (a_{n-1} + b_{n-1})\sqrt{2}, \end{aligned}$$

and the lemma is proved. ■

The first sequence (x_n) of interest to us in this Theme has as terms the ratios of a_n to b_n; thus $x_n = \frac{a_n}{b_n}$. The recurrence relations given in the above lemma yield a recurrence relation for the sequence (x_n):

$$x_n = \frac{a_n}{b_n} = \frac{a_{n-1} + 2b_{n-1}}{a_{n-1} + b_{n-1}} = \frac{\frac{a_{n-1}}{b_{n-1}} + 2}{\frac{a_{n-1}}{b_{n-1}} + 1} = \frac{x_{n-1} + 2}{x_{n-1} + 1} \text{ and } x_1 = 1.$$

Hence the first few terms of (x_n) are as follows:

$$1,\ \frac{3}{2},\ \frac{7}{5},\ \frac{17}{12},\ \frac{41}{29},\ \frac{99}{70},\ \frac{239}{169},\ \frac{577}{408},\ \frac{1393}{985},\ \frac{3363}{2378},\ \frac{8119}{5741}.$$

Note that the fractions $\frac{a_n}{b_n}$ are reduced as written since

$$\begin{aligned}\gcd(a_n, b_n) &= \gcd(a_{n-1} + 2b_{n-1}, a_{n-1} + b_{n-1}) = \gcd(b_{n-1}, a_{n-1} + b_{n-1}) \\ &= \gcd(b_{n-1}, a_{n-1}) = \cdots = \gcd(a_1, b_1) = 1.\end{aligned}$$

In order to get an idea of the behavior of the sequence (x_n), we compute the decimal expansion (to seven places) of the first few terms:

1.000000
1.500000
1.400000
1.416667
1.413793
1.414201
1.414216
1.414213
1.414214

There is no point in computing further terms since clearly they will all have the same digits in the first seven decimal places. It looks very much as if the sequence (x_n) converges to $\sqrt{2}$. With a view to proving this, we estimate $x_n^2 - 2$. Since

$$x_n^2 - 2 = \frac{(x_{n-1} + 2)^2 - 2(x_{n-1} + 1)^2}{(x_{n-1} + 1)^2} = \frac{2 - x_{n-1}^2}{(x_{n-1} + 1)^2}$$

and $x_{n-1} \geq 1$, we infer the inequality

$$\left|x_n^2 - 2\right| \leq \tfrac{1}{4} \left|x_{n-1}^2 - 2\right|,$$

so that $x_n^2 \to 2$, whence $x_n \to \sqrt{2}$.

We define our second sequence, having the same limit $\sqrt{2}$, by means of Newton's tangent method applied to the equation $x^2 - 2 = 0$. Thus we start with any number x_0, and, as the first step, find the equation of the tangent line to the graph of $y = x^2 - 2$ at the point on it with abscissa x_0. We obtain in the usual way $y = x_0^2 - 2 + 2x_0(x - x_0) = 2x_0x - x_0^2 - 2$. The abscissa of the point of intersection of this line with the x-axis is then the solution of the equation $2x_0x = x_0^2 + 2$, yielding $x = \frac{x_0^2+2}{2x_0}$. Figure 12 shows the graph of $y = x^2 - 2$ and its tangent line to the point (2, 2).

Hence the sequence defined by the recurrence relation $y_n = \frac{y_{n-1}^2+2}{2y_{n-1}}$ together with $y_1 = 1$, is a sequence of successive approximations to $\sqrt{2}$, obtained via Newton's tangent method. It is well known (and will be proved in connection with Theme 23) that this sequence does in fact converge (very

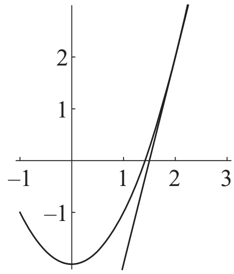

Figure 12

rapidly) to its limit. Here are the first few terms:

$$1,\ \frac{3}{2},\ \frac{17}{12},\ \frac{577}{408},\ \frac{665\,857}{470\,832}.$$

The first terms of our two sequences coincide: $y_1 = x_1$. However, the coincidences don't end there: we see that $y_2 = x_2$, $y_3 = x_4$ and $y_4 = x_8$ and calculation of further terms of the first sequence yields $y_5 = x_{16}$. Such coincidences might suggest the general hypothesis that $y_n = x_{2^{n-1}}$.

However, there are more peculiarities to come. Let's look at a third sequence (z_n) with the same first term $z_1 = 1$, this time given by the recurrence relation $z_n = \frac{z_{n-1}^2+4z_{n-1}+2}{z_{n-1}^2+2z_{n-1}+2}$. Its first few terms are

$$1,\ \frac{7}{5},\ \frac{239}{169},\ \frac{275\,807}{195\,025},\ \frac{367\,296\,043\,199}{259\,717\,522\,849}$$

from which we see that $z_2 = x_3$ and $z_3 = x_7$. Further calculation leads one to suspect that quite generally $z_n = x_{2^n-1}$.

In fact the sequence (z_n) can also be obtained via an application of Newton's tangent method—this time to the equation $x - 1 - \frac{1}{x+1} = 0$. Writing $f(x) = x - 1 - \frac{1}{x+1}$, one has $f'(x) = 1 + \frac{1}{(x+1)^2} = \frac{x^2+2x+2}{(x+1)^2}$. Hence

$$\begin{aligned}
z_n &= z_{n-1} - \frac{f(z_{n-1})}{f'(z_{n-1})}\\
&= z_{n-1} - \frac{z_{n-1}^2 - 2}{z_{n-1} + 1} \cdot \frac{(z_{n-1}+1)^2}{z_{n-1}^2 + 2z_{n-1} + 2}\\
&= \frac{z_{n-1}^3 + 2z_{n-1}^2 + 2z_{n-1} - z_{n-1}^3 - z_{n-1}^2 + 2z_{n-1} + 2}{z_{n-1}^2 + 2z_{n-1} + 2}\\
&= \frac{z_{n-1}^2 + 4z_{n-1} + 2}{z_{n-1}^2 + 2z_{n-1} + 2}.
\end{aligned}$$

The main result of this section is the following.

Theorem 5.2. *The following equalities are valid for all n:*

$$y_n = x_{2^{n-1}} \tag{1}$$

$$z_n = x_{2^n-1}. \tag{2}$$

These coincidences are surprising, and so also are their proofs, which verge on the obvious.

Proof. Define

$$c_n + d_n\sqrt{2} = (1+\sqrt{2})^{2^{n-1}} = a_{2^{n-1}} + b_{2^{n-1}}\sqrt{2}.$$

Then

$$c_n + d_n\sqrt{2} = (c_{n-1} + d_{n-1}\sqrt{2})^2 = c_{n-1}^2 + 2d_{n-1}^2 + 2c_{n-1}d_{n-1}\sqrt{2},$$

so that $c_n = c_{n-1}^2 + 2d_{n-1}^2$ and $d_n = 2c_{n-1}d_{n-1}$, whence

$$\frac{c_n}{d_n} = \frac{c_{n-1}^2 + 2d_{n-1}^2}{2c_{n-1}d_{n-1}} = \frac{\left(\frac{c_{n-1}}{d_{n-1}}\right)^2 + 2}{2\frac{c_{n-1}}{d_{n-1}}}.$$

Thus the sequence $(\frac{c_n}{d_n})$ satisfies the same recurrence relation as the sequence (y_n). And then since $y_1 = 1 = \frac{c_1}{d_1}$, it follows that

$$y_n = \frac{c_n}{d_n} = \frac{a_{2^{n-1}}}{b_{2^{n-1}}} = x_{2^{n-1}}.$$

Next define

$$u_n + v_n\sqrt{2} = (1+\sqrt{2})^{2^n-1} = a_{2^n-1} + b_{2^n-1}\sqrt{2}.$$

Since $2^n - 1 = 2(2^{n-1} - 1) + 1$, we have

$$\begin{aligned} u_n + v_n\sqrt{2} &= (1+\sqrt{2})(u_{n-1} + v_{n-1}\sqrt{2})^2 \\ &= (1+\sqrt{2})\big(u_{n-1}^2 + 2v_{n-1}^2 + 2u_{n-1}v_{n-1}\sqrt{2}\big) \\ &= u_{n-1}^2 + 4u_{n-1}v_{n-1} + 2v_{n-1}^2 + \big(u_{n-1}^2 + 2u_{n-1}v_{n-1} + 2v_{n-1}^2\big)\sqrt{2}, \end{aligned}$$

whence $z_n = \frac{u_n}{v_n} = x_{2^n-1}$. This concludes the proof of the theorem. ■

The fact that the sequences of this section are related to those arising in the solution of Problem 2 of Theme 1 is not accidental. This connection will be pursued further in the context of Theme 22.