A NOTE ON THE DIFFERENCE BETWEEN THE YEAR-2006 VERSION AND THE BOOK VERSION OF [FOOO].

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1. Introduction

In this document we explain the relationships between various versions of the monograph “Lagrangian intersection Floer theory: anomaly and obstructions, I & II”.

This document is written mainly for those who read some of the earlier versions of this book. The published version [FOOO4] itself is complete and does not rely on any of the previous versions. Hence readers of [FOOO4] do not need to read any of the earlier versions or this document, unless they would like to compare them with the published version.

Several people have helped us finding errors by asking questions throughout the time of writing this book. We thank all of them for their help.

There are 4 versions prior to the final published version [FOOO4]. There are several errors in the earlier versions of this book, which are corrected in later versions. There are a few places where we made some changes in the statements of some theorems in the earlier versions, which we found is necessary to accommodate the relevant proofs we have constructed in the course of time. The purpose of this document is to explain what kind of changes we have made in the final published version [FOOO4] from the earlier versions.

We started the project of writing this book in 1998 and a preliminary version was completed in December of the year 2000. This first version [FOOO1] was distributed as No.00-17 of the preprint series of Kyoto University, Department of Mathematics. We also put it on the first named authors home page (http://www.math.kyoto-u.ac.jp/~fukaya/) at the same time which is still there in the original form intact. This version is around 300 pages.

In the mean time, many of the results from this book have been further explained and improved in various articles of the authors since the appearance of [FOOO1]. (See [CO], [Fu03, Fu04], [Oht].)

The second version [FOOO2] was completed on March 2006 and was distributed privately to about 30-40 researchers or so. This version was also submitted to JSPS as a part of the official report of JSPS Grant-in-Aid for Scientific Research
K. FUKAYA, Y.-G. OH, H. OHTA, K. ONO

# 30165261, which the three Japanese authors received during the year 2001-2005. This version has been given out to a few more people upon request. The volume of this version [FOOO2], which contains Chapters 1-7, 9 and appendix, is about 800 pages with about 100 pages of figures in addition.

In 2007 November, we added two more chapters (Chapters 8 and 10) to [FOOO2] with some minor modifications made thereon. The added two chapters together with appendix and Chapter 9 (in a slightly modified form) are put on the first named author’s home page (http://www.math.kyoto-u.ac.jp/~fukaya/). We quote it as [FOOO3] in this document. Namely [FOOO3] consists of [FOOO2] and Chapters 8 and 10. (Chapter 9 and appendix of [FOOO2] are those on the home page which are slightly modified from [FOOO3].) The volume of the version [FOOO3] is around 1400 pages in total.

After some modification, this version [FOOO3] was sent to the publisher, International Press, for the publication in the series AMS/IP Studies of Advanced Mathematics. For the size restriction put on the volumes in this series, the publisher requested us to shorten the volume of the manuscript to 1000 pages. So we took out Chapters 8 and 10 from [FOOO3], and made some further necessary modifications on the rest of the manuscript to suit the publisher’s purely editorial request. (It is not because of any mathematical problem in the manuscript.) The revised manuscript then was sent to the publisher in March 2009. After some further non-mathematical changes, such as changing the numbering of the theorems, are made, the final version of the book was published as [FOOO4] in October 2009. The volume of the final version is approximately 800 pages. Actually the font size of [FOOO4] is smaller than one of [FOOO3]. So if one use the same font as [FOOO3], the book [FOOO4] would become 1000 pages. So what is removed from [FOOO3] is mainly Chapters 8 and 10.

We are currently in the process rewriting the material of the parts taken out into a separate research monograph and a few papers. We are writing a book [FOOO5] based on [FOOO3] Chapter 10. Actually we are improving the results of [FOOO3] Chapter 10 and adding new results. The final version of the book [FOOO5] will include new results in addition to those already in [FOOO3] Chapter 10. Furthermore [FOOO3] section 37 (which was in [FOOO3] Chapter 8) will be also included in [FOOO5]. Based on the material of Chapter 8 [FOOO3], we are writing three separate research articles [FOOO6, FOOO7]. We hope to complete these process in a near future. Meanwhile Chapters 8 and 10 of [FOOO3] are still kept in the first named author’s home page in the exactly the same form as we put in 2007. We will also put a slightly modified version of [FOOO3] Chapters 8. (See §5 of this document for the difference.) We quote it as [FOOO8] in this document.

We changed the numbering of theorems, lemmas, formulas etc. when we modified Chapters 1-7 of [FOOO2, FOOO3] to Chapters 1-7 of [FOOO4], and Chapter 9 of [FOOO2, FOOO3] to Chapter 8 of [FOOO4]. Especially the way of enumerating sections are changed. We put the table of the changes of the enumeration of sections at the first named author’s home page. The number of the theorems etc. are usually changed accordingly. For example Section 13 of [FOOO2] becomes Section 3.8 in [FOOO3]. So Theorem 13.32 of [FOOO2] becomes Theorem 3.8.32 in [FOOO3]. (However in some case there are more changes.)
We also remark that some of Theorems (etc.) A-Z of [FOOO2] are renamed in [FOOO4]. We also put the list of these changes at the first named author’s home page.

2. Errors in Theorems stated in the introduction [FOOO1].

Among Theorems A-K stated in the introduction of [FOOO1], there was an error in the statement of [FOOO1] Theorem H. In fact, we need an additional assumption in our proof of [FOOO1] Theorem H, which then becomes [FOOO2] Theorem M. We still believe that the original statement of [FOOO1] Theorem H is correct but have not been able to find its proof at the time of writing this document. The proof of [FOOO1] Theorem H has a serious gap, which was pointed out by U. Frauenfelder. More precisely, we overlooked the possibility that the involution on the moduli space of pseudo-holomorphic discs induced by an anti-symplectic involution of the given symplectic manifold may have a fixed point. This phenomenon is relevant to the proof of [FOOO1] Proposition 11.24. We can use [FOOO2] Proposition X to partially fix this gap. The main idea we use for this purpose is to inductively introduce a secondary involution \( \tau^{(1)} \) as well as higher involutions \( \tau^{(i)} \) on the neighborhoods of the fixed point sets. This idea is due to the fourth-named author, who communicated the idea with Frauenfelder, without working it out in detail. Frauenfelder used the idea to prove the Arnold-Givental conjecture for some particular cases of Lagrangian submanifold in a symplectic quotient [Fra]. This idea of using the secondary involutions is quite hard to apply at the boundary of the moduli space for the general semi-positive Fix \( \tau \), which is explained more in [FOOO3] §43.1. We further need to modify the statement of [FOOO2] Proposition X and Theorem M. See §5.

We emphasize that all the theorems stated in introduction of [FOOO1] other than Theorem H [FOOO1] are correct and proved in [FOOO4] (except Theorem G, which is proved in [FOOO6] or [FOOO8]).

3. Errors in the proofs of theorems in [FOOO1].

There was one error in the proof of [FOOO1] Theorem A: In [FOOO1], we asserted that the obstruction classes are contained in the kernel of the natural homomorphism \( H_*(L; \mathbb{Q}) \to H_*(M; \mathbb{Q}) \). This statement is equivalent to the statement in [FOOO2] Theorem C (= [FOOO4] Theorem C) that the obstruction class lies in the quotient of \( H^*(L; \mathbb{Q}) \) by the image of \( H^*(M; \mathbb{Q}) \). However for this statement to hold, we need to consider \( \mathcal{M}_{\text{def}}(L) \) in place of \( \mathcal{M}(L) \). This means that we need to deform the filtered \( A_\infty \) structures of \( L \) by cohomology classes from the ambient symplectic manifold. This deformation, which is called the bulk deformation in [FOOO2], [FOOO4], was not studied in [FOOO1]. The reason of this error in [FOOO1] is that we overlooked the fact that stable map “compactification” of the moduli space of marked pseudo-holomorphic discs, is actually not compact unless we put at least one marked point on the boundary. See [FOOO2] Proposition 13.27 (= [FOOO4] Proposition 3.8.27) and [FOOO2] §32.1 (= [FOOO4] §7.4.1). In summary, the proof of [FOOO1] Theorem A of that statement contains error and should be replaced by the proof of [FOOO2] and [FOOO4].

Another error in [FOOO1] lies in the study of transversality in [FOOO1] Chapter 5 which corresponds to [FOOO2] Chapter 7 or [FOOO4] Chapter 7. We describe where the error lies therein. Let \( P_1 \) be chains on \( L \) and let \( \mathcal{M}_{k+1}(\beta) \) be the moduli
space of pseudo-holomorphic discs with $k + 1$ marked points on the boundary in homology class $\beta \in H_2(M, L)$. To define the $m_k$-operator

$$(P_1, \cdots, P_k) \mapsto m_k(P_1, \cdots, P_k)$$

we need to take the fiber product $\mathcal{M}_{k+1}(\beta) \times_{L^k} (P_1 \times \cdots \times P_k)$ and to use its virtual fundamental chain. In [FOOO1] Chapter 5, we stated that we first perturb $\mathcal{M}_{k+1}(\beta)$ and then take a fiber product and claimed that we can achieve transversality in that way. However the argument for its proof has a gap. The gap lies in the proof of [FOOO1] Theorem 19.6 in [FOOO1] p.171. In [FOOO2] and [FOOO4], we first take the fiber product and then perturb. To make the choices, which we carry out strata-wise, compatible to one another in a precise sense, we need to use a delicate induction argument in our description of the order in which we make the perturbations. This procedure is given in [FOOO2] §30 (= [FOOO4] §7.2) in detail. (There is one more technical point for which we made some corrections in [FOOO4] §7.2 from the arguments used in [FOOO2] §30. See §6 of this document.)

We modify the proof of [FOOO1] Theorem B (= [FOOO2] Theorem I) in [FOOO1] Appendix §A4. The argument of [FOOO1] Appendix §A4 until [FOOO1] Definition A4.32 is correct and is in [FOOO2] §26. However there is a gap in the first line of the proof of [FOOO1] Lemma A4.45. So in [FOOO2] §28.2 we give two proofs of Theorem I (which are different from the argument in [FOOO1] §A4). One of these two proofs is in [FOOO4] subsection 6.5.2.

Another place where some proof is modified is in [FOOO1] Chapter 7 which corresponds to [FOOO3] Chapter 10. Now we are not sure whether the proof in [FOOO1] §29 of [FOOO1] Theorem 26.2 (which corresponds to [FOOO2] Theorem Z) works in general. It certainly works if $w$ in [FOOO2] Theorem Z has the multiplicity one at $p$ (See [FOOO3] Definition 54.19) as we assume in [FOOO2] Theorem Z. In [FOOO1] Theorem 26.2 this assumption was missing. We have checked (in [FOOO3] §56) that this assumption is satisfied for the examples that we apply [FOOO2] Theorem Z in [FOOO3] Chapter 10 and hence the statement in [FOOO2] Theorem Z is enough for the purpose of studying them. In our application in [FOOO3] Chapter 10, [FOOO3] Assumption 54.20 holds.

4. How theorems in [FOOO1] are subsumed, improved, modified, etc. In later version.

Here we make a list of correspondences between the theorems in the published version and the earlier versions:

- [FOOO1] Theorem A is subsumed in the combination of [FOOO2] Theorems C and G (= [FOOO4] Theorems C and G). The homology notation is used in [FOOO1], while we use the cohomology notation in [FOOO2] and [FOOO4]. They are of course equivalent.
• [FOOO1] Theorem G is [FOOO2] Theorem Ks. In [FOOO2] we have added its minor modification, [FOOO2] Theorem K (= [FOOO4] Theorem K), which in fact is an immediate consequence of what was written in [FOOO1].
• [FOOO1] Conjectures I and J are related to [FOOO2] Conjecture B (= [FOOO4] Conjecture R): We now explain the difference between them. In [FOOO1] Conjectures I and J, we made the statement under the assumption that the formal power series defining the boundary operator of Floer cohomology converges. On the other hand, [FOOO2] Conjecture B (= [FOOO4] Conjecture R) is written in the level of formal power series and so can be formulated without assuming the convergence. This is an idea due to Kontsevich-Soibelman [KS]. Their idea is first to prove a rigid analytic geometry version of the homological mirror symmetry conjecture, and then to use a version of GAGA ([Se]) in rigid analytic geometry to prove the convergence. We have learned this idea after we had finished writing [FOOO1]. Since we reckoned their proposal appealing and promising, we have rewritten [FOOO1] Conjectures I and J based on the proposal, whose outcome is [FOOO2] Conjecture B (= [FOOO4] Conjecture R). In summary, if Conjecture B (= [FOOO4] Conjecture R) is proved, then GAGA in rigid analytic geometry may imply [FOOO1] Conjectures I and J. Obviously [FOOO1] Conjectures I and J are stronger than their formal power series or the rigid analytic geometry versions.
• Statements like [FOOO2] Theorems A and F (= [FOOO4] Theorems A and F) were already scattered around in [FOOO1] Chapter 4. More specifically in [FOOO1] Theorem 13.22, [FOOO1] Theorem 14.4, [FOOO1] Theorem 15.41, [FOOO1] Theorem 15.50, [FOOO1] Theorem 16.12 and others. Especially filtered $A_\infty$ algebra and its bar complex were constructed in [FOOO1] without assuming $\mathcal{M}(L) \neq \emptyset$. ([FOOO2] Theorem A itself together with [FOOO2] Theorems T, U (= [FOOO4] Theorems V, W) were announced in [Fu03] with summary of its proof.) However in [FOOO1], we had not proven [FOOO2] Theorem U (= [FOOO4] Theorem W) yet, so the filtered $A_\infty$ algebra was constructed only on a countably generated sub-complex but not yet on the cohomology group of $L$ itself. And the homological algebra of filtered $A_\infty$ algebras used in [FOOO1] is rather limited compared to [FOOO2], [FOOO4]. However a version of [FOOO2] Theorem T (= [FOOO4] Theorem V) was already included in [FOOO1] as Theorem 19.8. In summary, although statements in [FOOO1] Chapter 4 are all correct, the discussion therein was less transparent than the one in the later versions.

There is another reason why we did not pursue in further developing the rather sophisticated homological algebra involved in [FOOO1] Chapters 2 and 3. Around the time of the year 2000 when [FOOO1] appeared, the language of homological algebra of $A_\infty$-structures was still rather unfamiliar to the symplectic geometry community to which we expect the main readers of our book would belong. We also wanted to have concrete applications of the machinery developed in the book visible to the researchers in the related areas, without long delay. Because of these, we took the short cut in [FOOO1] of giving direct geometric constructions rather
than attempting to optimize our exposition by separating the algebraic story of homological algebra of filtered $A_\infty$ algebras from the geometric construction.

At the time of writing [FOOO2] though, 7 years after since the appearance of [FOOO1], the mathematical landscape in symplectic geometry and related areas has undergone significant transformation and usage of higher algebraic structures in Floer theory has become a more common and popular practice in the area. Therefore it is much more natural to present the materials from [FOOO1] in the most logical and natural manner using the natural algebraic language of filtered $A_\infty$-algebras. Reflecting this climate change, we separate most of the algebraic story of homological algebra of filtered $A_\infty$ algebras from the corresponding geometric constructions associated to Lagrangian submanifolds. Geometry shall enter only when it is needed. This makes the whole picture of Lagrangian intersection Floer theory more transparent. The presentation of the [FOOO1] Chapter 4 is rather a mixture of the geometric story and the algebraic story. For example, we did not know in year 2000 how to define the notion of gauge equivalence of bounding cochains in purely algebraic terms and so we used the definition given in [FOOO1] Definition 15.37 of [FOOO1] Chapter 4 by blending geometry and algebra. In the final published version, we give the definition, [FOOO2] Definition 16.1 (= [FOOO4] Definition 4.3.1), in a purely algebraic manner. The bounding chains $B(\beta)$, which played a major role in [FOOO1] Chapter 2, almost disappeared in the later versions. More algebraic discussion of the bounding cochains are given [FOOO2] §11 (= [FOOO4] §3.6).

In [FOOO1] we mentioned that the argument on orientation related to the homotopy unit is deferred to the final version. This is now in [FOOO2] §53 (= [FOOO4] §8.10.1). (The argument in fact turns out to be not difficult.)

5. Corrections etc. related to [FOOO2] Chapters 8 and 10

The main result presented in [FOOO2] Chapter 8 is the Floer theory over the integer. The assumption we put in [FOOO6, FOOO8] for the construction of Floer homology over the integer is different from those stated in the introduction of [FOOO2]. Namely we assume that $(M, \omega)$ is spherically positive in [FOOO6, FOOO8] while semi-positivity of $L$ is assumed in [FOOO2]. On one hand, the result in the former is an improvement in that the new condition is imposed only on the ambient symplectic manifold but independent of $L$. On the other hand, it is weaker than the one claimed in [FOOO2] in that we need the unpleasant condition of existence of spherically positive almost complex structure. The reason why we need to put spherical positivity has something to do with the index calculation in [FOOO3] §35.5.

The proof of [FOOO2] Theorem X or [FOOO3] Theorem 34.11 (which is in [FOOO3] §35) is rather different from those given in [FOOO1] §A3. In [FOOO1] the normally polynomial perturbation is used, but in [FOOO3] §35 we use normally conical perturbation. A normally polynomial perturbation also provides a perturbed moduli space that carries a Whitney stratification, in the way that the dimension of each stratum is as expected. Since this result seems to carry independent interest on its own, we put the manuscript [FOOO9] explaining the story in the first named authors home page. The reason why we do not use normally polynomial perturbations in [FOOO3] §35, although the results and their proofs of this manuscript are correct, is explained at the beginning of [FOOO3] §35.
Chapter 10 becomes bulkier than we originally intended. This is because at some point we decided to include full details of the proof of [FOOO2] Theorem Z, whose proof was only sketched in [FOOO1]. To simplify our exposition, we put an extra hypothesis, [FOOO3] Condition 54.20, on [FOOO2] Theorem Z. We have no doubt that this condition can be removed. For all the cases to which we apply [FOOO2] Theorem Z in [FOOO3], this condition is satisfied. Also as far as applications to the Floer theory are concerned, we can reduce the general case to the case where [FOOO3] Condition 54.20 is satisfied. (See [FOOO3] Corollary 55.15.)

The definition of $\tau$-relatively spin structure in [FOOO3] Chapter 8 of the version has some defect. The correct definition is now given as Definition 44.17 in [FOOO8]. In [FOOO3], we said that $\mathbb{R}P^{2n+1}$ is always $\tau$-relatively spin, which is incorrect. In fact, $\mathbb{R}P^{2n+1}$ is $\tau$-relatively spin if and only if $n$ is odd. (See [FOOO8] Proposition 44.19.) This error is related to the condition whether two relatively spin structures determine the same orientation or not. (See [FOOO8] Definition 44.5.) These points are contained and further discussed in [FOOO7].

6. Corrections etc. from [FOOO2] to [FOOO4]

We added new materials to [FOOO2] §32 which becomes [FOOO4] subsections 7.4.4-7.4.11. Namely we clarify the algebraic background of the operations $p, q, r$ in [FOOO2] §13 (= [FOOO4] section 3.8). (We use the symbol $p_{\ell, k}$ in [FOOO4] instead of $pq_{\ell, k}$ which was used in [FOOO2].) One main outcome of these added sections is [FOOO4] Theorem W and various other results proved in [FOOO4] section 7.4.

There is an important change in the transversality argument. In [FOOO2] Chapter 7, we used a countably generated sub-complexes of the chain complex of currents, which are contained in the set of the currents induced by smooth singular chains. (In other words we identify two smooth singular chains if they define the same current.) In [FOOO4] we use a countably generated sub-complex of the chain complex of smooth singular chains. (In other words, we do not identify two smooth singular chains which induce the same current.) The reason we need this change is that it seems that if we divide the complex of smooth singular chains by the equivalence relation and identify two smooth singular chains which induce the same current, then the resulting complex may not give the usual (say singular) cohomology of our manifold. Namely Proposition A2.13 in page 803 of [FOOO2] does not seem to be correct. An error of its proof is at the end of page 808 of [FOOO2]. Namely the surjectivity of a chain map may not imply surjectivity of the map induced on homology.

The other reason is that it is not clear how to make sense of multi-valued perturbation of the fiber product of the moduli space of bordered stable maps with currents rather than smooth singular chains.

There was an error in [FOOO2] Lemma A1.58. So we proved [FOOO4] Lemma A1.58 in stead.

As we mentioned at the end of [FOOO4] subsection A1.4, the forgetful map of boundary marked point is not necessary smooth. We use the forgetful map of boundary marked point only during the discussion of homotopy unit in [FOOO4] §7.3. We can take care of this trouble in the way explained at end of [FOOO4] §A1.4. We made some changes to [FOOO4] §7.3 accordingly.
REFERENCES


[FOOO8] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, Modified version of Chapter 8 [FOOO3].


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