1 Uniquely Colorable Graphs

A complete $k$-partite graph has only one partition into $k$ color classes. Thus it has only one way to be colored, provided that we don’t care which particular colors are used.

**Definition 1.** A graph is uniquely $k$-colorable if any $k$-coloring produces the same vertex partition. A graph is uniquely colorable if any minimum coloring produces the same vertex partition.

Adding edges consistent with a minimum coloring of a graph limits the possible minimum colorings, until eventually the graph is uniquely colorable. Thus uniquely $k$-colorable graphs are a larger class containing maximal $k$-chromatic graphs. A sufficiently large size guarantees a graph is uniquely $k$-colorable.

**Theorem 2.** (Bollobas [1978]) If $G$ is a $k$-colorable graph with $\delta (G) > \frac{3k-5}{3k-2}n$, then $G$ is uniquely $k$-colorable.

**Proof.** We use induction on $k$. Let $k = 2$. If $G$ is not connected, let $H$ be a component of $G$ of order $m \leq \frac{n}{2}$. Then $\delta (H) > \frac{m}{2}$, so $H$ contains a triangle. But this is impossible, so $G$ is connected and uniquely 2-colorable.

Now let $k \geq 3$ and suppose the result holds for smaller values of $k$. Given $x \in V(G)$, let $G_x = G[N(x)]$. Denote the order of $G_x$ by $n_x$. Then $n_x > \frac{3k-5}{3k-2}n$, and

$$d_{G_x}(y) \geq \frac{3k-5}{3k-2}n - (n - n_x) = n_x - \frac{3}{3k-2}n > \frac{3(k-1)-5}{3(k-1)-2}n_x.$$  

Therefore by the induction hypothesis $G_x$ is uniquely $k-1$-colorable.

Now let $u_1$ and $u_2$ be vertices of $G$. As $d(u_i) > \frac{3k-5}{3k-2}n \geq \frac{3}{2}n > \frac{1}{7}n$, there is a vertex $x$ adjacent to both $u_1$ and $u_2$, so $u_1$ and $u_2$ belong to $G_x$. Now a $k$-coloring of $G$ always gives a $k-1$-coloring of $G_x$. As this $k-1$-coloring is unique, either $u_1$ and $u_2$ get the same color or they get different colors, independently of the $k$-coloring of $G$. Thus $G$ is uniquely colorable.

One way to produce a uniquely colorable graph is to have many overlapping cliques. Trees are uniquely colorable, as every edge corresponds to a 2-clique. If $G$ is uniquely $k$-colorable, and a vertex $v$ of degree $k-1$ is added so that it is adjacent to vertices in all but one color class, the new graph is uniquely $k$-colorable. Thus $k$-trees are uniquely colorable. Adding a vertex with degree less than $k-1$ yields more than one possible color for this vertex, so a uniquely $k$-colorable graph $G$ has $\delta(G) \geq k-1$.

**Lemma 3.** (HHR [1969]) In a uniquely colorable graph, any two color classes induce a connected graph.

**Proof.** If not, the colors could be exchanged on one component of the graph they induce.

Bollobas [1978] showed that any graph with $\delta(G) > \frac{k-2}{k-1}n$ so that each pair of color classes induce a connected subgraph is uniquely $k$-colorable.

**Theorem 4.** (Shao ji [1990]) A uniquely $k+1$-colorable graph has $m \geq kn - \frac{k(k+1)}{2}$.

**Proof.** Let $G$ be uniquely $k+1$-colorable with color classes $V_i$. Each edge of $G$ is in exactly one subgraph induced by two color classes. Thus

$$m(G) = \sum_{i \neq j} m(G[V_i \cup V_j])$$

$$\geq \sum_{i \neq j} (|V_i \cup V_j| - 1)$$

$$= \sum_{i \neq j} |V_i \cup V_j| - \binom{k+1}{2}$$

$$= kn - \frac{k(k+1)}{2}.$$  

Uniquely colorable maximal $k$-degenerate graphs have $m = kn - \frac{k(k+1)}{2}$. Note that any uniquely 4-colorable graph has $m \geq 3n - 6$. Thus any uniquely 4-colorable planar graph is maximal planar (Chartrand/Geller [1969]). A natural example of such graphs are planar 3-trees.

**Theorem 5.** (Fowler [1998]) Every uniquely 4-colorable planar graph is a 3-tree.
The proof of this theorem uses techniques from the proof of the Four Color Theorem. In fact, Fowler showed that any maximal planar graph that is not a 3-tree has at least two 4-colorings, which generalizes the Four Color Theorem.

Any 3-colorable maximal planar graph is uniquely 3-colorable, but maximality is not required. Aksionov [1977] showed that any uniquely 3-colorable planar graph with \( n \geq 5 \) has at least three triangles, and all such graphs are maximal 2-degenerate. LZSX [2017] proved that any uniquely 3-colorable planar graph with at most four triangles has two adjacent triangles.

Surprisingly, there are also triangle-free uniquely 3-colorable graphs. The example below has one edge deleted from the Chvátal graph (HHR [1969]).

![Graph](image)

This result has been generalized to large girths using a probabilistic argument.

**Theorem 6.** (Nesetril [1973], Erdős [1974], Bollobás/Sauer [1976]) For all \( k \geq 2 \) and \( g \geq 3 \) there is a uniquely \( k \)-colorable graph with girth at least \( g \).

The graph above has \( m > 2n - 3 \). AMS [2001] showed that the following graph is uniquely 3-colorable, triangle-free, and has \( m = 2n - 3 \).

![Graph](image)

Li/Xu [2016] found an example of such a graph with order 16.

**Exercises**

1. Characterize uniquely 2-colorable graphs.
2. Determine which graphs in the following classes are uniquely colorable.
   a. \( K_n \)
   b. \( C_n \)
   c. \( W_n \)
3. Show that if \( G \) is \( k \)-critical and uniquely \( k \)-colorable, then \( G = K_k \).
4. Prove or disprove: If \( G \) is \((k + 1)\)-critical, then \( G - e \) is uniquely \( k \)-colorable for any edge \( e \).
5. (Bollobás [1978]) Show that the bound in Theorem 2 is sharp for all \( k \).
6. (Bollobás [1978]) Show that the graph \( H \) formed from \( K_3 \square K_2 \) by substituting \( K_1 \) for each vertex. Form \( G \) by joining \( H \) to \( K_{3 \ldots, 3} \) (there are \( k - 3 \) partite sets). Show that this graph is not uniquely \( k \)-colorable and \( \delta(G) = \frac{k^2 + k}{2} n \). (Note: Bollobás showed that any graph with larger minimum degree so that each pair of color classes induce a connected subgraph is uniquely \( k \)-colorable.)
7. Prove or disprove: A maximal \( k \)-degenerate graph is uniquely colorable if and only if it is a \( k \)-tree.
8. Prove or disprove: A chordal graph is uniquely colorable if and only if it is a \( k \)-tree.

9. Let \( G \) be a uniquely \( k \)-colorable graph with \( d(v) = k - 1 \) for some vertex \( v \). Show that \( G - v \) is uniquely colorable.

10. Show that a uniquely \( k \)-colorable graph \( G \) has \( \kappa(G) \geq k - 1 \).

11. Show that \( G \) and \( H \) are uniquely colorable graphs if and only if \( G + H \) is uniquely colorable.

12. (Li/Xu [2016]) Let \( G_1 \) and \( G_2 \) be uniquely 3-colorable graphs and \( u_i \) and \( v_i \) be differently colored vertices in the unique coloring of \( G_i, i \in \{1, 2\} \).
   a. Form \( G \) by identifying \( u_1 \) with \( u_2 \) and \( v_1 \) with \( v_2 \). Show that \( G \) is uniquely 3-colorable.
   b. Form \( G \) by identifying \( u_1 \) with \( u_2 \) and adding edge \( v_1v_2 \). Show that \( G \) is uniquely 3-colorable.
   c. Describe a method for constructing a uniquely 3-colorable graph from \( G_1 \cup G_2 \) by adding three edges.

13. (Chartrand/Geller [1969]) Show that an outerplanar graph \( G \) of order \( n \geq 3 \) is uniquely 3-colorable if and only if \( G \) is maximal outerplanar.

14. (Chartrand/Geller [1969]) Show that if a 2-connected 3-chromatic plane graph \( G \) and at most one region of \( G \) is not a triangle, then \( G \) is uniquely 3-colorable. Show that this statement is false if there are two non-triangular regions.

15. Verify that the graph formed by deleting one edge from the outer 4-cycle of the Chvatal graph is uniquely 3-colorable.

16. (Chao/Chen [1993]) Show that for all \( n \geq 12 \), there is a uniquely 3-colorable triangle-free graph.

17. (Chao/Chen [1993]) Show that there is a 5-regular uniquely 3-colorable triangle-free graph of order 24.

18. Prove or disprove: There is a cubic uniquely 3-colorable graph.

19. + (Li/Xu [2016]) Show that the following graph is uniquely 3-colorable, triangle-free, and has \( m = 2n - 3 \). (Note: Li/Xu use a construction based on this graph to show that there are infinitely many 4-regular uniquely 3-colorable triangle-free graphs.)

20. + Verify that the graph of AMS [2001] is uniquely 3-colorable.

References


