These pages consist of some additional references, errata as of May 27, 2012, and a remark on the 7–dimensional case.

1. References on page 175.

It turned out that Hans Zassenhaus did rework his 1934 thesis "Über endliche Fastkörper" some fifty years after mistakes were found. See his papers

On Frobenius Groups, I, Results in Math. 8 (1985) 132-145

and

On Frobenius Groups, II, Results in Math. 11 (1987) 317-358.

My thanks to Prof. Ming-Chang Kang (National Taiwan University) for bringing this to my attention.

After I completed "Spaces of Constant Curvature" in 1965, including the classification of finite fixed point free groups, Daniel Gorenstein and I briefly discussed the problem of referring to Zassenhaus' paper. Some twenty years later, at Gorenstein's suggestion, Zassenhaus published his corrected version cited above.

A bit after the first publication of "Spaces of Constant Curvature", Donald Passman published a different proof of the classification of finite fixed point free groups in his book "Permutation Groups", W. A. Benjamin 1968 and Dover 2012.

In 1985 I received a manuscript "On the structure of finite groups with periodic cohomology" by C. B. Thomas and C. T. C. Wall, in which they gave yet another proof of the classification of finite fixed point free groups. It was not published at that time, but that paper was rewritten in late 2010 by C. T. C. Wall and should appear in 2012.

2. Errata for Section 7.5.

In passage from the second edition to the third edition of "Spaces of Constant Curvature", some entries from the list of 7–dimensional spherical space forms in Section 7.5 were lost due to an editing error. The general classification is correct in all editions, and Section 7.5 simply shows how it works out in low dimensions. The correct list is contained in the following pages, which replace the material of Section 7.5.

3. A comment on S^7.

A method using quaternions to classify the 3–dimensional spherical space forms is described near the end of the replacement pages here. At the end of those replacement pages we explain why this does not work very well for octonions and 7–dimensional spherical space forms.

7.5 Spherical space forms of low dimension

In order to illustrate our classification of the spherical space forms, we will list the spherical space forms of dimensions $q < 13$.

If q is even, one has only the sphere and the elliptic space. Now assume that q is odd. We will list the q–dimensional spherical space forms not isometric to the sphere nor to the elliptic space. We write $R(\theta)$ for the 2×2 matrix $\begin{bmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{bmatrix}$ and $S(\theta,\varphi)$ for the 4×4 matrix $\begin{bmatrix} 0 & R(\varphi) \\ R(\theta) & 0 \end{bmatrix}$, and $R(t_1,\dots t_s)$ for the $2s\times 2s$ matrix $diag\{R(t_1),\dots,R(t_s)\}$.

Let $\mathbf{q = 3}$. The divisors of $(q+1)/2$ are 1 and 2. That gives us

type of G	d	$\delta(G)$	resulting spherical space forms
I	1	1	$\mathbf{S}^3/\Gamma$, $\Gamma = \left\langle R(\frac{1}{n},\frac{a}{n})\right\rangle$ cyclic, $n>2$, $(a,n)=1$
I	2	2	$\mathbf{S}^3/\Gamma$ where $\Gamma = \hat{\pi}_{k,l}(\mathbf{Z}_u \times \mathbf{D}^*_{2v-1})$
II	1	2	$\mathbf{S}^3/\Gamma$ where $\Gamma = \hat{\alpha}_{k,\ell}(G)$
II	2	2	$\mathbf{S}^3/\Gamma$ where $\Gamma = \hat{\beta}_{k,l}(\mathbf{Z}_u \times \mathbf{D}^*_{2v})$
III	1	2	$\mathbf{S}^3/\Gamma$ where $\Gamma = (\hat{\beta}\otimes\hat{\tau})(\mathbf{Z}_u \times \mathbf{T}^*_v)$, $(6,u)=1$
IV	1	2	$\mathbf{S}^3/\Gamma$ where $\Gamma = (\hat{\beta}\otimes\hat{o})(\mathbf{Z}_u \times \mathbf{O}^*)$, $(6,u)=1$
V	1	2	$\mathbf{S}^3/\Gamma$ where $\Gamma = (\hat{\beta}\otimes\hat{\iota})(\mathbf{Z}_u \times \mathbf{I}^*)$, $(30,u)=1$

In the first entry of type II, Γ is generated $R(1/m,k/m)$, $S(2\ell/n)$, and either $R(1/4,-1/4)$ or $S(-\ell/n,\ell/n)$. In the entries for types III, IV and V the space form is independent of choice of $\beta\in\mathfrak{F}_C(\mathbf{Z}_u)$, and it is independent of choice of $\tau\in\mathfrak{F}_C(\mathbf{T}^*_v)$, $o\in\mathfrak{F}_C(\mathbf{O}^*)$, or $\iota\in\mathfrak{F}_C(\mathbf{I}^*)$, as well.

Let $\mathbf{q = 5}$. The divisors of $(q+1)/2$ are 1 and 3. If $d=1$ then G is of type I because otherwise $\delta(G)=3d$, which never happens. If $d=3$ then $\delta(G)=d$, so either G is of type I or G is of type II with $3=d\equiv 2(4)$. The latter is not possible. Now the five dimensional spherical space forms are

type of G	d	$\delta(G)$	resulting spherical space forms
I	1	1	$\mathbf{S}^5/\Gamma$, $\Gamma = \left\langle R(\frac{1}{n},\frac{a_1}{n},\frac{a_2}{n})\right\rangle$ cyclic, $n>2$, $(a_i,n)=1$
I	1	3	$\mathbf{S}^5/\Gamma$ where $\Gamma = \hat{\psi}_{k,\ell}(G)$

In the second case above, G is given by

$$A^m = B^n = 1\,, BAB^{-1} = A^r \text{ with } n\equiv 0\,(9), (n(r-1),m)=1, r\not\equiv r^3\equiv 1\,(m).$$

with Γ generated by $R(\frac{1}{m},\frac{r}{m},\frac{r^2}{m})$ and $\begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ R(\frac{3\ell}{n}) & 0 & 0 \end{bmatrix}$ where $(\ell,n/3)=1$.

Let $\mathbf{q} = \mathbf{7}$. The divisors of $(q+1)/2$ are 1, 2 and 4. That gives us

type of G	d	$\delta(G)$	resulting spherical space forms
I	1	1	$\mathbf{S}^7/\Gamma$, $\Gamma = \langle R(\frac{1}{n}, \frac{a_1}{n}, \frac{a_2}{n}, \frac{a_3}{n}) \rangle$ cyclic, $n > 2$ and each $(a_i, n) = 1$
I	2	2	$\mathbf{S}^7/\Gamma$ where $\Gamma = (\hat{\pi}_{k_1,\ell_1} \oplus \hat{\pi}_{k_2,\ell_2})(G)$
I	4	4	$\mathbf{S}^7/\Gamma$ where $\Gamma = \hat{\pi}_{k,\ell}(G)$
II	1	2	$\mathbf{S}^7/\Gamma$ where $\Gamma = (\hat{\alpha}_{k_1,\ell_1} \oplus \hat{\alpha}_{k_2,\ell_2})(G)$
II	2	2	$\mathbf{S}^7/\Gamma$ where $\Gamma = (\hat{\beta}_{k_1,\ell_1} \oplus \hat{\beta}_{k_2,\ell_2})(G)$
II	2	4	$\mathbf{S}^7/\Gamma$ where $\Gamma = \hat{\alpha}_{k,\ell}(G)$
III	1	2	$\mathbf{S}^7/\Gamma$ where $\Gamma = (\hat{\nu}_{k_1,\ell_1} \oplus \hat{\nu}_{k_2,\ell_2})(G)$
III	1	2	$\mathbf{S}^7/\Gamma$ where $\Gamma = (\hat{\nu}_{k_1,\ell_1,j_1} \oplus \hat{\nu}_{k_2,\ell_2,j_2})(G)$
III	2	4	$\mathbf{S}^7/\Gamma$ where $\Gamma = \hat{\nu}_{k,\ell}(G)$
III	2	4	$\mathbf{S}^7/\Gamma$ where $\Gamma = \hat{\nu}_{k,\ell,j}(G)$
IV	1	2	$\mathbf{S}^7/\Gamma$ where $\Gamma = (\hat{\psi}_{k_1,\ell_1,j_1} \oplus \hat{\psi}_{k_2,\ell_2,j_2})(G)$
IV	1	4	$\mathbf{S}^7/\Gamma$ where $\Gamma = \hat{\gamma}_{k,\ell,j}(G)$
IV	2	4	$\mathbf{S}^7/\Gamma$ where $\Gamma = \hat{\psi}_{k,\ell}(G)$
IV	2	4	$\mathbf{S}^7/\Gamma$ where $\Gamma = \hat{\psi}_{k,\ell,j}(G)$
IV	2	4	$\mathbf{S}^7/\Gamma$ where $\Gamma = \hat{\xi}_{k,\ell}(G)$
IV	2	4	$\mathbf{S}^7/\Gamma$ where $\Gamma = \hat{\xi}_{k,\ell,j}(G)$
V	1	2	$\mathbf{S}^7/\Gamma$ where $\Gamma = (\hat{\iota}_{k_1,\ell_1,j_1} \oplus \hat{\iota}_{k_2,\ell_2,j_2})(G)$
VI	1	4	$\mathbf{S}^7/\Gamma$ where $\Gamma = \hat{\kappa}_{k,\ell,j}(G)$

Let $\mathbf{q} = \mathbf{9}$. Then 1 and 5 are the divisors of $(q+1)/2$. That gives us

type of G	d	$\delta(G)$	resulting spherical space forms
I	1	1	$\mathbf{S}^9/\Gamma$, $\Gamma = \langle R(\frac{1}{n}, \frac{a_1}{n}, \frac{a_2}{n}, \frac{a_3}{n}, \frac{a_4}{n}) \rangle$ cyclic
I	5	5	$\mathbf{S}^9/\Gamma$, where $\Gamma = \hat{\psi}_{k,\ell}(G)$

where $n > 2$ and each $(a_i, n) = 1$ in the first entry.

Let $\mathbf{q} = \mathbf{11}$. Then 1, 2, 3 and 6 are the divisors of $(q+1)/2$. That gives

type of G	d	$\delta(G)$	resulting spherical space forms
I	1	1	$\mathbf{S}^{11}/\Gamma$, $\Gamma = \langle R(\frac{1}{n}, \frac{a_1}{n}, \frac{a_2}{n}, \frac{a_3}{n}), \frac{a_4}{n}, \frac{a_5}{n}) \rangle$, cyclic, $n > 2$ and each $(a_i, n) = 1$
I	2	2	$\mathbf{S}^{11}/\Gamma$ where $\Gamma = (\hat{\pi}_{k_1,\ell_1} \oplus \hat{\pi}_{k_2,\ell_2} \oplus \hat{\pi}_{k_3,\ell_3})(G)$
I	3	3	$\mathbf{S}^{11}/\Gamma$ where $\Gamma = (\hat{\pi}_{k_1,\ell_1} \oplus \hat{\pi}_{k_2,\ell_2})(G)$
I	6	6	$\mathbf{S}^{11}/\Gamma$ where $\Gamma = \hat{\pi}_{k,\ell}(G)$
II	1	2	$\mathbf{S}^{11}/\Gamma$ where $\Gamma = (\hat{\alpha}_{k_1,\ell_1} \oplus \hat{\alpha}_{k_2,\ell_2} \oplus \hat{\alpha}_{k_3,\ell_3})(G)$
II	3	6	$\mathbf{S}^{11}/\Gamma$ where $\Gamma = \hat{\alpha}_{k,\ell}(G)$
III	1	2	$\mathbf{S}^{11}/\Gamma$ where $\Gamma = (\hat{\nu}_{k_1,\ell_1,} \oplus \hat{\nu}_{k_2,\ell_2,} \oplus \hat{\nu}_{k_3,\ell_3,})(G)$
III	1	2	$\mathbf{S}^{11}/\Gamma$ where $\Gamma = (\hat{\nu}_{k_1,\ell_1,j_1} \oplus \hat{\nu}_{k_2,\ell_2,j_2} \oplus \hat{\nu}_{k_3,\ell_3,j_3})(G)$
III	3	6	$\mathbf{S}^{11}/\Gamma$ where $\Gamma = \hat{\mu}_{k,\ell}(G)$
IV	1	2	$\mathbf{S}^{11}/\Gamma$ where $\Gamma = (\hat{\psi}_{k_1,\ell_1,j_1} \oplus \hat{\psi}_{k_2,\ell_2,j_2} \oplus \hat{\psi}_{k_3,\ell_3,j_3})(G)$
V	1	2	$\mathbf{S}^{11}/\Gamma$ where $\Gamma = (\hat{\iota}_{k_1,\ell_1,j_1} \oplus \hat{\iota}_{k_2,\ell_2,j_2} \oplus \hat{\iota}_{k_3,\ell_3,j_3})(G)$

It is interesting to survey the earlier work on three dimensional spherical space forms, for in dimension $q = 3$ the classification has been known for some time and does not depend on our heavy machinery.

The first deep study of three dimensional spherical space forms was made by H. Hopf [1], who saw the connection with the binary dihedral and binary polyhedral groups.

The main step was taken by W. Threlfall and H. Seifert in their studies [1] and [2] of the topology of three dimensional spherical space forms. The first half of (Threlfall–Seifert [1]) is a classification of three dimensional spherical space forms based on the observation that the rotation group $\mathbf{SO}(4)$ is locally isomorphic to $\mathbf{SO}(3) \times \mathbf{SO}(3)$. Let $\pi : \mathbf{Spin}(n) \to \mathbf{SO}(n)$ denote the universal covering group for $n > 2$. Then π is a two to one epimorphism. In a manner which we will explain shortly, one has an isomorphism $\psi : \mathbf{Spin}(4) \to \mathbf{Spin}(3) \times \mathbf{Spin}(3)$. Let $\psi_i (i = 1, 2)$ denote the composition of ψ with projection onto the i-th factor. Given a finite subgroup $\Gamma \subset \mathbf{SO}(4)$, one has finite subgroups $\Gamma_i = \psi_i(\pi^{-1}\Gamma) \subset \mathbf{Spin}(3)$. As $\mathbf{Spin}(3) \cong \mathbf{SU}(2) \cong$ (unit quaternions), all of its finite subgroups are known. Thus one knows the possibilities for Γ_1 and Γ_2. Using this, one finds all finite subgroups $\Gamma \subset \mathbf{SO}(4)$. It is then easily decided which ones act freely on $\mathbf{S}^3$.

The final step on the classification of three dimensional spherical space forms was taken by A. Hattori [1] in his reformulation of the Threlfall-Seifert classification by means of quaternions. Given unit quaternions a and b one defines $F(a, b) : q \mapsto aqb^{-1}, q$ any quaternion. Then F is a two to one homomorphism of $\mathbf{Spin}(3) \times \mathbf{Spin}(3)$ onto $\mathbf{SO}(4)$, the kernel of F being $K = \{(1, 1), (-1, -1)\}$. The isomorphism ψ above is given by $F = \pi.\psi^{-1}$. If Γ is a finite subgroup of $\mathbf{SO}(4)$, then we define $\Gamma^* = F^{-1}(\Gamma)$ and we let Γ_i denote the projection of Γ^* to the i-th factor $\mathbf{Spin}(3)$. Then $F(a, b)q = aqb^{-1} = q$ if and only if $a = qbq^{-1}$, so $F(a, b)$ has a fixed point on $\mathbf{S}^3$ if and only if a is conjugate to b in $\mathbf{Spin}(3)$. Now Γ acts freely on $\mathbf{S}^3$ if and only if Γ^* has no element $(a, b) \notin K$ such that a is conjugate to b. For example, if Γ acts freely on $\mathbf{S}^3$ then a and b cannot both be of order 3 and cannot both be of order 4. As the possible groups Γ_i are known and the more complicated ones have many elements of orders 3 and 4, one can then give a quick classification of the conjugacy classes of finite subgroups of $\mathbf{SO}(4)$ which acts freely on $\mathbf{S}^3$.

This fails dramatically for the octonion algebra $\mathbb{O} = \mathbb{H} + \ell\mathbb{H}$. S^7 is the loop $\mathbb{O}' = \{x \in \mathbb{O} \mid x\bar{x} = 1\}$ of unit octonions. If Γ is a finite subloop then the cosets Γx, (= orbits $F(\Gamma, 1)x$ on S^7) need not be disjoint, so the quotient space need not be a manifold. This happens for $\Gamma = \{\pm 1, \pm i, \pm j, \pm k\}$; the orbits of $x = \frac{1}{\sqrt{2}}(1 + \ell)$ and $y = \frac{1}{\sqrt{2}}(i - \ell i)$ intersect but do not coincide.